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INEQUALITIES FOR A GENERALIZED FINITE HILBERT TRANSFORM OF CONVEX FUNCTIONS

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Abstract. In this paper we obtain some new inequalities for a generalized finite Hilbert transform of convex functions. Applications for particular instances of finite Hilbert transforms are given as well.

1. Introduction

Finite Hilbert transform on the open interval (a,b) is defined by

$$(Tf)(a,b;t) := \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau \tag{1.1}$$

for $t \in (a,b)$ and for various classes of functions f for which the above Cauchy Principal Value integral exists, see [14, Section 3.2] or [18, Lemma II.1.1].

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,\ y \in \mathring{I}$ and $x < y$, then $f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x - a)\varphi(a) \text{ for any } x,\ a \in I. \tag{1.2}$$

It is also well known that if f is convex on I, then ∂f is nonempty, $f'_-,\ f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \le \varphi(x) \le f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function. If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following result holds for the finite Hilbert transform of convex functions.

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Theorem 1 (Dragomir et al., 2001 [2])**.** *Let $f:(a,b)\to\mathbb{R}$ be a convex function on (a,b). Then we have*

$$\begin{aligned}(1.3)\qquad &\frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(t)-f(a)+f'_{+}(t)(b-t)\right]\\ &\leq (Tf)(a,b;t)\\ &\leq \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(b)-f(t)+f'_{-}(t)(t-a)\right],\end{aligned}$$

for all $t\in(a,b)$.

In particular, we have

$$\begin{aligned}(1.4)\qquad &\frac{1}{\pi}\left[f\left(\frac{a+b}{2}\right)-f(a)+f'_{+}\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)\right]\\ &\leq (Tf)\left(a,b;\frac{a+b}{2}\right)\\ &\leq \frac{1}{\pi}\left[f(b)-f\left(\frac{a+b}{2}\right)+f'_{-}\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)\right].\end{aligned}$$

For several recent papers devoted to inequalities for the finite Hilbert transform (Tf), see [3]-[11], [15]-[17] and [19]-[20].

We can naturally generalize the concept of Hilbert transform as follows.

For a *continuous strictly increasing function* $g:[a,b]\to[g(a),g(b)]$ that is *differentiable* on (a,b) we define the following generalization of the finite Hilbert transform of a function $f:(a,b)\to\mathbb{C}$ by

$$\begin{aligned}(1.5)\qquad (T_g f)(a,b;t) &:= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau\\ &:= \lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right]\frac{f(\tau)g'(\tau)}{\pi[g(\tau)-g(t)]}d\tau\\ &:= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau\right]\end{aligned}$$

for $t\in(a,b)$, provided the above PV exists.

For $[a,b]\subset(0,\infty)$ and $g(t)=\ln t$, $t\in[a,b]$ we have the *logarithmic finite Hilbert transform* defined by

$$(1.6)\qquad (T_{\ln}f)(a,b;t) := \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{\tau\ln\left(\frac{\tau}{t}\right)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)}{\tau\ln\left(\frac{\tau}{t}\right)}d\tau\right]$$

where $t\in(a,b)$.

For $g(t)=\exp(\alpha t)$, $t\in[a,b]\subset\mathbb{R}$ with $\alpha>0$ we have *exponential finite Hilbert transform* defined by

$$\begin{aligned}(1.7)\qquad &\left(T_{\exp(\alpha)}f\right)(a,b;t)\\ &:= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau\right]\end{aligned}$$

where $t\in(a,b)$.

For $[a,b] \subset (0,\infty)$ and $g(t) = t^r$, $t \in [a,b]$, $r > 0$, we have the *positive r-power finite Hilbert transform* defined by

$$(T_r f)(a,b;t) := \frac{r}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)\tau^{r-1}}{\tau^r - t^r} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)\tau^{r-1}}{\tau^r - t^r} d\tau \right], \tag{1.8}$$

where $t \in (a,b)$.

Similarly, we can consider the function $g(t) = -t^{-p}$, $t \in [a,b] \subset (0,\infty)$, $p > 0$, and then we have the *negative p-power finite Hilbert transform*

$$\begin{aligned} (T_{-p} f)(a,b;t) &:= \frac{p}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)\tau^{-p-1}}{t^{-p} - \tau^{-p}} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)\tau^{-p-1}}{t^{-p} - \tau^{-p}} d\tau \right] \\ &= \frac{pt^p}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau(\tau^p - t^p)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau(\tau^p - t^p)} d\tau \right], \end{aligned} \tag{1.9}$$

where $t \in (a,b)$.

For $[a,b] \subset \left[-\frac{\pi}{2\rho}, \frac{\pi}{2\rho}\right]$ and $g(t) = \sin(\rho t)$, $t \in [a,b]$ where $\rho > 0$, we have the *ρ-sine finite Hilbert transform*

$$\begin{aligned} &\left(T_{\sin(\rho)} f\right)(a,b;t) \\ &:= \frac{\rho}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)\cos(\rho\tau)}{\sin(\rho\tau) - \sin(\rho t)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)\cos(\rho\tau)}{\sin(\rho\tau) - \sin(\rho t)} d\tau \right] \end{aligned} \tag{1.10}$$

where $t \in (a,b)$.

