INEQUALITIES OF JENSEN'S TYPE FOR K-BOUNDED MODULUS CONVEX COMPLEX FUNCTIONS

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Abstract. Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. We say that the function $f : D \subset \mathbb{C} \to \mathbb{C}$ is called *K-bounded modulus convex*, for the given $K > 0$, if it satisfies the condition

$$|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)| \leq \frac{1}{2} K \lambda (1-\lambda) |x-y|^2$$

for any $x, y \in D$ and $\lambda \in [0,1]$.

In this paper we establish some new Jensen's type inequalities for the complex integral on γ, a smooth path from $\mathbb{C}$ and K-bounded modulus convex functions. Some examples for the complex exponential and complex logarithm are also given.

1. Introduction

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. In the recent paper [3] we introduced the following class of functions:

Definition 1. *A mapping $F : C \subset X \to Y$ is called K-bounded norm convex, for some given $K > 0$, if it satisfies the condition*

$$\|(1-\lambda) F(x) + \lambda F(y) - F((1-\lambda) x + \lambda y)\|_Y \leq \frac{1}{2} K \lambda (1-\lambda) \|x-y\|_X^2 \tag{1.1}$$

for any $x, y \in C$ and $\lambda \in [0,1]$. For simplicity, we denote this by $F \in \mathcal{BN}_K(C)$.

We have from (1.1) for $\lambda = \frac{1}{2}$ the Jensen's inequality

$$\left\| \frac{F(x) + F(y)}{2} - F\left(\frac{x+y}{2}\right) \right\|_Y \leq \frac{1}{8} K \|x-y\|_X^2$$

for any $x, y \in C$.

We observe that $\mathcal{BN}_K(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y.

In the same paper [3], we obtained the following result which provides a large class of examples of such functions.

Theorem 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $F : C \to Y$ a twice-differentiable mapping on C. Then for any $x, y \in C$ and $\lambda \in [0,1]$ we have*

$$\|(1-\lambda) F(x) + \lambda F(y) - F((1-\lambda) x + \lambda y)\|_Y \leq \frac{1}{2} K \lambda (1-\lambda) \|y-x\|_X^2, \tag{1.2}$$

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where

$$K_{F''} := \sup_{z\in C} \|F''(z)\|_{\mathcal{L}(X^2;Y)} \tag{1.3}$$

is assumed to be finite, namely $F \in \mathcal{BN}_{K_{F''}}(C)$.

We have the following inequalities of Hermite-Hadamard type [3]:

Theorem 2. *Let* $(X; \|\cdot\|_X)$ *and* $(Y; \|\cdot\|_Y)$ *be two normed linear spaces over the complex number field* $\mathbb{C}$ *with* Y *complete. Assume that the mapping* $F : C \subset X \to Y$ *is continuous on the convex set* C *in the norm topology. If* $F \in \mathcal{BN}_K(C)$ *for some* $K > 0$, *then we have*

$$\left\| \frac{F(x) + F(y)}{2} - \int_0^1 F((1-\lambda)x + \lambda y)\, d\lambda \right\|_Y \le \frac{1}{12} K \|x - y\|_X^2 \tag{1.4}$$

and

$$\left\| \int_0^1 F((1-\lambda)x + \lambda y)\, d\lambda - F\left(\frac{x+y}{2}\right) \right\|_Y \le \frac{1}{24} K \|x - y\|_X^2 \tag{1.5}$$

for any $x, y \in C$.

The constants $\frac{1}{12}$ *and* $\frac{1}{24}$ *are best possible.*

Following [1, p. 59], let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, Ω an open subset of X and $F : \Omega \to Y$. If $a \in \Omega$, $u \in X \setminus \{0\}$ and if the limit

$$\lim_{t\to 0} \frac{1}{t}[F(a + tu) - F(a)]$$

exists, then we denote this derivative $\partial_u F(a)$. It is called the directional derivative of F at a in the direction u. If the directional derivative is defined in all directions and there is a continuous linear mapping Φ from X into Y such that for all $u \in X$

$$\partial_u F(a) = \Phi(u),$$

then we say that F is Gâteaux-differentiable at a and that Φ is the Gâteaux differential of F at a. If a mapping F is differentiable at a point a, then clearly all its directional derivatives exist and we have

$$\partial_u F(a) = F'(a)u, \ u \in X.$$

Thus F is Gâteaux-differentiable at a. However, the Gâteaux differential may exist without the differential existing. The existence of directional derivatives at a point does not imply that the mapping is Gâteaux-differentiable. To distinguish the differential from the Gâteaux differential, the differential is often referred as the Fréchet differential.

