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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR COMPOSITE CONVEX FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we obtain some inequalities of Hermite-Hadamard type for composite convex functions. Applications for AG, AH-convex functions, GA, GG, GH-convex functions and HA, HG, HH-convex function are given. Applications for p, r-convex and $LogExp$ convex functions are presented as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [18]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [3]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [18]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality.*

In order to extend this result for other classes of functions, we need the following preparations.

Let $g:[a,b]\to[g(a),g(b)]$ be a *continuous strictly increasing function* that is *differentiable* on (a,b).

Definition 1. *A function $f:[a,b]\to\mathbb{R}$ will be called composite-g^{-1} convex (concave) on $[a,b]$ if the composite function $f\circ g^{-1}:[g(a),g(b)]\to\mathbb{R}$ is convex (concave) in the usual sense on $[g(a),g(b)]$.*

In this way, any concept of convexity (log-convexity, harmonic convexity, trigonometric convexity, hyperbolic convexity, h-convexity, quasi-convexity, s-convexity, s-Godunova-Levin convexity etc...) can be extended to the corresponding *composite-g^{-1} convexity.* The details however will not be presented here.

If $f:[a,b]\to\mathbb{R}$ is composite-g^{-1} convex on $[a,b]$ then we have the inequality

$$f\circ g^{-1}((1-\lambda)u+\lambda v) \le (1-\lambda)f\circ g^{-1}(u)+\lambda f\circ g^{-1}(v) \tag{1.2}$$

for any $u,\ v\in[g(a),g(b)]$ and $\lambda\in[0,1]$.

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This is equivalent to the condition

$$f\circ g^{-1}\left(\left(1-\lambda \right) g\left(t\right) +\lambda g\left(s\right) \right) \leq \left(1-\lambda \right) f\left(t\right) +\lambda f\left(s\right) \tag{1.3}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$.

If we take $g\left(t\right) =\ln t,\ t\in \left[a,b\right] \subset \left(0,\infty \right)$, then the condition (1.3) becomes

$$f\left(t^{1-\lambda }s^{\lambda }\right) \leq \left(1-\lambda \right) f\left(t\right) +\lambda f\left(s\right) \tag{1.4}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$, which is the concept of GA-*convexity* as considered in [1].

If we take $g\left(t\right) =-\frac{1}{t},\ t\in \left[a,b\right] \subset \left(0,\infty \right)$, then (1.3) becomes

$$f\left(\frac{ts}{\left(1-\lambda \right) s+\lambda t}\right) \leq \left(1-\lambda \right) f\left(t\right) +\lambda f\left(s\right) \tag{1.5}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$, which is the concept of HA-*convexity* as considered in [1].

If $p>0$ and we consider $g\left(t\right) =t^{p},\ t\in \left[a,b\right] \subset \left(0,\infty \right)$, then the condition (1.3) becomes

$$f\left[\left(\left(1-\lambda \right) t^{p}+\lambda s^{p}\right) ^{1/p}\right] \leq \left(1-\lambda \right) f\left(t\right) +\lambda f\left(s\right) \tag{1.6}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$, which is the concept of p-*convexity* as considered in [22].

If we take $g\left(t\right) =\exp t,\ t\in \left[a,b\right]$, then the condition (1.3) becomes

$$f\left[\ln \left(\left(1-\lambda \right) \exp \left(t\right) +\exp g\left(s\right) \right) \right] \leq \left(1-\lambda \right) f\left(t\right) +\lambda f\left(s\right) \tag{1.7}$$

which is the concept of *LogExp convex function* on $\left[a,b\right]$ as considered in [7].

Further, assume that $f:\left[a,b\right] \rightarrow J$, J an interval of real numbers and $k:J\rightarrow \mathbb{R}$ a continuous function on J that is *strictly increasing (decreasing)* on J.

Definition 2. *We say that the function $f:\left[a,b\right] \rightarrow J$ is k-composite convex (concave) on $\left[a,b\right]$, if $k\circ f$ is convex (concave) on $\left[a,b\right]$.*

In this way, any concept of convexity as mentioned above can be extended to the corresponding k-*composite convexity.* The details however will not be presented here.

With $g:\left[a,b\right] \rightarrow \left[g\left(a\right) ,g\left(b\right) \right]$ a *continuous strictly increasing function* that is *differentiable* on $\left(a,b\right)$, $f:\left[a,b\right] \rightarrow J$, J an interval of real numbers and $k:J\rightarrow \mathbb{R}$ a continuous function on J that is *strictly increasing (decreasing)* on J, we can also consider the following concept:

Definition 3. *We say that the function $f:\left[a,b\right] \rightarrow J$ is k-composite-g^{-1} convex (concave) on $\left[a,b\right]$, if $k\circ f\circ g^{-1}$ is convex (concave) on $\left[g\left(a\right) ,g\left(b\right) \right]$.*

This definition is equivalent to the condition

$$k\circ f\circ g^{-1}\left(\left(1-\lambda \right) g\left(t\right) +\lambda g\left(s\right) \right) \leq \left(1-\lambda \right) \left(k\circ f\right) \left(t\right) +\lambda \left(k\circ f\right) \left(s\right) \tag{1.8}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$.

If $k:J\rightarrow \mathbb{R}$ is *strictly increasing (decreasing)* on J, then the condition (1.8) is equivalent to:

$$f\circ g^{-1}\left(\left(1-\lambda \right) g\left(t\right) +\lambda g\left(s\right) \right) \leq \left(\geq \right) k^{-1}\left[\left(1-\lambda \right) \left(k\circ f\right) \left(t\right) +\lambda \left(k\circ f\right) \left(s\right) \right] \tag{1.9}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$.

If $k(t) = \ln t$, $t > 0$ and $f : [a, b] \to (0, \infty)$, then the fact that f is k-composite convex on $[a, b]$ is equivalent to the fact that f is *log-convex* or *multiplicatively convex* or AG-convex, namely, for all x, $y \in I$ and $t \in [0, 1]$ one has the inequality:

$$f(tx + (1 - t) y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{1.10}$$

A function $f : I \to \mathbb{R} \setminus \{0\}$ is called *AH-convex (concave)* on the interval I if the following inequality holds [1]

$$f((1 - \lambda) x + \lambda y) \leq (\geq) \frac{1}{(1 - \lambda) \frac{1}{f(x)} + \lambda \frac{1}{f(y)}} = \frac{f(x) f(y)}{(1 - \lambda) f(y) + \lambda f(x)} \tag{1.11}$$

for any x, $y \in I$ and $\lambda \in [0, 1]$.

An important case that provides many examples is that one in which the function is assumed to be positive for any $x \in I$. In that situation the inequality (1.11) is equivalent to

$$(1 - \lambda) \frac{1}{f(x)} + \lambda \frac{1}{f(y)} \leq (\geq) \frac{1}{f((1 - \lambda) x + \lambda y)}$$

for any x, $y \in I$ and $\lambda \in [0, 1]$.

Taking into account this fact, we can conclude that the function $f : I \to (0, \infty)$ is AH-convex (concave) on I if and only if f is k-composite concave (convex) on I with $k : (0, \infty) \to (0, \infty)$, $k(t) = \frac{1}{t}$.

