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SOME INEQUALITIES OF OSTROWSKI AND TRAPEZOID TYPE FOR HYPERBOLIC p-CONVEX FUNCTIONS

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Abstract. In this paper we establish some Ostrowski and Trapezoid type integral inequalities for hyperbolic p-convex functions.

1. Introduction

In 1938, A. Ostrowski [16], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1 (Ostrowski). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

The following result of Ostrowski type for convex functions holds.

Theorem 2 (Dragomir, 2002 [8]). *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in[a,b]$ one has the inequality*

$$\begin{aligned} &\frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \\ &\leq \int_a^b f(t)\,dt - (b-a) f(x) \\ &\leq \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right]. \end{aligned} \tag{1.2}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x=a$ or $x=b$.

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In particular, for $x=\frac{a+b}{2}$, we get the sharp inequalities

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[f_{+}^{\prime}\left(\frac{a+b}{2}\right)-f_{-}^{\prime}\left(\frac{a+b}{2}\right)\right](b-a) \\
&\leq \frac{1}{b-a}\int_{a}^{b} f(t)\,dt-f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{8}\left[f_{-}^{\prime}(b)-f_{+}^{\prime}(a)\right](b-a).
\end{aligned} \tag{1.3}
$$

For various Ostrowski type inequalities see the recent survey paper [11] and the references therein.

The following trapezoid type inequality for convex functions also holds.

Theorem 3 (Dragomir, 2002 [8])**.** *Let $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in[a,b]$ one has the inequality*

$$
\begin{aligned}
&\frac{1}{2}\left[(b-x)^{2} f_{+}^{\prime}(x)-(x-a)^{2} f_{-}^{\prime}(x)\right] \\
&\leq (x-a) f(a)+(b-x) f(b)-\int_{a}^{b} f(t)\,dt \\
&\leq \frac{1}{2}\left[(b-x)^{2} f_{-}^{\prime}(b)-(x-a)^{2} f_{+}^{\prime}(a)\right].
\end{aligned} \tag{1.4}
$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x=a$ or $x=b$.

In particular, for $x=\frac{a+b}{2}$, we get the sharp inequalities

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[f_{+}^{\prime}\left(\frac{a+b}{2}\right)-f_{-}^{\prime}\left(\frac{a+b}{2}\right)\right](b-a) \\
&\leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b} f(t)\,dt \\
&\leq \frac{1}{8}\left[f_{-}^{\prime}(b)-f_{+}^{\prime}(a)\right](b-a).
\end{aligned} \tag{1.5}
$$

Let I be a finite or infinite open interval of real numbers and $p\in\mathbb{R}$, $p\neq 0$.

In the following we present the basic definitions and results concerning the class of hyperbolic p-convex function, see [3]. For other concepts of modified convex functions see for example [13], [14], [4], [6], [7], [12], [15], [17] and [18].

We consider the hyperbolic functions of a real argument $x\in\mathbb{R}$ defined by

$$
\sinh x:=\frac{e^{x}-e^{-x}}{2}=\frac{e^{2x}-1}{2e^{x}},\quad \cosh x:=\frac{e^{x}+e^{-x}}{2}=\frac{e^{2x}+1}{2e^{x}},
$$

$$
\tanh x:=\frac{\sinh x}{\cosh x} \text{ and } \coth x:=\frac{\cosh x}{\sinh x}.
$$

We say that a function $f:I\rightarrow\mathbb{R}$ is *hyperbolic p-convex* (or sub H-function, according with [3]) on I, if for any closed subinterval $[a,b]$ of I we have

$$
f(x)\leq \frac{\sinh[p(b-x)]}{\sinh[p(b-a)]} f(a)+\frac{\sinh[p(x-a)]}{\sinh[p(b-a)]} f(b) \tag{1.6}
$$

for all $x\in[a,b]$.

If the inequality (1.6) holds with " $\geq$ ", then the function will be called *hyperbolic p-concave* on I.