In an earlier and more comprehensive version of [3], see [2], we also obtained the following Jensen's type discrete inequality:

Theorem 3. *Let* $(X; \|\cdot\|_X)$ *and* $(Y; \|\cdot\|_Y)$ *be two normed linear spaces over the complex number field* $\mathbb{C}$. *Assume that the mapping* $F : C \subset X \to Y$ *is defined on the open convex set* C *and* $F \in \mathcal{BN}_K(C)$ *for some* $K > 0$. *If* $x_k \in C$, $p_k \ge 0$ *for* $k \in \{1, ..., n\}$ *with* $\sum_{k=1}^n p_k = 1$ *and* F *is Gâteaux-differentiable at* $\sum_{k=1}^n p_k x_k \in$

C, then for any $y_j \in C$ and $q_j \geq 0$ for $j \in \{1, ..., m\}$ with $\sum_{j=1}^{m} q_j = 1$ and $\sum_{j=1}^{m} q_j y_j = \sum_{k=1}^{n} p_k x_k$ we have

$$\left\| \sum_{j=1}^{m} q_j F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \frac{1}{2} K \sum_{j=1}^{m} q_j \left\| y_j - \sum_{k=1}^{n} p_k x_k \right\|_X^2. \tag{1.6}$$

In particular, we have

$$\left\| \sum_{j=1}^{n} p_j F(x_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \frac{1}{2} K \sum_{j=1}^{n} p_j \left\| x_j - \sum_{k=1}^{n} p_k x_k \right\|_X^2. \tag{1.7}$$

If $(X; \langle \cdot, \cdot \rangle)$ is an inner product space, then

$$\sum_{j=1}^{n} p_j \left\| x_j - \sum_{k=1}^{n} p_k x_k \right\|_X^2 = \sum_{j=1}^{n} p_j \|x_j\|_X^2 - \left\| \sum_{k=1}^{n} p_k x_k \right\|_X^2$$

and by (1.7) we have

$$\left\| \sum_{j=1}^{n} p_j F(x_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \frac{1}{2} K \left[\sum_{j=1}^{n} p_j \|x_j\|_X^2 - \left\| \sum_{k=1}^{n} p_k x_k \right\|_X^2 \right]. \tag{1.8}$$

Corollary 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $F : C \to Y$ a twice-differentiable mapping on C. If $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$\begin{aligned} &\left\| \sum_{j=1}^{n} p_j F(x_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \\ &\qquad \leq \frac{1}{2} \sup_{z \in C} \|F''(z)\|_{\mathcal{L}(X^2;Y)} \sum_{j=1}^{n} p_j \left\| x_j - \sum_{k=1}^{n} p_k x_k \right\|_X^2. \end{aligned} \tag{1.9}$$

Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Following Definition 1, we say that the function $F : D \subset \mathbb{C} \to \mathbb{C}$ is called K-*bounded modulus convex*, for the given $K > 0$, if it satisfies the condition

$$|(1-\lambda) F(x) + \lambda F(y) - F((1-\lambda) x + \lambda y)| \leq \frac{1}{2} K \lambda (1-\lambda) |x - y|^2 \tag{1.10}$$

for any x, $y \in D$ and $\lambda \in [0, 1]$. For simplicity, we denote this by $F \in \mathcal{BM}_K(D)$.

All the above results can be translated for complex functions defined on convex subsets $D \subset \mathbb{C}$.

In the following, in order to obtain several inequalities for the complex integral, we need the following facts.