For $g(t) = \sinh(\sigma t)$, $t \in [a,b] \subset \mathbb{R}$ with $\sigma > 0$ we have *σ-sinh finite Hilbert transform*

$$\begin{aligned} &\left(T_{\sinh(\sigma)} f\right)(a,b;t) \\ &:= \frac{\sigma}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)\cosh(\sigma\tau)}{\sinh(\sigma\tau) - \sinh(\sigma t)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)\cosh(\sigma\tau)}{\sinh(\sigma\tau) - \sinh(\sigma t)} d\tau \right] \end{aligned} \tag{1.11}$$

where $t \in (a,b)$.

Similar transforms can be associated to the following functions as well:

$$g(t) = \tan(\rho t), \ t \in [a,b] \subset \left[-\frac{\pi}{2\rho}, \frac{\pi}{2\rho}\right] \text{ where } \rho > 0,$$

and

$$g(t) = \tanh(\sigma t), \ t \in [a,b] \subset \mathbb{R} \text{ with } \sigma > 0.$$

Motivated by the above results, we establish in this paper some inequalities for the generalized finite Hilbert transform of convex functions on an interval. Applications for some particular instances of finite Hilbert transforms such as the ones from (1.6)-(1.11) are given as well.

2. Main Results

Consider the function $\mathbf{1}(t) = 1$, $t \in (a,b)$. We need the following preliminary result:

Lemma 1. *For a continuous strictly increasing function* $g : [a,b] \to [g(a), g(b)]$ *that is differentiable on* (a,b) *we have*

$$(T_g\mathbf{1})(a,b;t) = \frac{1}{\pi}\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right),\ t\in(a,b). \tag{2.1}$$

We also have for $f : (a,b) \to \mathbb{C}$ *that*

$$(T_g f)(a,b;t) = \frac{1}{\pi} f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right) + \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau \tag{2.2}$$

for $t \in (a,b)$, *provided that the* PV *from the right hand side of the equality (2.2) exists.*

Proof. We have

$$\begin{aligned}(T_g\mathbf{1})(a,b;t) &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{g'(\tau)}{g(\tau)-g(t)}d\tau + \int_{t+\varepsilon}^b \frac{g'(\tau)}{g(\tau)-g(t)}d\tau\right] \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\ln|g(\tau)-g(t)|\big|_a^{t-\varepsilon} + \ln(g(\tau)-g(t))\big|_{t+\varepsilon}^b\right] \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\ln(g(t)-g(t-\varepsilon)) - \ln(g(t)-g(a))\right. \\ &\quad \left.+\ln(g(b)-g(t)) - \ln(g(t+\varepsilon)-g(t))\right] \\ &= \frac{1}{\pi}\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right) + \frac{1}{\pi}\lim_{\varepsilon\to 0+}\ln\left(\frac{g(t)-g(t-\varepsilon)}{g(t+\varepsilon)-g(t)}\right)\end{aligned} \tag{2.3}$$

for $t \in (a,b)$.

Since g is differentiable, we have

$$\lim_{\varepsilon\to 0+}\frac{g(t)-g(t-\varepsilon)}{g(t+\varepsilon)-g(t)} = \lim_{\varepsilon\to 0+}\frac{\frac{g(t)-g(t-\varepsilon)}{\varepsilon}}{\frac{g(t+\varepsilon)-g(t)}{\varepsilon}} = \frac{g'(t)}{g'(t)} = 1$$

for $t \in (a,b)$, and by (2.3) we get (2.1).

From the definition (1.5) we deduce

$$\begin{aligned}(T_g f)(a,b;t) &:= \frac{1}{\pi}PV\int_a^b \frac{(f(\tau)-f(t)+f(t))\,g'(\tau)}{g(\tau)-g(t)}d\tau \\ &= \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))\,g'(\tau)\,d\tau}{g(\tau)-g(t)} + \frac{1}{\pi}PV\int_a^b\frac{f(t)\,g'(\tau)\,d\tau}{g(\tau)-g(t)} \\ &= \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))\,g'(\tau)\,d\tau}{g(\tau)-g(t)} + \frac{1}{\pi}f(t)\,PV\int_a^b\frac{g'(\tau)\,d\tau}{g(\tau)-g(t)} \\ &= \frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right) + \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))\,g'(\tau)\,d\tau}{g(\tau)-g(t)}\end{aligned}$$

for $t \in (a,b)$, which proves the identity (2.2). □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right). \tag{2.4}$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean.* If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean.* If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean.* If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function.*