Geometrically speaking, this means that the graph of f on $[a,b]$ lies nowhere above the p-hyperbolic function determined by the equation

$$H(x) = H(x;a,b,f) := A\cosh(px) + B\sinh(px)$$

where A and B are chosen such that $H(a) = f(a)$ and $H(b) = f(b)$.

If we take $x = (1-t)a + tb \in [a,b]$, $t \in [0,1]$, then the condition (1.6) becomes

$$f((1-t)a+tb) \le \frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]} f(a) + \frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]} f(b) \tag{1.7}$$

for any $t \in [0,1]$.

We have the following properties of hyperbolic p-convex on I, [3].

(i) A hyperbolic p-convex function $f : I \to \mathbb{R}$ has finite right and left derivatives $f'_+(x)$ and $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \le f'_+(x)$. The function f is differentiable on I with the exception of an at most countable set.

(ii) A necessary and sufficient condition for the function $f : I \to \mathbb{R}$ to be hyperbolic p-convex function on I is that it satisfies the *gradient inequality*

$$f(y) \ge f(x)\cosh[p(y-x)] + K_{x,f}\sinh[p(y-x)] \tag{1.8}$$

for any x, $y \in I$ where $K_{x,f} \in \left[f'_-(x), f'_+(x)\right]$. If f is differentiable at the point x then $K_{x,f} = f'(x)$.

(iii) A necessary and sufficient condition for the function f to be a hyperbolic p-convex in I, is that the function

$$\varphi(x) = f'(x) - p^2 \int_a^x f(t)\,dt$$

is nondecreasing on I, where $a \in I$.

(iv) Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function on I. Then f is hyperbolic p-convex on I if and only if for all $x \in I$ we have

$$f''(x) - p^2 f(x) \ge 0. \tag{1.9}$$

For other properties of hyperbolic p-convex functions, see [3].

Consider the function $f_r : (0,\infty) \to (0,\infty)$, $f_r(x) = x^r$ with $p \in \mathbb{R}\backslash\{0\}$. If $r \in (-\infty,0) \cup [1,\infty)$ the function is convex and if $r \in (0,1)$ it is concave. We have for $r \in (-\infty,0) \cup [1,\infty)$

$$f_r''(x) - p^2 f_r(x) = r(r-1)x^{r-2} - p^2x^r = p^2x^{r-2}\left(\frac{r(r-1)}{p^2} - x^2\right),\ x > 0.$$

We observe that $f_r''(x) - p^2 f_r(x) > 0$ for $x \in \left(0, \frac{\sqrt{r(r-1)}}{|p|}\right)$ and $f_r''(x) - p^2 f_r(x) < 0$ for $x \in \left(\frac{\sqrt{r(r-1)}}{|p|}, \infty\right)$, which shows that the power function f_r for $r \in (-\infty,0) \cup [1,\infty)$ is hyperbolic p-convex on $\left(0, \frac{\sqrt{r(r-1)}}{|p|}\right)$ and hyperbolic p-concave on $\left(\frac{\sqrt{r(r-1)}}{|p|}, \infty\right)$.

If $r \in (0,1)$, then $f_r''(x) - p^2 f_r(x) < 0$ for any $x > 0$, which shows that f_r is hyperbolic p-concave on $(0,\infty)$.

Consider the exponential function $f_\alpha(x)=\exp(\alpha x)$ for $\alpha\neq 0$ and $x\in\mathbb{R}$. Then

$$f_\alpha''(x)-p^2 f_\alpha(x)=\alpha^2 e^{\alpha x}-p^2 e^{\alpha x}=\left(\alpha^2-p^2\right)e^{\alpha x},\ x>0.$$

If $|\alpha|>|p|$, then f_α is hyperbolic p-convex on $\mathbb{R}$ and if $|\alpha|<|p|$ then f_α is hyperbolic p-concave on $\mathbb{R}$.

In this paper we establish some Ostrowski and Trapezoid type integral inequalities for hyperbolic p-convex functions.