Suppose γ is a smooth path from $\mathbb{C}$ parametrized by $z(t)$, $t \in [a, b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with u, $w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := zz$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in D, and open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.11}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.12}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In the recent paper [5] we obtained the following results:

Theorem 4. *Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that f is holomorphic on D and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then*

$$\left|\int_\gamma f(z)\,dz - \left[f(v) + f'(v)\left(\frac{w+u}{2} - v\right)\right](w-u)\right| \le \frac{1}{2}K\int_\gamma |z-v|^2\,|dz| \tag{1.13}$$

and

$$\left|\frac{1}{2}\left[f(w)(w-v)+f(u)(v-u)+f(v)(w-u)\right]-\int_{\gamma}f(z)\,dz\right| \le \frac{1}{4}K\int_{\gamma}|z-v|^{2}|dz|. \tag{1.14}$$

Motivated by the above results, in this paper we establish some new Jensen's type inequalities for the complex integral on γ, a smooth path from $\mathbb{C}$ and K-bounded modulus convex functions. Some examples for the complex exponential and complex logarithm are also given.

2. General Integral Inequalities

We have:

Theorem 5. *Let $G\subset\mathbb{C}$ be a convex domain of complex numbers and $K>0$ and that F is holomorphic on G with $F\in\mathcal{BM}_K(G)$. Assume also that $f:D\to G$ is continuous on D, $\gamma\subset D$ parametrized by $z(t)$, $t\in[a,b]$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ with $w\neq u$ and $\frac{1}{w-u}\int_\gamma f(z)\,dz\in G$, then*

$$\left|\frac{1}{w-u}\int_{\gamma}(F\circ f)(v)\,dv-F\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\right| \le \frac{1}{2}K\frac{1}{|w-u|}\int_{\gamma}\left|f(v)-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right|^{2}|dv|. \tag{2.1}$$

Proof. Let x, $y\in G$. Since $F\in\mathcal{BM}_K(G)$, then we have

$$|F((1-\lambda)x+\lambda y)-F(x)+\lambda[F(x)-F(y)]|\le\frac{1}{2}K\lambda(1-\lambda)|x-y|^{2}$$

that implies that

$$\left|\frac{F(x+\lambda(y-x))-F(x)}{\lambda}+F(x)-F(y)\right|\le\frac{1}{2}K(1-\lambda)|x-y|^{2}$$

for $\lambda\in(0,1)$.

Since F is holomorphic on G, then by letting $\lambda\to0+$, we get

$$|F'(x)(y-x)+F(x)-F(y)|\le\frac{1}{2}K|x-y|^{2}$$

that is equivalent to

$$|F(y)-F(x)-F'(x)(y-x)|\le\frac{1}{2}K|y-x|^{2} \tag{2.2}$$

for all x, $y\in G$.

If we take in (2.2) $x=\frac{1}{w-u}\int_\gamma f(z)\,dz$, then we get

$$\left|F(y)-F\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)-F'\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\left(y-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)\right| \le\frac{1}{2}K\left|y-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right|^{2} \tag{2.3}$$

for all $y \in G$.

If we take in this inequality $y = f(v)$, $v \in \gamma$, then we get

$$\begin{aligned} (2.4)\quad & \left|(F \circ f)(v) - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right. \\ & \left. - F'\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\left(f(v) - \frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right| \\ & \leq \frac{1}{2}K\left|f(v) - \frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2 \end{aligned}$$

for all $v \in \gamma$.

We have

$$\begin{aligned} (2.5)\quad & \frac{1}{w-u}\int_\gamma \left[(F \circ f)(v) - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right. \\ & \left. - F'\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\left(f(v) - \frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right] dv \\ & = \frac{1}{w-u}\int_\gamma (F \circ f)(v)\,dv - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right) \\ & - F'\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\left(\frac{1}{w-u}\int_\gamma f(v)\,dv - \frac{1}{w-u}\int_\gamma f(z)\,dz\right) \\ & = \frac{1}{w-u}\int_\gamma (F \circ f)(v)\,dv - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right). \end{aligned}$$