Following [1], we can introduce the concept of *GH-convex (concave)* function $f : I \subset (0, \infty) \to \mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$f\left(x^{1-\lambda} y^{\lambda}\right) \leq (\geq) \frac{1}{(1 - \lambda) \frac{1}{f(x)} + \lambda \frac{1}{f(y)}} = \frac{f(x) f(y)}{(1 - \lambda) f(y) + \lambda f(x)}. \tag{1.12}$$

Since

$$f\left(x^{1-\lambda} y^{\lambda}\right) = f \circ \exp[(1 - \lambda) \ln x + \lambda \ln y]$$

and

$$\frac{f(x) f(y)}{(1 - \lambda) f(y) + \lambda f(x)} = \frac{f \circ \exp(\ln x) f \circ \exp(\ln y)}{(1 - \lambda) f \circ \exp(y) + \lambda f \circ \exp(x)}$$

then $f : I \subset (0, \infty) \to \mathbb{R}$ is GH-convex (concave) on I if and only if $f \circ \exp$ is *AH-convex (concave)* on $\ln I := \{x | \ x = \ln t, \ t \in I\}$. This is equivalent to the fact that f is k-composite-g^{-1} concave (convex) on I with $k : (0, \infty) \to (0, \infty)$, $k(t) = \frac{1}{t}$ and $g(t) = \ln t$, $t \in I$.

Following [1], we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to (0, \infty)$ is *HH-convex* if

$$f\left(\frac{xy}{tx + (1 - t) y}\right) \leq \frac{f(x) f(y)}{(1 - t) f(y) + t f(x)} \tag{1.13}$$

for all x, $y \in I$ and $t \in [0, 1]$. If the inequality in (1.13) is reversed, then f is said to be *HH-concave*.

We observe that the inequality (1.13) is equivalent to

$$(1 - t) \frac{1}{f(x)} + t \frac{1}{f(y)} \leq \frac{1}{f\left(\frac{xy}{tx + (1 - t) y}\right)} \tag{1.14}$$

for all x, $y \in I$ and $t \in [0, 1]$.

This is equivalent to the fact that f is k-composite-g^{-1} concave on $[a, b]$ with $k : (0, \infty) \to (0, \infty)$, $k(t) = \frac{1}{t}$ and $g(t) = -\frac{1}{t}$, $t \in [a, b]$.

The function $f : I \subset (0, \infty) \to (0, \infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [1]

$$f\left(x^{1-\lambda}y^{\lambda}\right) \leq [f(x)]^{1-\lambda}[f(y)]^{\lambda} \tag{1.15}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality is reversed in (1.15) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [19], however, the roots of the research in this area can be traced long before him [20]. It is easy to see that [20], the function $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is *GG-convex* if and only if the the function $g : [\ln a, \ln b] \to \mathbb{R}$, $g = \ln \circ f \circ \exp$ is convex on $[\ln a, \ln b]$. This is equivalent to the fact that f is k-composite-g^{-1} convex on $[a, b]$ with $k : (0, \infty) \to \mathbb{R}$, $k(t) = \ln t$ and $g(t) = \ln t$, $t \in [a, b]$.

Following [1] we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to (0, \infty)$ is *HG-convex* if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq [f(x)]^{1-t}[f(y)]^{t} \tag{1.16}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.3) is reversed, then f is said to be *HG-concave*.

Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ and define the associated functions $G_f : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $G_f(t) = \ln f\left(\frac{1}{t}\right)$. Then f is *HG-convex* on $[a, b]$ iff G_f is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$. This is equivalent to the fact that f is k-composite-g^{-1} convex on $[a, b]$ with $k : (0, \infty) \to \mathbb{R}$, $k(t) = \ln t$ and $g(t) = -\frac{1}{t}$, $t \in [a, b]$.

Following [21], we say that the function $f : [a, b] \to (0, \infty)$ is r-convex, for $r \neq 0$, if

$$f((1-\lambda)x + \lambda y) \leq [(1-\lambda)f^r(y) + \lambda f^r(x)]^{1/r} \tag{1.17}$$

for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

If $r > 0$, then the condition (1.17) is equivalent to

$$f^r((1-\lambda)x + \lambda y) \leq (1-\lambda)f^r(y) + \lambda f^r(x)$$

namely f is k-composite convex on $[a, b]$ where $k(t) = t^r$, $t \geq 0$.

If $r < 0$, then the condition (1.17) is equivalent to

$$f^r((1-\lambda)x + \lambda y) \geq (1-\lambda)f^r(y) + \lambda f^r(x)$$

namely f is k-composite concave on $[a, b]$ where $k(t) = t^r$, $t > 0$.

In this paper we obtain some inequalities of Hermite-Hadamard type for *composite convex functions*. Applications for various classes of convexity as provided above are given as well.

2. Some Refinements

We need the following refinement of Hermite-Hadamard inequality. This result was obtained for the first time by Barnett, Cerone & Dragomir in 2002 in the paper [2, p. 10, Eq. (2.2)] where various applications for the Hermite-Hadamard divergence measure in Information Theory were also given. The same result was also rediscovered by El Farissi in 2010 with a similar proof, see [16].

Lemma 1. *Assume that $h:[c,d]\to\mathbb{R}$ is convex on $[c,d]$. Then for any $\lambda\in[0,1]$ we have*

$$
\begin{aligned}
h\left(\frac{c+d}{2}\right) &\le \lambda h\left(\frac{\lambda d+(2-\lambda)c}{2}\right)+(1-\lambda)h\left(\frac{(1+\lambda)d+(1-\lambda)c}{2}\right)\\
&\le \frac{1}{d-c}\int_c^d h(u)\,du\\
&\le \frac{1}{2}\left[h((1-\lambda)c+\lambda d)+\lambda h(c)+(1-\lambda)h(d)\right]\le \frac{h(c)+h(d)}{2}.
\end{aligned}
\tag{2.1}
$$

Proof. For the sake of completeness, we give here a simple proof as in [2]. Applying the Hermite-Hadamard inequality on each subinterval $[c,(1-\lambda)c+\lambda d]$, $[(1-\lambda)c+\lambda d,d]$, where $\lambda\in(0,1)$, then we have,

$$
\begin{aligned}
&h\left(\frac{c+(1-\lambda)c+\lambda d}{2}\right)\times[(1-\lambda)c+\lambda d-c]\\
&\le \int_c^{(1-\lambda)c+\lambda d} h(u)\,du\\
&\le \frac{h((1-\lambda)c+\lambda d)+h(c)}{2}\times[(1-\lambda)c+\lambda d-c]
\end{aligned}
$$

and

$$
\begin{aligned}
&h\left(\frac{(1-\lambda)c+\lambda d+d}{2}\right)\times[d-(1-\lambda)c-\lambda d]\\
&\le \int_{(1-\lambda)c+\lambda d}^{d} h(u)\,du\\
&\le \frac{h(d)+h((1-\lambda)c+\lambda d)}{2}\times[d-(1-\lambda)c-\lambda d],
\end{aligned}
$$

which are clearly equivalent to

$$
\begin{aligned}
\lambda h\left(\frac{\lambda d+(2-\lambda)c}{2}\right) &\le \frac{1}{d-c}\int_c^{(1-\lambda)c+\lambda d} h(u)\,du\\
&\le \frac{\lambda h((1-\lambda)c+\lambda d)+\lambda h(c)}{2}
\end{aligned}
\tag{2.2}
$$

and

$$
\begin{aligned}
(1-\lambda)h\left(\frac{(1+\lambda)d+(1-\lambda)c}{2}\right) &\le \frac{1}{d-c}\int_{(1-\lambda)c+\lambda d}^{d} h(u)\,du\\
&\le \frac{(1-\lambda)h(d)+(1-\lambda)h((1-\lambda)c+\lambda d)}{2},
\end{aligned}
\tag{2.3}
$$

respectively.