2. Ostrowski Type Inequalities

We have:

Theorem 4. *Assume that the function $f:I\to\mathbb{R}$ is hyperbolic p-convex on I. Then for any $a,\ b\in I$ with $a<b$ and $x\in(a,b)$ we have*

$$\begin{aligned}(2.1)\quad &\frac{1}{2}\left[f_+'(x)(b-x)^2-f_-'(x)(x-a)^2\right]\\ &\le\int_a^b f(t)\,dt-\frac{1}{2}p^2\left[\int_a^x(t-a)^2 f(t)\,dt+\int_x^b(b-t)^2 f(t)\,dt\right]\\ &\qquad-f(x)(b-a)\\ &\le\frac{1}{2}\left[f_-'(b)(b-x)^2-f_+'(a)(x-a)^2\right]\\ &\qquad-\frac{1}{2}p^2\left[(x-a)^2\int_a^x f(t)\,dt+(b-x)^2\int_x^b f(t)\,dt\right].\end{aligned}$$

In particular, if f is differentiable in x, then we have

$$\begin{aligned}(2.2)\quad &f'(x)(b-a)\left(\frac{a+b}{2}-x\right)\\ &\le\int_a^b f(t)\,dt-\frac{1}{2}p^2\left[\int_a^x(t-a)^2 f(t)\,dt+\int_x^b(b-t)^2 f(t)\,dt\right]\\ &\qquad-f(x)(b-a)\\ &\le\frac{1}{2}\left[f_-'(b)(b-x)^2-f_+'(a)(x-a)^2\right]\\ &\qquad-\frac{1}{2}p^2\left[(x-a)^2\int_a^x f(t)\,dt+(b-x)^2\int_x^b f(t)\,dt\right].\end{aligned}$$

Proof. We use the *Montgomery identity* for an absolutely continuous function $g:[a,b]\to\mathbb{C}$ that says that

$$g(x)(b-a)-\int_a^b g(s)\,ds=\int_a^x(s-a)g'(s)\,ds-\int_x^b(b-s)g'(s)\,ds\tag{2.3}$$

for $x\in(a,b)$. This can be proved in one line by integrating by parts on the second term.

Using the property (iii) from Introduction we have that

$$f_+'(a)\le f'(s)-p^2\int_a^s f(t)\,dt\le f_-'(x)-p^2\int_a^x f(t)\,dt\tag{2.4}$$

for a.e. $s \in [a, x]$.

This implies that

$$f'_{+}(a)(s-a) \le \left[f'(s) - p^2 \int_a^s f(t)\, dt \right](s-a) \le \left[f'_{-}(x) - p^2 \int_a^x f(t)\, dt \right](s-a),$$

that is equivalent to

$$f'_{+}(a)(s-a) + p^2 (s-a) \int_a^s f(t)\, dt \le f'(s)(s-a) \le f'_{-}(x)(s-a) - p^2 (s-a) \int_a^x f(t)\, dt + p^2 (s-a) \int_a^s f(t)\, dt$$

for a.e. $s \in [a, x]$.

If we integrate this over $s \in [a, x]$ we get

$$f'_{+}(a) \int_a^x (s-a)\, ds + p^2 \int_a^x (s-a) \left(\int_a^s f(t)\, dt \right) ds \le \int_a^x f'(s)(s-a)\, ds \le f'_{-}(x) \int_a^x (s-a)\, ds - p^2 \int_a^x (s-a)\, ds \int_a^x f(t)\, dt + p^2 \int_a^x (s-a) \left(\int_a^s f(t)\, dt \right) ds,$$

that is equivalent to

$$\frac{1}{2} f'_{+}(a)(x-a)^2 + p^2 \int_a^x (s-a) \left(\int_a^s f(t)\, dt \right) ds \le \int_a^x f'(s)(s-a)\, ds \le \frac{1}{2} f'_{-}(x)(x-a)^2 - \frac{1}{2} p^2 (x-a)^2 \int_a^x f(t)\, dt + p^2 \int_a^x (s-a) \left(\int_a^s f(t)\, dt \right) ds, \tag{2.5}$$

for $x \in (a, b)$.