By using (2.4) and (2.5) we get

$$\begin{aligned} & \left|\frac{1}{w-u}\int_\gamma (F \circ f)(v)\,dv - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right| \\ & \leq \frac{1}{|w-u|}\int_\gamma \left|(F \circ f)(v) - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right. \\ & \left. - F'\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\left(f(v) - \frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right| |dv| \\ & \leq \frac{1}{2}K\frac{1}{|w-u|}\int_\gamma \left|f(v) - \frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2 |dv|, \end{aligned}$$

which proves the inequality (2.1). □

Corollary 2. *With the assumptions of Theorem 5 and if*

$$\|F''\|_{G,\infty} := \sup_{z \in G}|F''(z)| < \infty,$$

then

$$\begin{aligned} (2.6)\quad & \left|\frac{1}{w-u}\int_\gamma (F \circ f)(v)\,dv - F\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right| \\ & \leq \frac{1}{2}\|F''\|_{G,\infty}\frac{1}{|w-u|}\int_\gamma \left|f(v) - \frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2 |dv|. \end{aligned}$$

Remark 1. *If we take $D=G$, $\gamma\subset G$ and $f(z)=z$, then by (2.6) we get the Hermite-Hadamard type inequality (see also* [5]*)*

$$\left|\frac{1}{w-u}\int_{\gamma}F(v)\,dv-F\left(\frac{w+u}{2}\right)\right| \leq \frac{1}{2}\left\|F''\right\|_{G,\infty}\frac{1}{|w-u|}\int_{\gamma}\left|v-\frac{w+u}{2}\right|^{2}|dv|, \tag{2.7}$$

provided F is holomorphic on G and $\left\|F''\right\|_{G,\infty}:=\sup_{z\in G}\left|F''(z)\right|<\infty$.

We also have:

Theorem 6. *Let $G\subset\mathbb{C}$ be a convex domain of complex numbers and $K>0$ and that F is holomorphic on G with $F\in\mathcal{BM}_{K}(G)$. Assume also that $f:D\to G$ is continuous on D, $\gamma\subset D$ parametrized by $z(t)$, $t\in[a,b]$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ with $w\neq u$,*

$$\int_{\gamma}(F'\circ f)(v)\,dv\neq0 \text{ and } \frac{\int_{\gamma}(F'\circ f)(v)f(v)\,dv}{\int_{\gamma}(F'\circ f)(v)\,dv}\in G, \tag{2.8}$$

then

$$\left|F\left(\frac{\int_{\gamma}(F'\circ f)(v)f(v)\,dv}{\int_{\gamma}(F'\circ f)(v)\,dv}\right)-\frac{1}{w-u}\int_{\gamma}(F\circ f)(z)\,dz\right| \leq\frac{1}{2}K\frac{1}{|w-u|}\int_{\gamma}\left|\frac{\int_{\gamma}(F'\circ f)(v)f(v)\,dv}{\int_{\gamma}(F'\circ f)(v)\,dv}-f(z)\right|^{2}|dz|. \tag{2.9}$$

Proof. From (2.2) we get

$$\left|F(y)-F(f(v))-F'(f(v))(y-f(v))\right|\leq\frac{1}{2}K\left|y-f(v)\right|^{2} \tag{2.10}$$

for any $y\in G$ and for $v\in D$.

Taking the integral in (2.10) we get

$$\frac{1}{|w-u|}\int_{\gamma}\left|F(y)-F(f(v))-F'(f(v))(y-f(v))\right||dv| \leq\frac{1}{2}K\frac{1}{|w-u|}\int_{\gamma}\left|y-f(v)\right|^{2}|dv| \tag{2.11}$$

for $y\in G$.

Using the properties of integral and modulus, we also have

$$\left|\frac{1}{w-u}\int_{\gamma}\left[F(y)-F(f(w))-F'(f(w))(y-f(w))\right]dw\right| \leq\frac{1}{|w-u|}\int_{\gamma}\left|F(y)-F(f(w))-F'(f(w))(y-f(w))\right||dw| \tag{2.12}$$

for $y\in G$.