Theorem 2. *Assume that* $g:[a,b]\to[g(a),g(b)]$ *is a continuous strictly increasing function that is differentiable on* (a,b), f *a function such that* $f\circ g^{-1}:(g(a),g(b))\to\mathbb{R}$ *is a convex function on* $(g(a),g(b))$. *Then for* $t\in(a,b)$ *we have*

$$\begin{aligned}\text{(2.5)}\quad &\frac{1}{\pi}\left[f(t)-f(a)+[g(b)-g(t)]\frac{f'_+(t)}{g'(t)}\right]\\ &\leq (T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\\ &\leq \frac{1}{\pi}\left[f(b)-f(t)+[g(t)-g(a)]\frac{f'_-(t)}{g'(t)}\right].\end{aligned}$$

In particular, we have

$$\begin{aligned}\text{(2.6)}\quad &\frac{1}{\pi}\left[f(M_g(a,b))-f(a)+\frac{g(b)-g(a)}{2}\cdot\frac{f'_+(M_g(a,b))}{g'(M_g(a,b))}\right]\\ &\leq (T_g f)(a,b;M_g(a,b))\\ &\leq \frac{1}{\pi}\left[f(b)-f(M_g(a,b))+\frac{g(b)-g(a)}{2}\cdot\frac{f'_-(M_g(a,b))}{g'(M_g(a,b))}\right].\end{aligned}$$

Proof. For $t,\tau\in(a,b)$ with $t\neq\tau$ we then have

$$\frac{f(\tau)-f(t)}{g(\tau)-g(t)}=\frac{f\circ g^{-1}(g(\tau))-f\circ g^{-1}(g(t))}{g(\tau)-g(t)}. \tag{2.7}$$

By the convexity of $f\circ g^{-1}$ we can state that for all $g(a)\leq c<d\leq g(b)$ we have

$$\left(f\circ g^{-1}\right)'_-(d)\geq\frac{\left(f\circ g^{-1}\right)(d)-\left(f\circ g^{-1}\right)(c)}{d-c}\geq\left(f\circ g^{-1}\right)'_+(c). \tag{2.8}$$

Since $f\circ g^{-1}$ has lateral derivatives for $z\in(g(a),g(b))$ it follows f has lateral derivatives in each point of (a,b) and by the chain rule and the derivative of the inverse function,

$$\left(f\circ g^{-1}\right)'_\pm(z)=\left(f'_\pm\circ g^{-1}\right)(z)\left(g^{-1}\right)'(z)=\frac{\left(f'_\pm\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)}. \tag{2.9}$$

Let $t \in (a,b)$ and $t-a>\varepsilon>0$, then by (2.8) and (2.9) we have

$$\frac{f\circ g^{-1}(g(\tau))-f\circ g^{-1}(g(t))}{g(\tau)-g(t)}=\frac{f\circ g^{-1}(g(t))-f\circ g^{-1}(g(\tau))}{g(t)-g(\tau)} \geq \frac{\left(f'_{+}\circ g^{-1}\right)(g(\tau))}{\left(g'\circ g^{-1}\right)(g(\tau))}=\frac{f'_{+}(\tau)}{g'(\tau)} \tag{2.10}$$

for $\tau \in (a, t-\varepsilon)$.

If we integrate the inequality (2.10) over τ on $(a, t-\varepsilon)$, we get by (2.7) that

$$\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau \geq \int_a^{t-\varepsilon}\frac{f'_{+}(\tau)}{g'(\tau)}g'(\tau)\,d\tau = \int_a^{t-\varepsilon} f'_{+}(\tau)\,d\tau = f(t-\varepsilon)-f(a) \tag{2.11}$$

for $t\in(a,b)$ and $t-a>\varepsilon>0$.

Let $t\in(a,b)$ and $b-t>\varepsilon>0$, then

$$\frac{f\circ g^{-1}(g(\tau))-f\circ g^{-1}(g(t))}{g(\tau)-g(t)} \geq \frac{\left(f'_{+}\circ g^{-1}\right)(g(t))}{\left(g'\circ g^{-1}\right)(g(t))}=\frac{f'_{+}(t)}{g'(t)}$$

for $\tau\in(t+\varepsilon,b)$.

This implies that

$$\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau \geq \int_{t+\varepsilon}^{b}\frac{f'_{+}(t)}{g'(t)}g'(\tau)\,d\tau = \frac{f'_{+}(t)}{g'(t)}\left[g(b)-g(t+\varepsilon)\right] \tag{2.12}$$

for $t\in(a,b)$ and $b-t>\varepsilon>0$.

By adding the inequalities (2.11) and (2.12) we get

$$\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau+\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau \geq f(t-\varepsilon)-f(a)+\frac{f'_{+}(t)}{g'(t)}\left[g(b)-g(t+\varepsilon)\right] \tag{2.13}$$

for $t\in(a,b)$ and $\min\{b-t,t-a\}>\varepsilon>0$.