Summing (2.2) and (2.3), we obtain the second and first inequality in (2.1).

By the convexity property, we obtain

$$\begin{aligned}
&\lambda h\left(\frac{\lambda d+(2-\lambda) c}{2}\right)+(1-\lambda) h\left(\frac{(1+\lambda) d+(1-\lambda) c}{2}\right) \\
&\geq h\left[\lambda\left(\frac{\lambda d+(2-\lambda) c}{2}\right)+(1-\lambda)\left(\frac{(1+\lambda) d+(1-\lambda) c}{2}\right)\right] \\
&=h\left(\frac{c+d}{2}\right)
\end{aligned}$$

and the first inequality in (2.1) is proved. □

For various inequalities of Hermite-Hadamard type see the monograph online [8] and the more recent survey paper [6].

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\, b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right). \tag{2.4}$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right), \tag{2.5}$$

the *LogMeanExp function*.

Theorem 1. *Let $g:[a,b]\to[g(a),g(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If $f:[a,b]\to\mathbb{R}$ is composite-g^{-1} convex on $[a,b]$, then*

$$\begin{aligned}
f(M_g(a,b)) &\leq \lambda f\circ g^{-1}\left(\frac{\lambda g(b)+(2-\lambda) g(a)}{2}\right) \\
&\quad+(1-\lambda) f\circ g^{-1}\left(\frac{(1+\lambda) g(b)+(1-\lambda) g(a)}{2}\right) \\
&\leq \frac{1}{g(b)-g(a)}\int_a^b f(t) g'(t)\, dt \\
&\leq \frac{1}{2}\left[f\circ g^{-1}((1-\lambda) g(a)+\lambda g(b))+\lambda f(a)+(1-\lambda) f(b)\right] \\
&\leq \frac{f(a)+f(b)}{2}
\end{aligned} \tag{2.6}$$

for any $\lambda\in[0,1]$.

Proof. From the inequality (2.1) we have for the convex function $f\circ g^{-1}$ and $c, d\in[g(a), g(b)]$ that

$$\begin{aligned}
(2.7)\quad & f\circ g^{-1}\left(\frac{c+d}{2}\right)\\
&\leq \lambda f\circ g^{-1}\left(\frac{\lambda d+(2-\lambda)c}{2}\right)+(1-\lambda)f\circ g^{-1}\left(\frac{(1+\lambda)d+(1-\lambda)c}{2}\right)\\
&\leq \frac{1}{d-c}\int_c^d f\circ g^{-1}(u)\,du\\
&\leq \frac{1}{2}\left[f\circ g^{-1}((1-\lambda)c+\lambda d)+\lambda f\circ g^{-1}(c)+(1-\lambda)f\circ g^{-1}(d)\right]\\
&\leq \frac{f\circ g^{-1}(c)+f\circ g^{-1}(d)}{2}
\end{aligned}$$

for any $\lambda\in[0,1]$.

If we take $c=g(a)$ and $d=g(b)$, then we get

$$\begin{aligned}
(2.8)\quad & f\circ g^{-1}\left(\frac{g(a)+g(b)}{2}\right)\\
&\leq \lambda f\circ g^{-1}\left(\frac{\lambda g(b)+(2-\lambda)g(a)}{2}\right)\\
&+(1-\lambda)f\circ g^{-1}\left(\frac{(1+\lambda)g(b)+(1-\lambda)g(a)}{2}\right)\\
&\leq \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\circ g^{-1}(u)\,du\\
&\leq \frac{1}{2}\left[f\circ g^{-1}((1-\lambda)g(a)+\lambda g(b))+\lambda f(a)+(1-\lambda)f(b)\right]\\
&\leq \frac{f(a)+f(b)}{2}
\end{aligned}$$

for any $\lambda\in[0,1]$.

Using the change of variable $g^{-1}(u)=t$, $t\in[a,b]$ we have $u=g(t)$, $du=g'(t)\,dt$ and

$$\int_{g(a)}^{g(b)} f\circ g^{-1}(u)\,du=\int_a^b f(t)\,g'(t)\,dt$$

and by (2.8) we get the desired result (2.6). □

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
(2.9)\quad f(M_g(a,b)) &\leq \frac{1}{2}\left[f\circ g^{-1}\left(\frac{g(b)+3g(a)}{4}\right)+f\circ g^{-1}\left(\frac{g(a)+3g(b)}{4}\right)\right]\\
&\leq \frac{1}{g(b)-g(a)}\int_a^b f(t)\,g'(t)\,dt\\
&\leq \frac{1}{2}\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\leq \frac{f(a)+f(b)}{2}.
\end{aligned}$$

Remark 1. *Using the change of variable* $u = (1-s)c + sd$, $s \in [0,1]$, *then we have* $du = (d-c)\,ds$, *which gives that*

$$\frac{1}{d-c}\int_c^d h(u)\,du = \int_0^1 h((1-s)c+sd)\,ds.$$

Using this fact, we have from Theorem 1 the following inequality

$$\begin{aligned} f(M_g(a,b)) &\le \lambda f\circ g^{-1}\left(\frac{\lambda g(b)+(2-\lambda)g(a)}{2}\right) \\ &\quad + (1-\lambda) f\circ g^{-1}\left(\frac{(1+\lambda)g(b)+(1-\lambda)g(a)}{2}\right) \\ &\le \frac{b-a}{g(b)-g(a)}\int_0^1 f((1-s)a+sb)\,g'((1-s)a+sb)\,ds \\ &= \int_0^1 f\circ g^{-1}((1-\tau)g(a)+\tau g(b))\,d\tau \\ &\le \frac{1}{2}\left[f\circ g^{-1}((1-\lambda)g(a)+\lambda g(b)) + \lambda f(a) + (1-\lambda)f(b)\right] \\ &\le \frac{f(a)+f(b)}{2} \end{aligned} \tag{2.10}$$

for all $\lambda \in [0,1]$.