Using the property (iii) from Introduction we also have that

$$f'_{+}(x) - p^2 \int_a^x f(t)\, dt \le f'(s) - p^2 \int_a^s f(t)\, dt \le f'_{-}(b) - p^2 \int_a^b f(t)\, dt \tag{2.6}$$

for a.e. $s \in [x, b]$.

This implies that

$$f'_{+}(x)(b-s) - p^2 (b-s) \int_a^x f(t)\, dt \le \left(f'(s) - p^2 \int_a^s f(t)\, dt \right)(b-s) \le f'_{-}(b)(b-s) - p^2 (b-s) \int_a^b f(t)\, dt$$

for a.e $s \in [x, b]$.

If we integrate this over $s\in[x,b]$, we get

$$\begin{aligned}&\frac{1}{2}f_{+}'(x)(b-x)^{2}-\frac{1}{2}p^{2}(b-x)^{2}\int_{a}^{x}f(t)\,dt\\&\qquad\le\int_{x}^{b}\left(f'(s)-p^{2}\int_{a}^{s}f(t)\,dt\right)(b-s)\,ds\\&\qquad\qquad\le\frac{1}{2}f_{-}'(b)(b-x)^{2}-\frac{1}{2}p^{2}(b-x)^{2}\int_{a}^{b}f(t)\,dt,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{1}{2}f_{+}'(x)(b-x)^{2}-\frac{1}{2}p^{2}(b-x)^{2}\int_{a}^{x}f(t)\,dt+p^{2}\int_{x}^{b}(b-s)\left(\int_{a}^{s}f(t)\,dt\right)ds\\&\qquad\le\int_{x}^{b}f'(s)(b-s)\,ds\\&\le\frac{1}{2}f_{-}'(b)(b-x)^{2}-p^{2}\frac{1}{2}(b-x)^{2}\int_{a}^{b}f(t)\,dt+p^{2}\int_{x}^{b}(b-s)\left(\int_{a}^{s}f(t)\,dt\right)ds\end{aligned}$$

or to

$$\begin{aligned}&-\frac{1}{2}f_{-}'(b)(b-x)^{2}+p^{2}\frac{1}{2}(b-x)^{2}\int_{a}^{b}f(t)\,dt\\&\qquad-p^{2}\int_{x}^{b}(b-s)\left(\int_{a}^{s}f(t)\,dt\right)ds\\&\qquad\le-\int_{x}^{b}f'(s)(b-s)\,ds\\&-\frac{1}{2}f_{+}'(x)(b-x)^{2}+\frac{1}{2}p^{2}(b-x)^{2}\int_{a}^{x}f(t)\,dt-p^{2}\int_{x}^{b}(b-s)\left(\int_{a}^{s}f(t)\,dt\right)ds.\end{aligned}\tag{2.7}$$

Now, if we add (2.5) with (2.7) and use Montgomery identity (2.3) we get

$$\begin{aligned}&\frac{1}{2}f_{+}'(a)(x-a)^{2}-p^{2}\int_{a}^{x}(s-a)\left(\int_{a}^{s}f(t)\,dt\right)ds\\&-\frac{1}{2}f_{-}'(b)(b-x)^{2}-p^{2}\frac{1}{2}(b-x)^{2}\int_{a}^{b}f(t)\,dt+p^{2}\int_{x}^{b}(b-s)\left(\int_{a}^{s}f(t)\,dt\right)ds\\&\qquad\le f(x)(b-a)-\int_{a}^{b}f(s)\,ds\\&\le\frac{1}{2}f_{-}'(x)(x-a)^{2}+\frac{1}{2}p^{2}(x-a)^{2}\int_{a}^{x}f(t)\,dt-p^{2}\int_{a}^{x}(s-a)\left(\int_{a}^{s}f(t)\,dt\right)ds\\&-\frac{1}{2}f_{+}'(x)(b-x)^{2}-\frac{1}{2}p^{2}(b-x)^{2}\int_{a}^{x}f(t)\,dt+p^{2}\int_{x}^{b}(b-s)\left(\int_{a}^{s}f(t)\,dt\right)ds.\end{aligned}\tag{2.8}$$