By taking the limit over $\varepsilon\to 0+$ in (2.13) we get

$$PV\int_a^b\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau \geq f(t)-f(a)+\frac{f'_{+}(t)}{g'(t)}\left[g(b)-g(t)\right] \tag{2.14}$$

for $t\in(a,b)$.

By using the identity (2.2) we get the first inequality in (2.5).

Let $t \in (a,b)$ and $t-a>\varepsilon>0$, then by (2.8) and (2.9) we also have

$$\frac{f\circ g^{-1}(g(\tau))-f\circ g^{-1}(g(t))}{g(\tau)-g(t)}=\frac{f\circ g^{-1}(g(t))-f\circ g^{-1}(g(\tau))}{g(t)-g(\tau)} \leq \frac{\left(f'_{-}\circ g^{-1}\right)(g(t))}{\left(g'\circ g^{-1}\right)(g(t))}=\frac{f'_{-}(t)}{g'(t)} \tag{2.15}$$

for $\tau \in (a, t-\varepsilon)$.

If we integrate the inequality (2.15) over τ on $(a, t-\varepsilon)$, we get by (2.7) that

$$\int_a^{t-\varepsilon} \frac{f(\tau) - f(t)}{g(\tau) - g(t)} g'(\tau)\, d\tau \le \int_a^{t-\varepsilon} \frac{f'_-(t)}{g'(t)} g'(\tau)\, d\tau = \frac{f'_-(t)}{g'(t)} [g(t-\varepsilon) - g(a)] \tag{2.16}$$

for $t \in (a, b)$ and $t - a > \varepsilon > 0$.

Let $t \in (a, b)$ and $b - t > \varepsilon > 0$, then

$$\frac{f \circ g^{-1}(g(\tau)) - f \circ g^{-1}(g(t))}{g(\tau) - g(t)} \le \frac{\left(f'_- \circ g^{-1}\right)(g(\tau))}{\left(g' \circ g^{-1}\right)(g(\tau))} = \frac{f'_-(\tau)}{g'(\tau)} \tag{2.17}$$

for $\tau \in (t+\varepsilon, b)$.

If we integrate the inequality (2.17) over τ on $(t+\varepsilon, b)$, we get

$$\int_{t+\varepsilon}^{b} \frac{f(\tau) - f(t)}{g(\tau) - g(t)} g'(\tau)\, d\tau \le \int_{t+\varepsilon}^{b} \frac{f'_-(\tau)}{g'(\tau)} g'(\tau)\, d\tau = f(b) - f(t+\varepsilon) \tag{2.18}$$

for $t \in (a, b)$ and $b - t > \varepsilon > 0$.

By adding the inequalities (2.16) and (2.18) we get

$$\int_a^{t-\varepsilon} \frac{f(\tau) - f(t)}{g(\tau) - g(t)} g'(\tau)\, d\tau + \int_{t+\varepsilon}^{b} \frac{f(\tau) - f(t)}{g(\tau) - g(t)} g'(\tau)\, d\tau \le \frac{f'_-(t)}{g'(t)} [g(t-\varepsilon) - g(a)] + f(b) - f(t+\varepsilon) \tag{2.19}$$

for $t \in (a, b)$ and $\min\{b-t, t-a\} > \varepsilon > 0$.

By taking the limit over $\varepsilon \to 0+$ in (2.19) we get

$$PV \int_a^b \frac{f(\tau) - f(t)}{g(\tau) - g(t)} g'(\tau)\, d\tau \le f(b) - f(t) + \frac{f'_-(t)}{g'(t)} [g(t) - g(a)]$$

for $t \in (a, b)$.

By using the identity (2.2) we obtain the second inequality in (2.5). □

Remark 1. *With the assumptions of Theorem 2, and if f is differentiable on (a, b), then we have*

$$\frac{1}{\pi} \left[f(t) - f(a) + [g(b) - g(t)] \frac{f'(t)}{g'(t)} \right] \le (T_g f)(a, b; t) - \frac{1}{\pi} f(t) \ln\left(\frac{g(b) - g(t)}{g(t) - g(a)} \right) \le \frac{1}{\pi} \left[f(b) - f(t) + [g(t) - g(a)] \frac{f'(t)}{g'(t)} \right] \tag{2.20}$$

for all $t \in (a, b)$.