Corollary 2. *Let* $g:[a,b]\to[g(a),g(b)]$ *be a continuous strictly increasing function that is differentiable on* (a,b), $f:[a,b]\to J$, J *an interval of real numbers and* $k:J\to\mathbb{R}$ *a continuous function on* J *that is strictly increasing (decreasing) on* J. *If the function* $f:[a,b]\to J$ *is* k*-composite-*g^{-1} *convex on* $[a,b]$, *then*

$$\begin{aligned} & f(M_g(a,b)) \\ &\le (\ge)\, k^{-1}\left\{\lambda k\circ f\circ g^{-1}\left(\frac{\lambda g(b)+(2-\lambda)g(a)}{2}\right)\right. \\ &\quad \left. + (1-\lambda) k\circ f\circ g^{-1}\left(\frac{(1+\lambda)g(b)+(1-\lambda)g(a)}{2}\right)\right\} \\ &\le (\ge)\, k^{-1}\left(\frac{1}{g(b)-g(a)}\int_a^b k\circ f(t)\,g'(t)\,dt\right) \\ &\le (\ge)\, k^{-1}\left\{\frac{1}{2}\left[k\circ f\circ g^{-1}((1-\lambda)g(a)+\lambda g(b)) + \lambda k\circ f(a) + (1-\lambda)k\circ f(b)\right]\right\} \\ &\le (\ge)\, k^{-1}\left(\frac{k\circ f(a)+k\circ f(b)}{2}\right) \end{aligned} \tag{2.11}$$

for any $\lambda \in [0,1]$.

Proof. From (2.6) we have

$$\begin{aligned}
k \circ f\left(M_g(a,b)\right) & \\
&\leq \lambda k \circ f \circ g^{-1}\left(\frac{\lambda g(b)+(2-\lambda) g(a)}{2}\right) \\
&+(1-\lambda) k \circ f \circ g^{-1}\left(\frac{(1+\lambda) g(b)+(1-\lambda) g(a)}{2}\right) \\
&\leq \frac{1}{g(b)-g(a)} \int_a^b k \circ f(t) g'(t)\, dt \\
&\leq \frac{1}{2}\left[k \circ f \circ g^{-1}((1-\lambda) g(a)+\lambda g(b))+\lambda k \circ f(a)+(1-\lambda) k \circ f(b)\right] \\
&\leq \frac{k \circ f(a)+k \circ f(b)}{2}
\end{aligned} \tag{2.12}$$

for any $\lambda \in [0,1]$.

Taking k^{-1} in (2.12) we obtain the desired result (2.11). □

In 1906, Fejér [17], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 2 (Fejér's Inequality)**.** *Consider the integral* $\int_a^b h(x) w(x)\, dx$*, where* h *is a convex function in the interval* (a,b) *and* w *is a positive function in the same interval such that*

$$w(x)=w(a+b-x), \text{ for any } x \in [a,b]$$

i.e., $y=w(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b), 0\right)$ *and is normal to the* x*-axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right) \int_a^b w(x)\, dx \leq \int_a^b h(x) w(x)\, dx \leq \frac{h(a)+h(b)}{2} \int_a^b w(x)\, dx. \tag{2.13}$$

If h *is concave on* (a,b)*, then the inequalities reverse in (2.13).*

If $w:[a,b] \rightarrow \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b] \rightarrow [0,\infty)$ is strictly increasing and differentiable on (a,b) and the inverse $W^{-1}:\left[a, \int_a^b w(s)\, ds\right] \rightarrow [a,b]$ exists.

Corollary 3. *Assume that* $w:[a,b] \rightarrow \mathbb{R}$ *is continuous and positive on the interval* $[a,b]$ *and* $f:[a,b] \rightarrow \mathbb{R}$ *is composite-*W^{-1} *convex on* $[a,b]$*, then we have the* *following Fejér's type inequality*

$$
\begin{aligned}
(2.14)\quad & f\left[W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right)\right] \\
&\leq \lambda f\left[W^{-1}\left(\frac{1}{2}\lambda\int_a^b w(s)\,ds\right)\right]+(1-\lambda)f\left[W^{-1}\left(\frac{1}{2}(1+\lambda)\int_a^b w(s)\,ds\right)\right] \\
&\leq \frac{1}{\int_a^b w(s)}\int_a^b f(t)\,w(t)\,dt \\
&\leq \frac{1}{2}\left[f\left[W^{-1}\left(\lambda\int_a^b w(s)\,ds\right)\right]+\lambda f(a)+(1-\lambda)f(b)\right]\leq\frac{f(a)+f(b)}{2}
\end{aligned}
$$

for all $\lambda\in[0,1]$.

In particular, we have

$$
\begin{aligned}
(2.15)\quad & f\left[W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right)\right] \\
&\leq \frac{1}{2}f\left[W^{-1}\left(\frac{1}{4}\int_a^b w(s)\,ds\right)\right]+\frac{1}{2}f\left[W^{-1}\left(\frac{3}{4}\int_a^b w(s)\,ds\right)\right] \\
&\leq \frac{1}{\int_a^b w(s)}\int_a^b f(t)\,w(t)\,dt \\
&\leq \frac{1}{2}\left[f\left[W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right)\right]+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2}.
\end{aligned}
$$

Remark 2. *Assume that* $w:[a,b]\to\mathbb{R}$ *is continuous and positive on the interval* $[a,b]$, $f:[a,b]\to J$, J *an interval of real numbers and* $k:J\to\mathbb{R}$ *a continuous function on* J *that is strictly increasing (decreasing) on* J. *If the function* $f:[a,b]\to J$ *is* k*-composite-*W^{-1} *convex on* $[a,b]$, *then*

$$
\begin{aligned}
(2.16)\quad & f\left[W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right)\right] \\
&\leq(\geq)\,k^{-1}\left\{\lambda k\circ f\left[W^{-1}\left(\frac{1}{2}\lambda\int_a^b w(s)\,ds\right)\right]\right. \\
&\qquad\left.+(1-\lambda)\,k\circ f\left[W^{-1}\left(\frac{1}{2}(1+\lambda)\int_a^b w(s)\,ds\right)\right]\right\} \\
&\leq(\geq)\,k^{-1}\left(\frac{1}{\int_a^b w(s)}\int_a^b k\circ f(t)\,w(t)\,dt\right) \\
&\leq(\geq)\,k^{-1}\left\{\frac{1}{2}\left[k\circ f\left[W^{-1}\left(\lambda\int_a^b w(s)\,ds\right)\right]+\lambda k\circ f(a)+(1-\lambda)\,k\circ f(b)\right]\right\} \\
&\leq(\geq)\,k^{-1}\left(\frac{k\circ f(a)+k\circ f(b)}{2}\right)
\end{aligned}
$$

for all $\lambda\in[0,1]$.

In particular, we have

$$f\left[W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right)\right] \tag{2.17}$$
$$\leq (\geq)\, k^{-1}\left\{\frac{1}{2}k\circ f\left[W^{-1}\left(\frac{1}{4}\int_a^b w(s)\,ds\right)\right] + \frac{1}{2}k\circ f\left[W^{-1}\left(\frac{3}{4}\int_a^b w(s)\,ds\right)\right]\right\}$$
$$\leq (\geq)\, k^{-1}\left(\frac{1}{\int_a^b w(s)}\int_a^b k\circ f(t)\,w(t)\,dt\right)$$
$$\leq (\geq)\, k^{-1}\left\{\frac{1}{2}\left[k\circ f\left[W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right)\right] + \frac{1}{2}k\circ f(a) + \frac{1}{2}k\circ f(b)\right]\right\}$$
$$\leq (\geq)\, k^{-1}\left(\frac{k\circ f(a) + k\circ f(b)}{2}\right).$$

3. Reverse Inequalities

The following reverse inequalities may be stated:

Theorem 3. *Let $g : [a,b] \to [g(a), g(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If $f : [a,b] \to \mathbb{R}$ is composite-g^{-1} convex on $[a,b]$, then*

$$0 \leq \frac{1}{g(b)-g(a)}\int_a^b f(t)\,g'(t)\,dt - f(M_g(a,b)) \leq \frac{1}{8}(g(b)-g(a))\left[\frac{f'_-(b)}{g'_-(b)} - \frac{f'_+(a)}{g'_+(a)}\right] \tag{3.1}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{g(b)-g(a)}\int_a^b f(t)\,g'(t)\,dt \leq \frac{1}{8}(g(b)-g(a))\left[\frac{f'_-(b)}{g'_-(b)} - \frac{f'_+(a)}{g'_+(a)}\right], \tag{3.2}$$

provided that the lateral derivatives $f'_+(a)$, $g'_+(a)$, $f'_-(b)$ and $g'_-(b)$ are finite.