Now, observe that

$$\frac{1}{w-u}\int_{\gamma}\left[F\left(y\right)-F\left(f\left(v\right)\right)-F'\left(f\left(v\right)\right)\left(y-f\left(v\right)\right)\right]dv$$
$$=F\left(y\right)-\frac{1}{w-u}\int_{\gamma}\left(F\circ f\right)\left(v\right)dv$$
$$-y\frac{1}{w-u}\int_{\gamma}\left(F'\circ f\right)\left(v\right)dv+\frac{1}{w-u}\int_{\gamma}\left(F'\circ f\right)\left(v\right)f\left(v\right)dv$$

and by (2.11) and (2.12) we get the following inequality of interest

$$\left|F\left(y\right)-\frac{1}{w-u}\int_{\gamma}\left(F\circ f\right)\left(v\right)dv\right.$$
$$\left.-y\frac{1}{w-u}\int_{\gamma}\left(F'\circ f\right)\left(v\right)dv+\frac{1}{w-u}\int_{\gamma}\left(F'\circ f\right)\left(v\right)f\left(v\right)dv\right|$$
$$\leq\frac{1}{2}K\frac{1}{\left|w-u\right|}\int_{\gamma}\left|y-f\left(z\right)\right|^{2}\left|dz\right| \tag{2.13}$$

for $y\in G$.

If we take in (2.13)

$$y=\frac{\int_{\gamma}\left(F'\circ f\right)\left(v\right)f\left(v\right)dv}{\int_{\gamma}\left(F'\circ f\right)\left(v\right)dv}\in G,$$

then we get the desired result (2.9). □

Corollary 3. *With the assumptions of Corollary 2 and Theorem 6 we have*

$$\left|F\left(\frac{\int_{\gamma}\left(F'\circ f\right)\left(v\right)f\left(v\right)dv}{\int_{\gamma}\left(F'\circ f\right)\left(v\right)dv}\right)-\frac{1}{w-u}\int_{\gamma}\left(F\circ f\right)\left(z\right)dz\right|$$
$$\leq\frac{1}{2}\left\|F''\right\|_{G,\infty}\frac{1}{\left|w-u\right|}\int_{\gamma}\left|\frac{\int_{\gamma}\left(F'\circ f\right)\left(v\right)f\left(v\right)dv}{\int_{\gamma}\left(F'\circ f\right)\left(v\right)dv}-f\left(z\right)\right|^{2}\left|dz\right|. \tag{2.14}$$

We have by the integration by parts formula (1.11) that

$$\int_{\gamma}F'\left(v\right)vdv=F\left(w\right)w-F\left(u\right)u-\int_{\gamma}F\left(v\right)dv$$

and

$$\int_{\gamma}F'\left(v\right)dv=F\left(w\right)-F\left(u\right).$$

Therefore we can state the following result as well:

Remark 2. *Let $G\subset\mathbb{C}$ be a convex domain of complex numbers and that F is holomorphic on G with $\left\|F''\right\|_{G,\infty}:=\sup_{z\in G}\left|F''\left(z\right)\right|<\infty$. Assume also that $\gamma\subset D$ parametrized by $z\left(t\right)$, $t\in\left[a,b\right]$ is a piecewise smooth path from $z\left(a\right)=u$ to $z\left(b\right)=w$ with $w\neq u$, $F\left(w\right)\neq F\left(u\right)$ and*

$$\frac{F\left(w\right)w-F\left(u\right)u-\int_{\gamma}F\left(v\right)dv}{F\left(w\right)-F\left(u\right)}\in G, \tag{2.15}$$

then by (2.14) we get

$$\left|F\left(\frac{F(w)w-F(u)u-\int_\gamma F(v)\,dv}{F(w)-F(u)}\right)-\frac{1}{w-u}\int_\gamma F(z)\,dz\right| \le \frac{1}{2}\|F''\|_{G,\infty}\frac{1}{|w-u|}\int_\gamma\left|\frac{F(w)w-F(u)u-\int_\gamma F(v)\,dv}{F(w)-F(u)}-z\right|^2|dz|. \tag{2.16}$$