Using the integration by parts, we have

$$\begin{aligned}\int_a^x (s-a)\left(\int_a^s f(t)\,dt\right) ds &= \frac{1}{2}\int_a^x \left(\int_a^s f(t)\,dt\right) ds\left((s-a)^2\right)\\ &= \frac{1}{2}\left(\int_a^s f(t)\,dt\right)(s-a)^2\Big|_a^x - \frac{1}{2}\int_a^x (s-a)^2 f(s)\,ds\\ &= \frac{1}{2}(x-a)^2\int_a^x f(t)\,dt - \frac{1}{2}\int_a^x (s-a)^2 f(s)\,ds\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-s)\left(\int_a^s f(t)\,dt\right) ds &= -\frac{1}{2}\int_x^b \left(\int_a^s f(t)\,dt\right) d\left((b-s)^2\right)\\ &= -\frac{1}{2}\left(\int_a^s f(t)\,dt\right)(b-s)^2\Big|_x^b + \frac{1}{2}\int_x^b (b-s)^2 f(s)\,ds\\ &= \frac{1}{2}\int_x^b (b-s)^2 f(s)\,ds + \frac{1}{2}(b-x)^2\int_a^x f(t)\,dt.\end{aligned}$$

Then by (2.8) we get

$$\begin{aligned}(2.9)\quad &\frac{1}{2}f'_+(a)(x-a)^2 - p^2\left[\frac{1}{2}(x-a)^2\int_a^x f(t)\,dt - \frac{1}{2}\int_a^x (s-a)^2 f(s)\,ds\right]\\ &- \frac{1}{2}f'_-(b)(b-x)^2 - p^2\frac{1}{2}(b-x)^2\int_a^b f(t)\,dt\\ &+ p^2\left[\frac{1}{2}\int_x^b (b-s)^2 f(s)\,ds + \frac{1}{2}(b-x)^2\int_a^x f(t)\,dt\right]\\ &\le f(x)(b-a) - \int_a^b f(s)\,ds\\ &\le \frac{1}{2}f'_-(x)(x-a)^2 + \frac{1}{2}p^2(x-a)^2\int_a^x f(t)\,dt\\ &- p^2\left[\frac{1}{2}(x-a)^2\int_a^x f(t)\,dt - \frac{1}{2}\int_a^x (s-a)^2 f(s)\,ds\right]\\ &- \frac{1}{2}f'_+(x)(b-x)^2 - \frac{1}{2}p^2(b-x)^2\int_a^x f(t)\,dt\\ &+ p^2\left[\frac{1}{2}\int_x^b (b-s)^2 f(s)\,ds + \frac{1}{2}(b-x)^2\int_a^x f(t)\,dt\right]\end{aligned}$$

or, equivalently

$$\begin{aligned}(2.10)\quad &\frac{1}{2}f'_{+}(a)(x-a)^2-\frac{1}{2}f'_{-}(b)(b-x)^2\\ &+\frac{1}{2}(b-x)^2p^2\int_a^x f(t)\,dt-\frac{1}{2}(x-a)^2p^2\int_a^x f(t)\,dt-\frac{1}{2}(b-x)^2p^2\int_a^b f(t)\,dt\\ &\quad+\frac{1}{2}p^2\left[\int_x^b (b-s)^2 f(s)\,ds+\int_a^x (s-a)^2 f(s)\,ds\right]\\ &\leq f(x)(b-a)-\int_a^b f(s)\,ds\\ &\leq \frac{1}{2}f'_{-}(x)(x-a)^2-\frac{1}{2}f'_{+}(x)(b-x)^2\\ &\quad+\frac{1}{2}p^2\left[\int_a^x (s-a)^2 f(s)\,ds+\int_x^b (b-s)^2 f(s)\,ds\right]\end{aligned}$$

for $x\in(a,b)$.