In particular, we have

$$\frac{1}{\pi} \left[f(M_g(a, b)) - f(a) + \frac{g(b) - g(a)}{2} \frac{f'(M_g(a, b))}{g'(M_g(a, b))} \right] \le (T_g f)(a, b; M_g(a, b)) \le \frac{1}{\pi} \left[f(b) - f(M_g(a, b)) + \frac{g(b) - g(a)}{2} \frac{f'(M_g(a, b))}{g'(M_g(a, b))} \right]. \tag{2.21}$$

We also have:

Theorem 3. *Assume that* $g:[a,b]\to[g(a),g(b)]$ *is a continuous strictly increasing function that is differentiable on* (a,b) *and* $g'_+(a)$ *and* $g_-(b)$ *are finite. If* $f\circ g^{-1}:(g(a),g(b))\to\mathbb{R}$ *is a convex function on* (a,b) *and* f *has finite lateral derivatives* $f'_+(a)$ *and* $f_-(b)$, *then for* $t\in(a,b)$ *we have*

$$\begin{aligned}(2.22)\quad \frac{f'_+(a)}{\pi g'_+(a)}[g(b)-g(a)] &\le \frac{f(t)-f(a)}{\pi[g(t)-g(a)]}[g(b)-g(a)]\\ &\le (T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\\ &\le \frac{f(b)-f(t)}{\pi[g(b)-g(t)]}[g(b)-g(a)]\le\frac{f'_-(b)}{\pi g'_-(b)}[g(b)-g(a)].\end{aligned}$$

In particular, for $t=M_g(a,b)$ *we get*

$$\begin{aligned}(2.23)\quad \frac{f'_+(a)}{\pi g'_+(a)}[g(b)-g(a)] &\le \frac{2}{\pi}[f(M_g(a,b))-f(a)]\\ &\le (T_g f)(a,b;M_g(a,b))\\ &\le \frac{2}{\pi}[f(b)-f(M_g(a,b))]\le\frac{f'_-(b)}{\pi g'_-(b)}[g(b)-g(a)].\end{aligned}$$

Proof. We recall that if $\Phi:I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I and $\alpha\in I$ then the divided difference function $\Phi_\alpha:I\setminus\{\alpha\}\to\mathbb{R}$,

$$\Phi_\alpha(t):=[\alpha,t;\Phi]:=\frac{\Phi(t)-\Phi(\alpha)}{t-\alpha}$$

is monotonic nondecreasing on $I\setminus\{\alpha\}$.

Using this property for the function $\Phi:(c,d)\to\mathbb{R}$, we have for $t\in(c,d)$ that

$$\frac{\Phi(c)-\Phi(t)}{c-t}\le\frac{\Phi(\tau)-\Phi(t)}{\tau-t}\le\frac{\Phi(d)-\Phi(t)}{d-t}$$

for any $\tau\in(c,d)$, $\tau\neq t$.

By the gradient inequality for the convex function Φ we also have

$$\frac{\Phi(t)-\Phi(c)}{t-c}\ge\Phi'_+(c)\ \text{ for } t\in(c,d)$$

and

$$\frac{\Phi(d)-\Phi(t)}{d-t}\le\Phi_-(d)\ \text{ for } t\in(c,d).$$

Therefore we have the following inequality

$$\begin{aligned}(2.24)\quad \Phi'_+(c)\le\frac{\Phi(t)-\Phi(c)}{t-c}&\le\frac{\Phi(\tau)-\Phi(t)}{\tau-t}\\ &\le\frac{\Phi(d)-\Phi(t)}{d-t}\le\Phi_-(d)\end{aligned}$$

for t, $\tau\in(c,d)$ and $\tau\neq t$.

If we write the inequality (2.24) for the convex function $\Phi = f \circ g^{-1}$ and the interval $(g(a), g(b))$, we get

$$\begin{aligned} \frac{\left(f'_{+} \circ g^{-1}\right)(g(a))}{\left(g'_{+} \circ g^{-1}\right)(g(a))} &\leq \frac{\left(f \circ g^{-1}\right)(g(t)) - \left(f \circ g^{-1}\right)(g(a))}{g(t) - g(a)} \\ &\leq \frac{\left(f \circ g^{-1}\right)(g(\tau)) - \left(f \circ g^{-1}\right)(g(t))}{g(\tau) - g(t)} \\ &\leq \frac{\left(f \circ g^{-1}\right)(g(b)) - \left(f \circ g^{-1}\right)(g(t))}{g(b) - g(t)} \\ &\leq \frac{\left(f'_{-} \circ g^{-1}\right)(g(b))}{\left(g'_{-} \circ g^{-1}\right)(g(b))} \end{aligned} \tag{2.25}$$

for $t, \tau \in (a, b)$ and $\tau \neq t$.

This is equivalent to

$$\frac{f'_{+}(a)}{g'_{+}(a)} \leq \frac{f(t) - f(a)}{g(t) - g(a)} \leq \frac{f(\tau) - f(t)}{g(\tau) - g(t)} \leq \frac{f(b) - f(t)}{g(b) - g(t)} \leq \frac{f'_{-}(b)}{g'_{-}(b)} \tag{2.26}$$

for $t, \tau \in (a, b)$ and $\tau \neq t$.