3. Some Examples

If we consider the function $F(z)=\exp z$, $z\in\mathbb{C}$ and $\gamma\subset\mathbb{C}$ parametrized by $z(t)$, $t\in[a,b]$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ with $w\neq u$, then by (2.6) we have for continuous function $f:\gamma\to\mathbb{C}$

$$\left|\frac{1}{w-u}\int_\gamma(\exp\circ f)(v)\,dv-\exp\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right| \le \frac{1}{2}\|\exp\|_{G,\infty}\frac{1}{|w-u|}\int_\gamma\left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2|dv|, \tag{3.1}$$

while from (2.6) we obtain

$$\left|\frac{\exp w-\exp u}{w-u}-\exp\left(\frac{w+u}{2}\right)\right| \le \frac{1}{2}\|\exp\|_{G,\infty}\frac{1}{|w-u|}\int_\gamma\left|v-\frac{w+u}{2}\right|^2|dv|. \tag{3.2}$$

From (2.14) we get

$$\left|\exp\left(\frac{\int_\gamma(\exp\circ f)(v)f(v)\,dv}{\int_\gamma(\exp\circ f)(v)\,dv}\right)-\frac{1}{w-u}\int_\gamma(\exp\circ f)(z)\,dz\right| \le \frac{1}{2}\|\exp\|_{G,\infty}\frac{1}{|w-u|}\int_\gamma\left|\frac{\int_\gamma(\exp\circ f)(v)f(v)\,dv}{\int_\gamma(\exp\circ f)(v)\,dv}-f(z)\right|^2|dz|, \tag{3.3}$$

while from (2.15) we get

$$\left|\exp\left(\frac{(w-1)\exp w-(u-1)\exp u}{\exp w-\exp u}\right)-\frac{\exp w-\exp u}{w-u}\right| \le \frac{1}{2}\|\exp\|_{G,\infty}\frac{1}{|w-u|}\int_\gamma\left|\frac{(w-1)\exp w-(u-1)\exp u}{\exp w-\exp u}-z\right|^2|dz|. \tag{3.4}$$

Consider the function $F(z)=\operatorname{Log}(z)$ where $\operatorname{Log}(z)=\ln|z|+i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $0<\operatorname{Arg}(z)<2\pi$. Log is called the *"principal branch"* of the complex logarithmic function. F is analytic on all of $\mathbb{L}:=\mathbb{C}\setminus\{x+iy:x\geq 0,\ y=0\}$ and $F'(z)=\frac{1}{z}$ on this set.

If we consider $g:D\to\mathbb{C}$, $g(z)=\frac{1}{z}$ where $D\subset\mathbb{L}$, then F is a primitive of g on D and if $\gamma\subset D$ parametrized by $z(t)$, $t\in[a,b]$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$, then

$$\int_\gamma\frac{dz}{z}=\operatorname{Log}(w)-\operatorname{Log}(u).$$

Also, the function $G:\mathbb{L}\to\mathbb{C}$, $G(z)=z\operatorname{Log}(z)-z$ is analytic on $\mathbb{L}$ and $G'(z)=\operatorname{Log}(z)$, $z\in\mathbb{L}$.

Assume also that $f:D\to\mathbb{L}$ is continuous on D, $\gamma\subset D$ parametrized by $z(t)$, $t\in[a,b]$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ with $w\neq u$ and $\frac{1}{w-u}\int_\gamma f(z)\,dz\in\mathbb{L}$, then from (2.1) for $F(z)=\operatorname{Log} z$, we get

$$\left|\frac{1}{w-u}\int_\gamma(\operatorname{Log}\circ f)(v)\,dv-\operatorname{Log}\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)\right| \le\frac{1}{2}\frac{1}{d_\gamma^2|w-u|}\int_\gamma\left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2|dv|, \tag{3.5}$$

where $d_\gamma:=\inf_{z\in\gamma}|z|$ is assumed to be positive and finite.