Proof. Let $h : [c,d] \to \mathbb{R}$ be a convex function on $[c,d]$. We use the inequality that has been established in [4]

$$0 \leq \frac{1}{d-c}\int_c^d h(u)\,du - h\left(\frac{c+d}{2}\right) \leq \frac{1}{8}(d-c)\left[h'_-(d) - h'_+(c)\right] \tag{3.3}$$

and the inequality obtained in [5]

$$0 \leq \frac{h(c)+h(d)}{2} - \frac{1}{d-c}\int_c^d h(u)\,du \leq \frac{1}{8}(d-c)\left[h'_-(d) - h'_+(c)\right]. \tag{3.4}$$

The constant $\frac{1}{8}$ is best possible in both (3.3) and (3.4).

From the inequalities (3.3) and (3.4) we have for the convex function $h = f \circ g^{-1}$ and $c,\ d \in [g(a), g(b)]$ that

$$0 \leq \frac{1}{d-c}\int_{c}^{d}\left(f\circ g^{-1}\right)(u)\,du-\left(f\circ g^{-1}\right)\left(\frac{c+d}{2}\right) \leq \frac{1}{8}(d-c)\left[\left(f\circ g^{-1}\right)_{-}^{\prime}(d)-\left(f\circ g^{-1}\right)_{+}^{\prime}(c)\right] \tag{3.5}$$

and

$$0 \leq \frac{\left(f\circ g^{-1}\right)(c)+\left(f\circ g^{-1}\right)(d)}{2}-\frac{1}{d-c}\int_{c}^{d}\left(f\circ g^{-1}\right)(u)\,du \leq \frac{1}{8}(d-c)\left[\left(f\circ g^{-1}\right)_{-}^{\prime}(d)-\left(f\circ g^{-1}\right)_{+}^{\prime}(c)\right]. \tag{3.6}$$

Since $f \circ g^{-1}$ has lateral derivatives for $z \in (g(a), g(b))$ it follows f has lateral derivatives in each point of (a, b) and by the chain rule and the derivative of the inverse function, we have

$$\left(f\circ g^{-1}\right)_{\pm}^{\prime}(z)=\left(f_{\pm}^{\prime}\circ g^{-1}\right)(z)\left(g^{-1}\right)^{\prime}(z)=\frac{\left(f_{\pm}^{\prime}\circ g^{-1}\right)(z)}{\left(g^{\prime}\circ g^{-1}\right)(z)}. \tag{3.7}$$

Therefore, by (3.5) and (3.6) we get

$$0 \leq \frac{1}{d-c}\int_{c}^{d}\left(f\circ g^{-1}\right)(u)\,du-\left(f\circ g^{-1}\right)\left(\frac{c+d}{2}\right) \leq \frac{1}{8}(d-c)\left[\frac{\left(f_{-}^{\prime}\circ g^{-1}\right)(d)}{\left(g^{\prime}\circ g^{-1}\right)(d)}-\frac{\left(f_{+}^{\prime}\circ g^{-1}\right)(c)}{\left(g^{\prime}\circ g^{-1}\right)(c)}\right] \tag{3.8}$$

and

$$0 \leq \frac{\left(f\circ g^{-1}\right)(c)+\left(f\circ g^{-1}\right)(d)}{2}-\frac{1}{d-c}\int_{c}^{d}\left(f\circ g^{-1}\right)(u)\,du \leq \frac{1}{8}(d-c)\left[\frac{\left(f_{-}^{\prime}\circ g^{-1}\right)(d)}{\left(g^{\prime}\circ g^{-1}\right)(d)}-\frac{\left(f_{+}^{\prime}\circ g^{-1}\right)(c)}{\left(g^{\prime}\circ g^{-1}\right)(c)}\right] \tag{3.9}$$

and by taking $c = g(a)$ and $d = g(b)$ in (3.8) and (3.9), then we get the desired results (3.1) and (3.2). □

Corollary 4. *Assume that $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$. If $f : [a, b] \to \mathbb{R}$ is composite-W^{-1} convex on $[a, b]$, then we have the following weighted reverse integral inequalities*

$$0 \leq \frac{1}{\int_{a}^{b}w(s)}\int_{a}^{b}f(t)\,w(t)\,dt-f\left[W^{-1}\left(\frac{1}{2}\int_{a}^{b}w(s)\,ds\right)\right] \leq \frac{1}{8}\left[\frac{f_{-}^{\prime}(b)}{w(b)}-\frac{f_{+}^{\prime}(a)}{w(a)}\right]\int_{a}^{b}w(s)\,ds \tag{3.10}$$

and

$$0 \leq \frac{f(a)+f(b)}{2}-\frac{1}{\int_{a}^{b}w(s)}\int_{a}^{b}f(t)\,w(t)\,dt \leq \frac{1}{8}\left[\frac{f_{-}^{\prime}(b)}{w(b)}-\frac{f_{+}^{\prime}(a)}{w(a)}\right]\int_{a}^{b}w(s)\,ds, \tag{3.11}$$

provided that $f'_-(b)$ *and* $f'_+(a)$ *are finite.*

Remark 3. *Let* $g:[a,b]\to[g(a),g(b)]$ *be a continuous strictly increasing function that is differentiable on* (a,b), $f:[a,b]\to J$, J *an interval of real numbers and* $k:J\to\mathbb{R}$ *a continuous function on* J *that is strictly increasing on* J *and differentiable on the interior of* J. *If the function* $f:[a,b]\to J$ *is* k*-composite-*g^{-1} *convex on* $[a,b]$ *and* $f'_+(a)$, $g'_+(a)$, $f'_-(b)$, $g'_-(b)$, $k'(f(a))$ *and* $k'(f(b))$ *are finite, then by Theorem 3 we have*

$$0\le\frac{1}{g(b)-g(a)}\int_a^b(k\circ f)(t)\,g'(t)\,dt-k\circ f(M_g(a,b))\le\frac{1}{8}(g(b)-g(a))\left[\frac{k'(f(b))\,f'_-(b)}{g'_-(b)}-\frac{k'(f(a))\,f'_+(a)}{g'_+(a)}\right]\tag{3.12}$$

and

$$0\le\frac{k\circ f(a)+k\circ f(b)}{2}-\frac{1}{g(b)-g(a)}\int_a^b(k\circ f)(t)\,g'(t)\,dt\le\frac{1}{8}(g(b)-g(a))\left[\frac{k'(f(b))\,f'_-(b)}{g'_-(b)}-\frac{k'(f(a))\,f'_+(a)}{g'_+(a)}\right].\tag{3.13}$$