The inequality (2.10) can also be written as

$$\begin{aligned}&\frac{1}{2}f'_{+}(a)(x-a)^2-\frac{1}{2}f'_{-}(b)(b-x)^2\\ &\quad-\frac{1}{2}p^2\left[(x-a)^2\int_a^x f(t)\,dt+(b-x)^2\int_x^b f(t)\,dt\right]\\ &\qquad\leq f(x)(b-a)-\int_a^b f(s)\,ds\end{aligned}$$

$$\begin{aligned}&-\frac{1}{2}p^2\left[\int_a^x (s-a)^2 f(s)\,ds+\int_x^b (b-s)^2 f(s)\,ds\right]\\ &\qquad\leq \frac{1}{2}f'_{-}(x)(x-a)^2-\frac{1}{2}f'_{+}(x)(b-x)^2\end{aligned}$$

for $x\in(a,b)$, which proves the desired inequality (2.1). □

Corollary 1. *With the assumptions of Theorem 4 we have*

$$\begin{aligned}(2.11)\quad 0&\leq \frac{1}{8}(b-a)^2\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right]\\ &\leq \int_a^b f(t)\,dt-\frac{1}{2}p^2\left[\int_a^{\frac{a+b}{2}}(t-a)^2 f(t)\,dt+\int_{\frac{a+b}{2}}^b (b-t)^2 f(t)\,dt\right]\\ &\quad-f\left(\frac{a+b}{2}\right)(b-a)\\ &\leq \frac{1}{8}(b-a)^2\left[f'_{-}(b)-f'_{+}(a)\right]-\frac{1}{8}p^2(b-a)^2\int_a^b f(t)\,dt.\end{aligned}$$

Remark 1. *From the first inequality in (2.11) we have the following midpoint inequality*

$$f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f(t)\,dt - \frac{1}{2}p^2\left[\int_a^{\frac{a+b}{2}} (t-a)^2 f(t)\,dt + \int_{\frac{a+b}{2}}^b (b-t)^2 f(t)\,dt\right]. \tag{2.12}$$

3. Trapezoid Type Inequalities

We have:

Theorem 5. *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic p-convex on I. Then for any a, $b \in I$ with $a < b$ and $x \in (a,b)$ we have*

$$\begin{aligned} &\frac{1}{2}f'_+(x)(b-x)^2 - \frac{1}{2}f'_-(x)(x-a)^2 \\ &\quad + \frac{1}{2}p^2\left[(x-a)^2\int_a^x f(t)\,dt + (b-x)^2\left(\int_x^b f(t)\,dt\right)\right] \\ &\leq (x-a)f(a) + (b-x)f(b) - \int_a^b f(s)\,dt + \frac{1}{2}p^2\int_a^b (x-s)^2 f(s)\,ds \\ &\leq \frac{1}{2}(b-x)^2 f'_-(b) - \frac{1}{2}f'_+(a)(x-a)^2. \end{aligned} \tag{3.1}$$

In particular, if f is differentiable in x, then we have

$$\begin{aligned} &f'(x)(b-a)\left(\frac{a+b}{2}-x\right) \\ &\quad + \frac{1}{2}p^2\left[(x-a)^2\int_a^x f(t)\,dt + (b-x)^2\left(\int_x^b f(t)\,dt\right)\right] \\ &\leq (x-a)f(a) + (b-x)f(b) - \int_a^b f(s)\,dt + \frac{1}{2}p^2\int_a^b (x-s)^2 f(s)\,ds \\ &\leq \frac{1}{2}(b-x)^2 f'_-(b) - \frac{1}{2}f'_+(a)(x-a)^2. \end{aligned} \tag{3.2}$$

Proof. We use the following identity that holds for the absolutely continuous function $g : [a,b] \to \mathbb{C}$

$$\begin{aligned} &(x-a)g(a) + (b-x)g(b) - \int_a^b g(s)\,dt \\ &= \int_a^b (s-x)g'(s)\,dt = \int_x^b (s-x)g'(s)\,ds - \int_a^x (x-s)g'(s)\,ds \end{aligned} \tag{3.3}$$

for any $x \in [a,b]$. This can can be proved by integrating by parts in the second term.