If we multiply with $g'(\tau) \geq 0$ and take the PV in (2.26), then we get

$$\begin{aligned} \frac{f'_{+}(a)}{g'_{+}(a)} \int_a^b g'(\tau)\, d\tau &\leq \frac{f(t) - f(a)}{g(t) - g(a)} \int_a^b g'(\tau)\, d\tau \\ &\leq PV \int_a^b \frac{f(\tau) - f(t)}{g(\tau) - g(t)} g'(\tau)\, d\tau \\ &\leq \int_a^b \frac{f(b) - f(t)}{g(b) - g(t)} g'(\tau)\, d\tau \leq \frac{f'_{-}(b)}{g'_{-}(b)} \int_a^b g'(\tau)\, d\tau \end{aligned}$$

for $t \in (a, b)$, which is equivalent to

$$\begin{aligned} \frac{f'_{+}(a)}{g'_{+}(a)} [g(b) - g(a)] &\leq \frac{f(t) - f(a)}{g(t) - g(a)} [g(b) - g(a)] \\ &\leq PV \int_a^b \frac{f(\tau) - f(t)}{g(\tau) - g(t)} g'(\tau)\, d\tau \\ &\leq \frac{f(b) - f(t)}{g(b) - g(t)} [g(b) - g(a)] \leq \frac{f'_{-}(b)}{g'_{-}(b)} [g(b) - g(a)] \end{aligned}$$

for $t \in (a, b)$.

By the use of the identity (2.2) we obtain the desired result (2.22). □

3. Applications for GA-Convex Functions

Let $I \subset (0, \infty)$ be an interval; a real-valued function $f : I \to \mathbb{R}$ is said to be *GA-convex* (concave) on I if

$$f\left(x^{1-\lambda} y^{\lambda}\right) \leq (\geq) (1 - \lambda) f(x) + \lambda f(y) \tag{3.1}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Since the condition (3.1) can be written as

$$f \circ \exp((1 - \lambda) \ln x + \lambda \ln y) \leq (\geq) (1 - \lambda) f \circ \exp(\ln x) + \lambda f \circ \exp(\ln y), \tag{3.2}$$

then we observe that $f : I \to \mathbb{R}$ is *GA-convex* (concave) on I if and only if $f \circ \exp$ is convex (concave) on $\ln I := \{\ln z, z \in I\}$. If $I = [a, b]$ then $\ln I = [\ln a, \ln b]$.

It is known that the function $f(x) = \ln(1 + x)$ is GA-convex on $(0, \infty)$ [1].

For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [21] that the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is GA-concave on $(0, \infty)$ while the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is GA-convex on $(0, \infty)$.

If $[a, b] \subset (0, \infty)$ and the function $g : [\ln a, \ln b] \to \mathbb{R}$ is convex (concave) on $[\ln a, \ln b]$, then the function $f : [a, b] \to \mathbb{R}$, $f(t) = g(\ln t)$ is GA-convex (concave) on $[a, b]$.

Indeed, if $x, y \in [a, b]$ and $\lambda \in [0, 1]$, then

$$\begin{aligned} f\left(x^{1-\lambda} y^{\lambda}\right) &= g\left(\ln\left(x^{1-\lambda} y^{\lambda}\right)\right) = g\left[(1-\lambda)\ln x + \lambda \ln y\right] \\ &\leq (\geq)(1-\lambda) g(\ln x) + \lambda g(\ln y) = (1-\lambda) f(x) + \lambda f(y) \end{aligned}$$

showing that f is GA-convex (concave) on $[a, b]$.

Consider the following *logarithmic finite Hilbert transform*

$$(T_{\ln} f)(a, b; t) := \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau \ln\left(\frac{\tau}{t}\right)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau \ln\left(\frac{\tau}{t}\right)} d\tau\right] \tag{3.3}$$

where $t \in (a, b) \subset (0, \infty)$.

Proposition 1. *Assume that $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ is GA-convex on $[a, b]$, then*

$$\begin{aligned} &\frac{1}{\pi}\left[f(t) - f(a) + t f'_+(t) \ln\left(\frac{b}{t}\right)\right] \\ &\leq (T_{\ln} f)(a, b; t) - \frac{1}{\pi} f(t) \ln\left(\frac{\ln\left(\frac{b}{t}\right)}{\ln\left(\frac{t}{a}\right)}\right) \\ &\leq \frac{1}{\pi}\left[f(b) - f(t) + t f'_-(t) \ln\left(\frac{t}{a}\right)\right] \end{aligned} \tag{3.4}$$

for all $t \in (a, b)$.

In particular,

$$\begin{aligned} &\frac{1}{\pi}\left[f(G(a, b)) - f(a) + G(a, b) \ln\left(\sqrt{\frac{b}{a}}\right) f'_+(G(a, b))\right] \\ &\leq (T_g f)(a, b; G(a, b)) \\ &\leq \frac{1}{\pi}\left[f(b) - f(G(a, b)) + G(a, b) \ln\left(\sqrt{\frac{b}{a}}\right) f'_-(G(a, b))\right], \end{aligned} \tag{3.5}$$

where $G(a, b) := \sqrt{ab}$ is the geometric mean of a, $b > 0$.

The proof follows by Theorem 2 for $g(t) = \ln t,\ t \in (a,b)$.