Assume that $w:[a,b]\to\mathbb{R}$ *is continuous and positive on the interval* $[a,b]$, $f:[a,b]\to J$, J *an interval of real numbers and* $k:J\to\mathbb{R}$ *a continuous function on* J *that is strictly increasing on* J *and differentiable on the interior of* J. *If the function* $f:[a,b]\to J$ *is* k*-composite-*W^{-1} *convex on* $[a,b]$ *and* $f'_+(a)$, $f'_-(b)$, $k'(f(a))$ *and* $k'(f(b))$ *are finite, then we have the weighted inequalities*

$$0\le\frac{1}{g(b)-g(a)}\int_a^b(k\circ f)(t)\,w(t)\,dt-k\circ f\left(W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right)\right)\le\frac{1}{8}(g(b)-g(a))\left[\frac{k'(f(b))\,f'_-(b)}{w(b)}-\frac{k'(f(a))\,f'_+(a)}{w(a)}\right]\tag{3.14}$$

and

$$0\le\frac{k\circ f(a)+k\circ f(b)}{2}-\frac{1}{g(b)-g(a)}\int_a^b(k\circ f)(t)\,w(t)\,dt\le\frac{1}{8}(g(b)-g(a))\left[\frac{k'(f(b))\,f'_-(b)}{w(b)}-\frac{k'(f(a))\,f'_+(a)}{w(a)}\right].\tag{3.15}$$

4. Applications for AG and AH-Convex Functions

The function $f:[a,b]\to(0,\infty)$ is AG-convex means that f is k-composite convex on $[a,b]$ with $k(t)=\ln t$, $t>0$. By making use of Corollary 2 for $g(t)=t$, we get

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\le f^{\lambda}\left(\frac{\lambda b+(2-\lambda)a}{2}\right)f^{1-\lambda}\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\\&\le\exp\left(\frac{1}{b-a}\int_a^b\ln f(t)\,dt\right)\\&\le\sqrt{f((1-\lambda)a+\lambda b)\,f^{\lambda}(a)\,f^{1-\lambda}(b)}\le\sqrt{f(a)\,f(b)}\end{aligned}\tag{4.1}$$

for any $\lambda\in[0,1]$, see also [9].

If we use Remark 3 for $g(t) = t$, then we get

$$0 \le \frac{1}{b-a}\int_a^b \ln f(t)\,dt - \ln f\left(\frac{a+b}{2}\right) \le \frac{1}{8}(b-a)\left[\frac{f'_-(b)}{f(b)} - \frac{f'_+(a)}{f(a)}\right] \tag{4.2}$$

and

$$0 \le \frac{\ln f(a) + \ln f(b)}{2} - \frac{1}{b-a}\int_a^b \ln f(t)\,dt \le \frac{1}{8}(b-a)\left[\frac{f'_-(b)}{f(b)} - \frac{f'_+(a)}{f(a)}\right]. \tag{4.3}$$

By taking the exponential in (4.2) and (4.3) we get the equivalent inequalities

$$1 \le \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)}{f\left(\frac{a+b}{2}\right)} \le \exp\left\{\frac{1}{8}(b-a)\left[\frac{f'_-(b)}{f(b)} - \frac{f'_+(a)}{f(a)}\right]\right\} \tag{4.4}$$

and

$$1 \le \frac{\sqrt{f(a)f(b)}}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)} \le \exp\left\{\frac{1}{8}(b-a)\left[\frac{f'_-(b)}{f(b)} - \frac{f'_+(a)}{f(a)}\right]\right\} \tag{4.5}$$

that was obtained in [9].

The function $f : [a,b] \to (0,\infty)$ is AH-convex on $[a,b]$ means that f is k-composite concave on $[a,b]$ with $k : (0,\infty) \to (0,\infty)$, $k(t) = \frac{1}{t}$. By making use of Corollary 2 for $g(t) = t$, we get

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) \\ &\le \left\{\lambda f^{-1}\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f^{-1}\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)\right\}^{-1} \\ &\le \left(\frac{1}{b-a}\int_a^b f^{-1}(t)\,dt\right)^{-1} \\ &\le \left\{\frac{1}{2}\left[f^{-1}((1-\lambda)a + \lambda b) + \lambda f^{-1}(a) + (1-\lambda) f^{-1}(b)\right]\right\}^{-1} \\ &\le \left(\frac{f^{-1}(a) + f^{-1}(b)}{2}\right)^{-1} \end{aligned} \tag{4.6}$$

for any $\lambda \in [0,1]$.

By taking the power -1, this inequality is equivalent to

$$\begin{aligned} &f^{-1}\left(\frac{a+b}{2}\right) \\ &\ge \lambda f^{-1}\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f^{-1}\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right) \\ &\ge \frac{1}{b-a}\int_a^b f^{-1}(t)\,dt \\ &\ge \frac{1}{2}\left[f^{-1}((1-\lambda)a + \lambda b) + \lambda f^{-1}(a) + (1-\lambda) f^{-1}(b)\right] \ge \frac{f^{-1}(a) + f^{-1}(b)}{2} \end{aligned} \tag{4.7}$$

for any $\lambda \in [0,1]$.

If we use Remark 3 for $g(t) = t$, then we get

$$0 \le f^{-1}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f^{-1}(t)\,dt \le \frac{1}{8}(b-a)\left[\frac{f'_-(b)}{f^2(b)} - \frac{f'_+(a)}{f^2(a)}\right] \tag{4.8}$$

and

$$0 \le \frac{1}{b-a}\int_a^b f^{-1}(t)\,dt - \frac{f^{-1}(a)+f^{-1}(b)}{2} \le \frac{1}{8}(b-a)\left[\frac{f'_-(b)}{f^2(b)} - \frac{f'_+(a)}{f^2(a)}\right]. \tag{4.9}$$

5. Applications for GA, GG and GH-Convex Functions

If we take $g(t) = \ln t$, $t \in [a,b] \subset (0,\infty)$, then $f : [a,b] \to \mathbb{R}$ is GA-convex on $[a,b]$ means that that $f : [a,b] \to \mathbb{R}$ composite-g^{-1} convex on $[a,b]$. By making use of Corollary 2 for $k(t) = t$, we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\le \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f(t)}{t}dt \\ &\le \frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right) + (1-\lambda) f(b) + \lambda f(a)\right] \le \frac{f(a)+f(b)}{2} \end{aligned} \tag{5.1}$$

for any $\lambda \in [0,1]$. This result was obtained in [10].

If we use Remark 3 for $k(t) = t$, then we get

$$0 \le \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f(t)}{t}dt - f\left(\sqrt{ab}\right) \le \frac{1}{8}\ln\left(\frac{b}{a}\right)\left[bf'_-(b) - af'_+(a)\right] \tag{5.2}$$

and

$$0 \le \frac{f(a)+f(b)}{2} - \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f(t)}{t}dt \le \frac{1}{8}\ln\left(\frac{b}{a}\right)\left[bf'_-(b) - af'_+(a)\right]. \tag{5.3}$$

These results were also obtained in [10].