Using the inequality (2.4) we get

$$f'_{+}(a)(x-s) \leq f'(s)(x-s) - p^2(x-s)\int_a^s f(t)\,dt$$
$$\leq f'_{-}(x)(x-s) - p^2(x-s)\int_a^x f(t)\,dt$$

for a.e. $s \in [a,x]$.

Integrating on $[a,x]$, we have

$$\frac{1}{2}f'_{+}(a)(x-a)^2$$
$$\leq \int_a^x f'(s)(x-s)\,ds - p^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds$$
$$\leq \frac{1}{2}f'_{-}(x)(x-a)^2 - \frac{1}{2}p^2(x-a)^2\int_a^x f(t)\,dt,$$

which is equivalent to

$$-p^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds - \frac{1}{2}f'_{-}(x)(x-a)^2 + \frac{1}{2}p^2(x-a)^2\int_a^x f(t)\,dt \tag{3.4}$$
$$\leq -\int_a^x f'(s)(x-s)\,ds \leq -p^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds - \frac{1}{2}f'_{+}(a)(x-a)^2$$

for any $x \in (a,b)$.

From (2.6) we have

$$f'_{+}(x)(s-x) - p^2(s-x)\int_a^x f(t)\,dt \leq (s-x)f'(s) - p^2(s-x)\int_a^s f(t)\,dt$$
$$\leq (s-x)f'_{-}(b) - p^2(s-x)\int_a^b f(t)\,dt$$

for a.e. $s \in [x,b]$.

Integrating on $[x,b]$ we get

$$\frac{1}{2}f'_{+}(x)(b-x)^2 - \frac{1}{2}p^2(b-x)^2\int_a^x f(t)\,dt$$
$$\leq \int_x^b (s-x)f'(s)\,ds - p^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds$$
$$\leq \frac{1}{2}(b-x)^2 f'_{-}(b) - \frac{1}{2}p^2(b-x)^2\int_a^b f(t)\,dt,$$

which is equivalent to

$$\begin{aligned}(3.5)\quad & \frac{1}{2}f'_{+}(x)(b-x)^2 \\ & -\frac{1}{2}p^2(b-x)^2\int_a^x f(t)\,dt+p^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds \\ & \le \int_x^b (s-x)f'(s)\,ds \\ & \le \frac{1}{2}(b-x)^2 f'_{-}(b)-\frac{1}{2}p^2(b-x)^2\int_a^b f(t)\,dt+p^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds,\end{aligned}$$

for any $x\in(a,b)$.

Adding (3.4) and (3.5) and using the identity (3.3) we get

$$\begin{aligned} & -p^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds-\frac{1}{2}f'_{-}(x)(x-a)^2+\frac{1}{2}p^2(x-a)^2\int_a^x f(t)\,dt \\ & +\frac{1}{2}f'_{+}(x)(b-x)^2-\frac{1}{2}p^2(b-x)^2\int_a^x f(t)\,dt+p^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds \\ & \le (x-a)f(a)+(b-x)f(b)-\int_a^b f(s)\,dt \\ & \le -p^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds-\frac{1}{2}f'_{+}(a)(x-a)^2 \\ & +\frac{1}{2}(b-x)^2 f'_{-}(b)-\frac{1}{2}p^2(b-x)^2\int_a^b f(t)\,dt+p^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds\end{aligned}$$

that is equivalent to

$$\begin{aligned}(3.6)\quad & \frac{1}{2}f'_{+}(x)(b-x)^2-\frac{1}{2}f'_{-}(x)(x-a)^2+\frac{1}{2}p^2(x-a)^2\int_a^x f(t)\,dt \\ & -\frac{1}{2}p^2(b-x)^2\int_a^x f(t)\,dt-p^2\int_a^b (x-s)\left(\int_a^s f(t)\,dt\right)ds \\ & \le (x-a)f(a)+(b-x)f(b)-\int_a^b f(s)\,dt \\ & \le \frac{1}{2}(b-x)^2 f'_{-}(b)-\frac{1}{2}f'_{+}(a)(x-a)^2-\frac{1}{2}p^2(b-x)^2\int_a^b f(t)\,dt \\ & -p^2\int_a^b (x-s)\left(\int_a^s f(t)\,dt\right)ds.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\int_a^b (x-s)\left(\int_a^s f(t)\,dt\right)ds & = -\frac{1}{2}\int_a^b \left(\int_a^s f(t)\,dt\right)d\left((x-s)^2\right) \\ & = -\frac{1}{2}\left[(x-s)^2\left(\int_a^s f(t)\,dt\right)\Big|_a^b-\int_a^b (x-s)^2 f(s)\,ds\right] \\ & = \frac{1}{2}\int_a^b (x-s)^2 f(s)\,ds-\frac{1}{2}(b-x)^2\left(\int_a^b f(t)\,dt\right)\end{aligned}$$

and by (3.6) we get

$$\frac{1}{2}f'_{+}(x)(b-x)^2-\frac{1}{2}f'_{-}(x)(x-a)^2+\frac{1}{2}p^2(x-a)^2\int_a^x f(t)\,dt$$
$$-\frac{1}{2}p^2(b-x)^2\int_a^x f(t)\,dt-\frac{1}{2}p^2\int_a^b (x-s)^2 f(s)\,ds+\frac{1}{2}(b-x)^2p^2\left(\int_a^b f(t)\,dt\right)$$
$$\leq (x-a)f(a)+(b-x)f(b)-\int_a^b f(s)\,dt$$
$$\leq \frac{1}{2}(b-x)^2 f'_{-}(b)-\frac{1}{2}f'_{+}(a)(x-a)^2-\frac{1}{2}p^2(b-x)^2\int_a^b f(t)\,dt$$
$$-\frac{1}{2}p^2\int_a^b (x-s)^2 f(s)\,ds+\frac{1}{2}(b-x)^2p^2\left(\int_a^b f(t)\,dt\right),$$

namely

$$\frac{1}{2}f'_{+}(x)(b-x)^2-\frac{1}{2}f'_{-}(x)(x-a)^2$$
$$+\frac{1}{2}p^2(x-a)^2\int_a^x f(t)\,dt+\frac{1}{2}(b-x)^2p^2\left(\int_x^b f(t)\,dt\right)$$
$$\leq (x-a)f(a)+(b-x)f(b)-\int_a^b f(s)\,dt+\frac{1}{2}p^2\int_a^b (x-s)^2 f(s)\,ds$$
$$\leq \frac{1}{2}(b-x)^2 f'_{-}(b)-\frac{1}{2}f'_{+}(a)(x-a)^2,$$

which proves the desired result (3.1). □

Corollary 2. *With the assumptions of Theorem 5, we have*

$$0\leq \frac{1}{8}(b-a)^2\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right] \tag{3.7}$$
$$\leq (b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(s)\,ds-\frac{1}{2}p^2\int_a^b (b-s)(s-a)f(s)\,ds$$
$$\leq \frac{1}{8}(b-a)^2\left[f'_{-}(b)-f'_{+}(a)\right]-\frac{1}{8}p^2(b-a)^2\int_a^b f(s)\,ds.$$

Remark 2. *From the first inequality in (3.7) we get the following trapezoid type inequality*

$$\int_a^b f(s)\,ds+\frac{1}{2}p^2\int_a^b (b-s)(s-a)f(s)\,ds\leq (b-a)\frac{f(a)+f(b)}{2}. \tag{3.8}$$

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