Proposition 2. *With the assumptions of Proposition 1 and if $f'_+(a)$ and $f'_-(b)$ are finite, then*

$$\begin{aligned}(3.6)\qquad \frac{af'_+(a)}{\pi}\ln\left(\frac{b}{a}\right) &\le \frac{f(t)-f(a)}{\pi}\frac{\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{t}{a}\right)} \\ &\le (T_{\ln} f)(a,b;t) - \frac{1}{\pi} f(t)\ln\left(\frac{\ln\left(\frac{b}{t}\right)}{\ln\left(\frac{t}{a}\right)}\right) \\ &\le \frac{f(b)-f(t)}{\pi}\frac{\ln\left(\frac{b}{a}\right)}{\ln\left(\frac{b}{t}\right)} \le \frac{bf'_-(b)}{\pi}\ln\left(\frac{b}{a}\right),\end{aligned}$$

for any $t \in (a,b)$.

In particular,

$$\begin{aligned}(3.7)\qquad \frac{af'_+(a)}{\pi}\ln\left(\frac{b}{a}\right) &\le \frac{2}{\pi}\left[f(G(a,b)) - f(a)\right] \\ &\le (T_{\ln} f)(a,b;G(a,b)) \\ &\le \frac{2}{\pi}\left[f(b) - f(G(a,b))\right] \le \frac{bf'_-(b)}{\pi}\ln\left(\frac{b}{a}\right).\end{aligned}$$

The proof follows by Theorem 3 for $g(t) = \ln t,\ t \in (a,b)$.

4. Application for *LogExp* Convex Function

We say that the function $f : [a,b] \to \mathbb{R}$ is a *LogExp convex function* on $[a,b]$ if $f \circ \ln$ is convex on the interval $[\exp a, \exp b]$, namely

$$(4.1)\qquad (f\circ\ln)((1-\lambda)u + \lambda v) \le (1-\lambda)(f\circ\ln)(u) + \lambda(f\circ\ln)(v)$$

for any $\lambda \in [0,1]$ and $u,\ v \in [\exp a, \exp b]$.

By taking $u = \exp t,\ v = \exp s,\ t,\ s \in [a,b]$, this is equivalent to

$$(4.2)\qquad f\left[\ln((1-\lambda)\exp t + \lambda \exp s)\right] \le (1-\lambda) f(t) + \lambda f(s)$$

for any $\lambda \in [0,1]$ and $t,\ s \in [a,b]$.

For $g(t) = \exp(t),\ t \in [a,b] \subset \mathbb{R}$ we have the *exponential finite Hilbert transform*

$$\begin{aligned}(4.3)\qquad &(T_{\exp} f)(a,b;t) \\ &:= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\exp(\tau)}{\exp(\tau)-\exp(t)}d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)\exp(\tau)}{\exp(\tau)-\exp(t)}d\tau\right] \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{1-\exp(t-\tau)}d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{1-\exp(t-\tau)}d\tau\right],\end{aligned}$$

where $t \in (a,b)$.

Proposition 3. *Assume that $f:[a,b]\to\mathbb{R}$ is LogExp convex function on $[a,b]$, then*

$$\begin{aligned}(4.4)\quad &\frac{1}{\pi}\left[f(t)-f(a)+[\exp(b-t)-1]f'_+(t)\right]\\ &\leq (T_{\exp}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\exp(b-t)-1}{1-\exp(a-t)}\right)\\ &\leq \frac{1}{\pi}\left[f(b)-f(t)+[1-\exp(a-t)]f'_-(t)\right]\end{aligned}$$

for any $t\in(a,b)$.

In particular,

$$\begin{aligned}(4.5)\quad &\frac{1}{\pi}\left[f(LME(a,b))-f(a)+\frac{\exp(b)-\exp(a)}{\exp(b)+\exp(a)}f'_+(LME(a,b))\right]\\ &\leq (T_g f)(a,b;LME(a,b))\\ &\leq \frac{1}{\pi}\left[f(b)-f(LME(a,b))+\frac{\exp(b)-\exp(a)}{\exp(b)+\exp(a)}f'_-(LME(a,b))\right],\end{aligned}$$

where $LME(a,b)=\ln\left(\frac{\exp a+\exp b}{2}\right)$ is the the LogMeanExp function of a, b.

The proof follows by Theorem 2 for $g(t)=\exp t$, $t\in(a,b)$.

Proposition 4. *With the assumptions of Proposition 3 and if $f'_+(a)$ and $f'_-(b)$ are finite, then*

$$\begin{aligned}(4.6)\quad &\frac{f'_+(a)}{\pi}[\exp(b-a)-1]\leq\frac{f(t)-f(a)}{\pi}\left[\frac{\exp(b-a)-1}{\exp(t-a)-1}\right]\\ &\leq (T_{\exp}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\exp(b-t)-1}{1-\exp(a-t)}\right)\\ &\leq \frac{f(b)-f(t)}{\pi}\left[\frac{1-g(a-b)}{1-g(t-b)}\right]\leq\frac{f'_-(b)}{\pi}[1-\exp(a-b)]\end{aligned}$$

for $t\in(a,b)$.

In particular,

$$\begin{aligned}(4.7)\quad &\frac{f'_+(a)}{\pi}[\exp(b-a)-1]\leq\frac{2}{\pi}[f(LME(a,b))-f(a)]\\ &\leq (T_{\exp}f)(a,b;LME(a,b))\\ &\leq\frac{2}{\pi}[f(b)-f(LME(a,b))]\leq\frac{f'_-(b)}{\pi}[1-\exp(a-b)].\end{aligned}$$

The proof follows by Theorem 3 for $g(t)=\exp t$, $t\in(a,b)$.

5. Application for Positive p-Convex Function

Let $p>0$. We say that the function $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ is a *positive p-convex function* on $[a,b]$ if $f\circ(\cdot)^{1/p}$ is convex on the interval $[a^p,b^p]$, namely

$$(5.1)\qquad f\left[((1-\lambda)u+\lambda v)^{1/p}\right]\leq(1-\lambda)f\left(u^{1/p}\right)+\lambda f\left(v^{1/p}\right)$$

for any $\lambda\in[0,1]$ and u, $v\in[a^p,b^p]$.

By taking $u=t^p$, $v=s^p$, $t,\ s\in[a,b]$, this is equivalent to, see also [22]

$$f\left[\left((1-\lambda)t^p+\lambda s^p\right)^{1/p}\right]\le(1-\lambda)f(t)+\lambda f(s) \tag{5.2}$$

for any $\lambda\in[0,1]$ and $t,\ s\in[a,b]$.

For $[a,b]\subset(0,\infty)$ and $g(t)=t^p$, $t\in[a,b]$, $p>0$, we consider the *positive p-power Hilbert transform*

$$(T_pf)(a,b;t):=\frac{p}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\tau^{p-1}}{\tau^p-t^p}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\tau^{p-1}}{\tau^p-t^p}d\tau\right], \tag{5.3}$$

where $t\in(a,b)$.

Proposition 5. *Assume that $f:[a,b]\to\mathbb{R}$ is positive p-convex function on $[a,b]$, then*

$$\begin{aligned}\frac{1}{\pi}&\left[f(t)-f(a)+\frac{b^p-t^p}{pt^{p-1}}f'_+(t)\right]\\&\le(T_pf)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{b^p-t^p}{t^p-a^p}\right)\\&\le\frac{1}{\pi}\left[f(b)-f(t)+\frac{t^p-a^p}{pt^{p-1}}f'_-(t)\right]\end{aligned} \tag{5.4}$$

for $t\in(a,b)$.

In particular, we have

$$\begin{aligned}\frac{1}{\pi}&\left[f(M_p(a,b))-f(a)+\frac{b^p-a^p}{2pM_p^{p-1}(a,b)}f'_+(M_p(a,b))\right]\\&\le(T_gf)(a,b;M_p(a,b))\\&\le\frac{1}{\pi}\left[f(b)-f(M_p(a,b))+\frac{b^p-a^p}{2pM_p^{p-1}(a,b)}f'_-(M_p(a,b))\right],\end{aligned} \tag{5.5}$$

where $M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$.

The proof follows by Theorem 2 for $g(t)=t^p$, $t\in[a,b]$.

Proposition 6. *With the assumptions of Proposition 5 and if $f'_+(a)$ and $f'_-(b)$ are finite, then*

$$\begin{aligned}\frac{b^p-a^p}{p\pi a^{p-1}}f'_+(a)&\le\frac{f(t)-f(a)}{\pi}\left(\frac{b^p-a^p}{t^p-a^p}\right)\\&\le(T_pf)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{b^p-t^p}{t^p-a^p}\right)\\&\le\frac{f(b)-f(t)}{\pi}\left(\frac{b^p-a^p}{b^p-t^p}\right)\le\frac{b^p-a^p}{p\pi a^{p-1}}f'_-(b).\end{aligned} \tag{5.6}$$

In particular,

$$\begin{aligned}\frac{b^p-a^p}{p\pi a^{p-1}}f'_+(a)&\le\frac{2}{\pi}\left[f(M_p(a,b))-f(a)\right]\\&\le(T_pf)(a,b;M_p(a,b))\\&\le\frac{2}{\pi}\left[f(b)-f(M_p(a,b))\right]\le\frac{b^p-a^p}{p\pi a^{p-1}}f'_-(b).\end{aligned} \tag{5.7}$$

The proof follows by Theorem 3 for $g(t) = t^p$, $t \in (a,b)$.

Similar results may be stated for *negative p-power convex functions*, namely for $g(t) = -\frac{1}{t^p}$, $t \in [a,b] \subset (0,\infty)$. The details are omitted.

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