For $\gamma\subset\mathbb{L}$ and $f(z)=z$, we get from (3.5) that

$$\left|\frac{w\operatorname{Log}(w)-u\operatorname{Log}(u)}{w-u}-\operatorname{Log}\left(\frac{w+u}{2}\right)-1\right| \le\frac{1}{2}\frac{1}{d_\gamma^2|w-u|}\int_\gamma\left|v-\frac{w+u}{2}\right|^2|dv|, \tag{3.6}$$

where $d_\gamma:=\inf_{z\in\gamma}|z|$ is assumed to be positive and finite.

Further, for $F(z)=\operatorname{Log} z$ we have

$$\begin{aligned}&\frac{w\operatorname{Log} w-u\operatorname{Log} u-\int_\gamma\operatorname{Log} z dz}{\operatorname{Log} w-\operatorname{Log} u}\\&=\frac{w\operatorname{Log} w-u\operatorname{Log} u-w\operatorname{Log}(w)+w+u\operatorname{Log}(u)-u}{\operatorname{Log} w-\operatorname{Log} u}\\&=\frac{w-u}{\operatorname{Log} w-\operatorname{Log} u}.\end{aligned}$$

So, if $\operatorname{Log} w\neq\operatorname{Log} u$ and

$$\frac{w-u}{\operatorname{Log} w-\operatorname{Log} u}\in\mathbb{L},$$

then by (2.16) we get

$$\left|\operatorname{Log}\left(\frac{w-u}{\operatorname{Log} w-\operatorname{Log} u}\right)-\frac{w\operatorname{Log}(w)-u\operatorname{Log}(u)}{w-u}+1\right| \le\frac{1}{2}\frac{1}{d_\gamma^2|w-u|}\int_\gamma\left|\frac{w-u}{\operatorname{Log} w-\operatorname{Log} u}-z\right|^2|dz|. \tag{3.7}$$

Assume also that $f:D\to\mathbb{L}$ is continuous on D, $\gamma\subset D$ parametrized by $z(t)$, $t\in[a,b]$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ with $w\neq u$ and $\frac{1}{w-u}\int_\gamma f(z)\,dz\in\mathbb{L}$, then from (2.1) for $F(z)=z^{-1}$, we get

$$\left|\frac{1}{w-u}\int_\gamma[f(v)]^{-1}\,dv-\left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)^{-1}\right| \le\frac{1}{d_\gamma^3|w-u|}\int_\gamma\left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right|^2|dv|, \tag{3.8}$$

where $d_\gamma:=\inf_{z\in\gamma}|z|$ is assumed to be positive and finite.

For $\gamma \subset \mathbb{L}$ and $f(z) = z$, we get from (3.8) that

$$\left| \frac{\operatorname{Log}(w) - \operatorname{Log}(u)}{w-u} - \left(\frac{w+u}{2} \right)^{-1} \right| \leq \frac{1}{d_\gamma^3 |w-u|} \int_\gamma \left| v - \frac{w+u}{2} \right|^2 |dv| . \tag{3.9}$$

Further, for $F(z) = z^{-1}$ we have

$$\begin{aligned} \frac{F(w)w - F(u)u - \int_\gamma F(v)\,dv}{F(w) - F(u)} &= \frac{-\operatorname{Log}(w) + \operatorname{Log}(u)}{\frac{1}{w} - \frac{1}{u}} \\ &= \frac{\operatorname{Log}(w) - \operatorname{Log}(u)}{w-u} wu \end{aligned}$$

for $w \neq u$ and $u, w \in \mathbb{L}$.

If $w \neq u$ and $u, w \in \mathbb{L}$ with

$$\frac{\operatorname{Log}(w) - \operatorname{Log}(u)}{w-u} wu \in \mathbb{L},$$

then by (2.16) we get

$$\begin{aligned} &\left| \left(\frac{\operatorname{Log}(w) - \operatorname{Log}(u)}{w-u} wu \right)^{-1} - \frac{\operatorname{Log}(w) - \operatorname{Log}(u)}{w-u} \right| \\ &\qquad \leq \frac{1}{d_\gamma^3 |w-u|} \int_\gamma \left| \frac{\operatorname{Log}(w) - \operatorname{Log}(u)}{w-u} wu - z \right|^2 |dz| . \end{aligned} \tag{3.10}$$

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