The function $f : I \subset (0,\infty) \to (0,\infty)$ is *GG-convex* means that f is k-composite-g^{-1} convex on $[a,b]$ with $k : (0,\infty) \to \mathbb{R}$, $k(t) = \ln t$ and $g(t) = \ln t$, $t \in [a,b]$. By making use of Corollary 2 we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le f^{\lambda}\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) f^{1-\lambda}\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) \\ &\le \exp\left(\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{\ln f(t)}{t}dt\right) \\ &\le \sqrt{f\left(a^{1-\lambda}b^{\lambda}\right) f^{\lambda}(a) f^{1-\lambda}(b)} \le \sqrt{f(a) f(b)} \end{aligned} \tag{5.4}$$

for any $\lambda \in [0,1]$. This result was obtained in [11], see also [12].

If we use Remark 3, then we have the inequalities

$$1 \le \frac{\sqrt{f(a) f(b)}}{\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)} \le \left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b)b}{f(b)} - \frac{f'_+(a)a}{f(a)}\right]} \tag{5.5}$$

and

$$1 \leq \frac{\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right)}{f\left(\sqrt{ab}\right)} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b)b}{f(b)} - \frac{f'_+(a)a}{f(a)}\right]}. \tag{5.6}$$

These results were obtained in [11], see also [12].

We also have that $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is GH-convex on $[a,b]$ is equivalent to the fact that f is k-composite-g^{-1} concave on $[a,b]$ with $k : (0,\infty) \to (0,\infty)$, $k(t) = \frac{1}{t}$ and $g(t) = \ln t$, $t \in I$. By making use of Corollary 2 we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \left[\lambda f^{-1}\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) + (1-\lambda) f^{-1}\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right)\right]^{-1} \\ &\leq \left(\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f^{-1}(t)}{t} dt\right)^{-1} \\ &\leq \left\{\frac{1}{2}\left[f^{-1}\left(a^{1-\lambda}b^{\lambda}\right) + \lambda f^{-1}(a) + (1-\lambda) f^{-1}(b)\right]\right\}^{-1} \\ &\leq \left(\frac{f^{-1}(a) + f^{-1}(b)}{2}\right)^{-1} \end{aligned} \tag{5.7}$$

for any $\lambda \in [0,1]$.

This is equivalent to

$$\begin{aligned} f^{-1}\left(\sqrt{ab}\right) &\geq \lambda f^{-1}\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) + (1-\lambda) f^{-1}\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) \\ &\geq \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f^{-1}(t)}{t} dt \\ &\geq \frac{1}{2}\left[f^{-1}\left(a^{1-\lambda}b^{\lambda}\right) + \lambda f^{-1}(a) + (1-\lambda) f^{-1}(b)\right] \\ &\geq \frac{f^{-1}(a) + f^{-1}(b)}{2}. \end{aligned} \tag{5.8}$$

If we use Remark 3, then we get

$$0 \leq f^{-1}\left(\sqrt{ab}\right) - \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f^{-1}(t)}{t} dt \leq \frac{1}{8}\ln\left(\frac{b}{a}\right)\left[\frac{bf'_-(b)}{f^2(b)} - \frac{af'_+(a)}{f^2(a)}\right] \tag{5.9}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f^{-1}(t)}{t} dt - \frac{f^{-1}(a) + f^{-1}(b)}{2} \\ &\leq \frac{1}{8}\ln\left(\frac{b}{a}\right)\left[\frac{bf'_-(b)}{f^2(b)} - \frac{af'_+(a)}{f^2(a)}\right]. \end{aligned} \tag{5.10}$$

6. Applications for HA, HG and HH-Convex Functions

Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. This is equivalent to the fact that f is composite-g^{-1} convex on $[a,b]$ with the increasing function $g(t) = -\frac{1}{t}$. Then by applying Corollary 2 for $k(t) = t$, we have the inequalities

$$f\left(\frac{2ab}{a+b}\right) \leq (1-\lambda) f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \tag{6.1}$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt$$
$$\leq \frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)\right]$$
$$\leq \frac{f(a)+f(b)}{2}$$

for any $\lambda \in [0,1]$. This result was obtained in [13].

If we use Remark 3, then we get

$$0 \leq \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt - f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{8}\left[\frac{f'_-(b)\,b^2 - f'_+(a)\,a^2}{ab}\right](b-a) \tag{6.2}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \leq \frac{1}{8}\left[\frac{f'_-(b)\,b^2 - f'_+(a)\,a^2}{ab}\right](b-a). \tag{6.3}$$

This results were obtained in [13].

Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be an HG-convex function on the interval $[a,b]$. This is equivalent to the fact that f is k-composite-g^{-1} convex on $[a,b]$ with $k:(0,\infty)\to\mathbb{R}$, $k(t)=\ln t$ and $g(t)=-\frac{1}{t}$, $t\in[a,b]$. Then by applying Corollary 2, we have the inequalities

$$f\left(\frac{2ab}{a+b}\right) \leq f^{1-\lambda}\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)f^{\lambda}\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \tag{6.4}$$
$$\leq \exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right)$$
$$\leq \sqrt{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)[f(a)]^{1-\lambda}[f(b)]^{\lambda}} \leq \sqrt{f(a)\,f(b)}$$

for any $\lambda\in[0,1]$. This result was obtained in [14].

If we use Remark 3, then we get

$$1 \leq \frac{\exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right)}{f\left(\frac{2ab}{a+b}\right)} \leq \exp\left(\frac{1}{8}\left[\frac{f'_-(b)\,b^2}{f(b)} - \frac{f'_+(a)\,a^2}{f(a)}\right]\frac{b-a}{ab}\right) \tag{6.5}$$

and

$$1 \leq \frac{\sqrt{f(a)\,f(b)}}{\exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right)} \leq \exp\left(\frac{1}{8}\left[\frac{f'_-(b)\,b^2}{f(b)} - \frac{f'_+(a)\,a^2}{f(a)}\right]\frac{b-a}{ab}\right). \tag{6.6}$$

These results were obtained in [14].

Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be an HH-convex function on the interval $[a,b]$. This is equivalent to the fact that f is k-composite-g^{-1} concave on $[a,b]$

with $k:(0,\infty)\to(0,\infty)$, $k(t)=\frac{1}{t}$ and $g(t)=-\frac{1}{t}$, $t\in[a,b]$. Then by applying Corollary 2, we have the inequalities

$$
\begin{aligned}
(6.7)\quad & f\left(\frac{2ab}{a+b}\right) \\
&\leq\left\{\lambda f^{-1}\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)+(1-\lambda)f^{-1}\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)\right\}^{-1} \\
&\leq\left(\frac{ab}{b-a}\int_a^b\frac{f^{-1}(t)}{t^2}dt\right)^{-1} \\
&\leq\left\{\frac{1}{2}\left[f^{-1}\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+\lambda f^{-1}(a)+(1-\lambda)f^{-1}(b)\right]\right\}^{-1}\leq\left(\frac{f^{-1}(a)+f^{-1}(b)}{2}\right)^{-1}
\end{aligned}
$$

for any $\lambda\in[0,1]$.

By taking the power -1 in (6.7), then we get

$$
\begin{aligned}
(6.8)\quad & f^{-1}\left(\frac{2ab}{a+b}\right) \\
&\geq\lambda f^{-1}\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)+(1-\lambda)f^{-1}\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right) \\
&\geq\frac{ab}{b-a}\int_a^b\frac{f^{-1}(t)}{t^2}dt \\
&\geq\frac{1}{2}\left[f^{-1}\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+\lambda f^{-1}(a)+(1-\lambda)f^{-1}(b)\right]\geq\frac{f^{-1}(a)+f^{-1}(b)}{2}
\end{aligned}
$$

for any $\lambda\in[0,1]$.

If we use Remark 3, then we get

$$
\begin{aligned}
(6.9)\quad 0 &\leq f^{-1}\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f^{-1}(t)}{t^2}dt \\
&\leq\frac{1}{8}\left[\frac{b^2f'_-(b)}{f^2(b)}-\frac{a^2f'_+(a)}{f^2(a)}\right]\frac{ab}{b-a}
\end{aligned}
$$

and

$$
\begin{aligned}
(6.10)\quad 0 &\leq\frac{ab}{b-a}\int_a^b\frac{f^{-1}(t)}{t^2}dt-\frac{f^{-1}(a)+f^{-1}(b)}{2} \\
&\leq\frac{1}{8}\left[\frac{b^2f'_-(b)}{f^2(b)}-\frac{a^2f'_+(a)}{f^2(a)}\right]\frac{ab}{b-a}.
\end{aligned}
$$

For related results, see [15].

7. Applications for p, r-Convex and *LogExp* Convex Functions

If $p>0$ and we consider $g(t)=t^p$, $t\in[a,b]\subset(0,\infty)$, then $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is p-convex on $[a,b]$ is equivalent to the fact that f is composite-g^{-1} convex on $[a,b]$. Using Corollary 2 for $k(t)=t$ we get

$$\begin{aligned}(7.1)\quad & f\left(M_p(a,b)\right)\\ &\le \lambda f\left[\left(\frac{\lambda b^p+(2-\lambda)a^p}{2}\right)^{1/p}\right]+(1-\lambda)f\left[\left(\frac{(1+\lambda)b^p+(1-\lambda)a^p}{2}\right)^{1/p}\right]\\ &\le \frac{p}{b^p-a^p}\int_a^b f(t)\,t^{p-1}dt\\ &\le \frac{1}{2}\left\{f\left[\left((1-\lambda)a^p+\lambda b^p\right)^{1/p}\right]+\lambda f(a)+(1-\lambda)f(b)\right\}\le \frac{f(a)+f(b)}{2}\end{aligned}$$

for any $\lambda\in[0,1]$, where $M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$. This improves the corresponding result from [22].

If we use Remark 3, then we get

$$(7.2)\quad 0\le \frac{p}{b^p-a^p}\int_a^b f(t)\,t^{p-1}dt-f\left(M_p(a,b)\right)\le \frac{1}{8p}\left(b^p-a^p\right)\left[\frac{f'_-(b)}{b^{p-1}}-\frac{f'_+(a)}{a^{p-1}}\right]$$

and

$$(7.3)\quad 0\le \frac{a^p+b^p}{2}-\frac{p}{b^p-a^p}\int_a^b f(t)\,t^{p-1}dt\le \frac{1}{8p}\left(b^p-a^p\right)\left[\frac{f'_-(b)}{b^{p-1}}-\frac{f'_+(a)}{a^{p-1}}\right].$$

Assume that the function $f:[a,b]\to(0,\infty)$ is r-convex, for $r>0$. This is equivalent to the fact that f is k-composite convex with $k(t)=t^r$, $t>0$, and by Corollary 2 for $g(t)=t$ we get

$$\begin{aligned}(7.4)\quad & f\left(\frac{a+b}{2}\right)\\ &\le \left\{\lambda f^r\left(\frac{\lambda a+(2-\lambda)b}{2}\right)+(1-\lambda)f^r\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right\}^{1/r}\\ &\le \left(\frac{1}{b-a}\int_a^b f^r(t)\,dt\right)^{1/r}\\ &\le \left\{\frac{1}{2}\left[f^r\left((1-\lambda)a+\lambda b\right)+\lambda f^r(a)+(1-\lambda)f^r(b)\right]\right\}^{1/r}\le\left(\frac{f^r(a)+f^r(b)}{2}\right)^{1/r}\end{aligned}$$

for any $\lambda\in[0,1]$.

By taking the power $r>0$, we get the equivalent inequality

$$\begin{aligned}(7.5)\quad & f^r\left(\frac{a+b}{2}\right)\\ &\le \lambda f^r\left(\frac{\lambda a+(2-\lambda)b}{2}\right)+(1-\lambda)f^r\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\\ &\le \frac{1}{b-a}\int_a^b f^r(t)\,dt\\ &\le \frac{1}{2}\left[f^r\left((1-\lambda)a+\lambda b\right)+\lambda f^r(a)+(1-\lambda)f^r(b)\right]\le \frac{f^r(a)+f^r(b)}{2}\end{aligned}$$

for any $\lambda\in[0,1]$.

From Remark 3, we get for $g(t) = t$ that

$$0 \leq \frac{1}{b-a}\int_a^b f^r(t)\,dt - f^r\left(\frac{a+b}{2}\right) \leq \frac{r}{8}(b-a)\left[f^{r-1}(b)\,f'_-(b) - f^{r-1}(a)\,f'_+(a)\right] \tag{7.6}$$

and

$$0 \leq \frac{f^r(a)+f^r(b)}{2} - \frac{1}{b-a}\int_a^b f^r(t)\,dt \leq \frac{r}{8}(b-a)\left[f^{r-1}(b)\,f'_-(b) - f^{r-1}(a)\,f'_+(a)\right]. \tag{7.7}$$

Assume that $f:[a,b]\to\mathbb{R}$ is *LogExp convex function* on $[a,b]$ as considered in [7]. This is equivalent to the fact that f is composite-g^{-1} with $g(t) = \exp t$. By utilising Corollary 2 for $k(t) = t$ we get,

$$\begin{aligned} &f(LME(a,b)) \\ &\leq \lambda f\left[\ln\left(\frac{\lambda\exp b + (2-\lambda)\exp a}{2}\right)\right] + (1-\lambda)\,f\left[\ln\left(\frac{(1+\lambda)\exp b + (1-\lambda)\exp a}{2}\right)\right] \\ &\leq \frac{1}{\exp b - \exp a}\int_a^b f(t)\exp t dt \\ &\leq \frac{1}{2}\left[f\left[\ln\left((1-\lambda)\exp(a) + \lambda\exp(b)\right)\right] + \lambda f(a) + (1-\lambda)\,f(b)\right] \leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{7.8}$$

for $\lambda \in [a,b]$, where $LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right)$.

If we use Remark 3, then we get

$$0 \leq \frac{1}{\exp b - \exp a}\int_a^b f(t)\exp t dt - f(LME(a,b)) \leq \frac{1}{8}(\exp b - \exp a)\left[\exp(-b)\,f'_-(b) - \exp(-a)\,f'_+(a)\right] \tag{7.9}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{\exp b - \exp a}\int_a^b f(t)\exp t dt \leq \frac{1}{8}(\exp b - \exp a)\left[\exp(-b)\,f'_-(b) - \exp(-a)\,f'_+(a)\right]. \tag{7.10}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa