ON SOME PROPERTIES AND INEQUALITIES OF THE SIGMOID FUNCTION

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Abstract. In this paper, we study some analytic properties of the sigmoid function, which is frequently applied in neural networks as well as other scientific disciplines. Specifically, by using analytical techniques, we establish several inequalities involving the function. Some of these inequalities connect the sigmoid function to the softplus function.

1. Introduction

The sigmoid function, which is also known as the standard logistic function is defined as

$$S(x) = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}}, \quad x \in (-\infty, \infty), \tag{1}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right), \quad x \in (-\infty, \infty). \tag{2}$$

Its first and second derivatives are given as

$$S'(x) = \frac{e^x}{(1+e^x)^2} = S(x)\left(1 - S(x)\right), \tag{3}$$

$$S''(x) = \frac{e^x(1-e^x)}{(1+e^x)^3} = S(x)(1-S(x))(1-2S(x)), \tag{4}$$

for all $x \in (-\infty, \infty)$. It follows swiftly from (3) that $S(x)$ is increasing on $(-\infty, \infty)$. Also, in view of (3), $y = S(x)$ is a solution to the autonomous differential equation

$$\frac{dy}{dx} = y(1-y), \tag{5}$$

with initial condition $y(0) = 0.5$. Furthermore, the sigmoid function satisfies the following properties.

$$S(x) + S(-x) = 1, \tag{6}$$

$$S'(x) = S(x)S(-x), \tag{7}$$

$$S'(x) = S'(-x), \tag{8}$$

$$\lim_{x\to\infty} S(x) = 1, \tag{9}$$

$$\lim_{x\to 0} S(x) = \frac{1}{2}, \tag{10}$$

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$$\lim_{x\to-\infty} S(x) = 0, \tag{11}$$

$$\lim_{x\to\pm\infty} S'(x) = 0, \tag{12}$$

$$\lim_{x\to 0} S'(x) = \frac{1}{4}, \tag{13}$$

$$\int S(x)\,dx = \ln(1+e^x) + C, \tag{14}$$

where C is a constant of integration. The function $\ln(1+e^x)$ is known in the literature as softplus function [9]. It is clear from (14) that, the derivative of the softplus function gives the sigmoid function.

The sigmoid function has found useful applications in many scientific disciplines including machine learning, probability and statistics, biology, ecology, population dynamics, demography, and mathematical psychology (see [3] , [14], and the references therein).

In particular, the function is widely used in artificial neural networks, where it serves as an activation function at the output of each neuron (see [4], [5], [6], [10], [15]). Also, in the business field, the function been applied to study performance growth in manufacturing and service management (see [13]). Another area of application is in the field of medicine, where the function is used to model the growth of tumors or to study pharmacokinetic reactions (see [14]). It is also applied in forestory. For example in [7], a generalized form of the function is applied to predict the site index of unmanaged loblolly and slash pine plantations in East Texas. Furthermore, it also applied in computer graphics or image processing to enhance image contrast (see [8], [12]).

The above important roles of the function makes its properties a matter of interest and hence worth studying. In the recent work [11], the authors studied some analytic properties of the function such as starlikeness and convexity in a unit disc.

In this paper, we continue the investigation. We establish among other things, properties such as inequalities, subadditivity, convexity and supermultiplicativity of the function. We begin with the following definitions and lemmas.

2. Auxiliary Definitions and Lemmas

Definition 2.1. A function $M : (0,\infty)\times(0,\infty) \to (0,\infty)$ is called a mean function if it satisfies the following conditions.

(i) $M(x,y) = M(y,x)$,
(ii) $M(x,x) = x$,
(iii) $x < M(x,y) < y$, for $x < y$,
(iv) $M(\lambda x, \lambda y) = \lambda M(x,y)$, for $\lambda > 0$.

There are several well-known mean functions in the literature. Amongst these are the following.

(i) Arithmetic mean: $A(x,y) = \frac{x+y}{2}$,
(ii) Geometric mean: $G(x,y) = \sqrt{xy}$,
(iii) Harmonic mean: $H(x,y) = \frac{1}{A(\frac{1}{x},\frac{1}{y})} = \frac{2xy}{x+y}$,

(iv) Logarithmic mean: $L(x,y) = \frac{x-y}{\ln x - \ln y}$ for $x \neq y$, and $L(x,x) = x$.

(v) Identric mean: $I(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$ for $x \neq y$, and $I(x,x) = x$.

Definition 2.2 ([1])**.** Let $f : I \subseteq (0,\infty) \to (0,\infty)$ be a continuous function and M and N be any two mean functions. Then f is said to be MN-convex (MN-concave) if

$$f(M(x,y)) \leq (\geq) N(f(x), f(y)),$$

for all $x, y \in I$.

Lemma 2.3 ([2])**.** *Let $f : (a,\infty) \to (-\infty,\infty)$ with $a \geq 0$. If the function defined by $g(x) = \frac{f(x)-1}{x}$ is increasing on (a,∞), then the function $h(x) = f(x^2)$ satisfies the Grumbaum-type inequality*

$$1 + h(z) \geq h(x) + h(y), \tag{15}$$

where $x, y \geq a$ and $z^2 = x^2 + y^2$. If g is decreasing, then the inequality (15) is reversed.

Lemma 2.4 ([16])**.** *Let $-\infty \leq a < b \leq \infty$ and f and g be continuous functions that are differentiable on (a,b), with $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$. Suppose that $g(x)$ and $g'(x)$ are nonzero for all $x \in (a,b)$. If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing) on (a,b), then $\frac{f(x)}{g(x)}$ is also increasing (or decreasing) on (a,b).*

3. Main Results

Theorem 3.1. *The function $S(x)$ is subadditive on $(-\infty,\infty)$. In other words, the function satisfies the inequality*

$$S(x+y) < S(x) + S(y), \tag{16}$$

for all $x, y \in (-\infty,\infty)$.

Proof. The case where $x = y = 0$ is trivial. Hence we only prove the result for the case where $x, y \in (0,\infty)$ and the case where $x, y \in (-\infty, 0)$. Let $u(x) = \frac{e^x}{(1+e^x)^2}$ for $x \in (0,\infty)$. Then $u'(x) = \frac{e^x(1-e^x)}{(1+e^x)^3} < 0$ which implies that $u(x)$ is decreasing. Next let

$$\begin{aligned} f(x,y) &= S(x+y) - S(x) - S(y) \\ &= \frac{e^{x+y}}{1+e^{x+y}} - \frac{e^x}{1+e^x} - \frac{e^y}{1+e^y}, \end{aligned}$$

for $x, y \in (0,\infty)$. Without loss of generality, let y be fixed. Then

$$\frac{\partial}{\partial x} f(x,y) = \frac{e^{x+y}}{(1+e^{x+y})^2} - \frac{e^x}{(1+e^x)^2} < 0,$$

since $u(x)$ is decreasing. Hence $f(x,y)$ is decreasing. Then for $x \in (0,\infty)$ we have

$$f(x,y) < f(0,y) = \lim_{x\to 0} f(x,y) = -\frac{1}{2} < 0,$$

which gives the desired result (16).

Likewise, let $w(x)=\frac{e^x}{(1+e^x)^2}$ for $x\in(-\infty,0)$. Then $w'(x)=\frac{e^x(1-e^x)}{(1+e^x)^3}>0$ which implies that $w(x)$ is increasing. Furthermore, let

$$g(x,y)=S(x+y)-S(x)-S(y),$$

for $x,y\in(-\infty,0)$. Then for a fixed y we have

$$\frac{\partial}{\partial x}g(x,y)=\frac{e^{x+y}}{(1+e^{x+y})^2}-\frac{e^x}{(1+e^x)^2}>0,$$

since $w(x)$ is increasing. Hence $g(x,y)$ is increasing. Then for $x\in(-\infty,0)$ we have

$$g(x,y)<g(0,y)=\lim_{x\to 0}g(x,y)<0.$$

Therefore, inequality (16) holds for all $x,y\in(-\infty,\infty)$.

Theorem 3.2. *The function $S(x)$ satisfies the following inequalities.*

$$1<\frac{S(x+1)}{S(x)}<e,\quad x\in(-\infty,\infty), \tag{17}$$

$$\frac{2e}{1+e}<\frac{S(x+1)}{S(x)}<e,\quad x\in(-\infty,0), \tag{18}$$

$$1<\frac{S(x+1)}{S(x)}<\frac{2e}{1+e},\quad x\in(0,\infty). \tag{19}$$

Proof. Note that $\left(\frac{S'(x)}{S(x)}\right)'=-\frac{e^x}{(1+e^x)^2}<0$, for all $x\in(-\infty,\infty)$. Thus, the function $\frac{S'(x)}{S(x)}$ is decreasing for all $x\in(-\infty,\infty)$. Now, for each $x\in(-\infty,\infty)$, let $Q(x)=\frac{S(x+1)}{S(x)}$ and $u(x)=\ln Q(x)$. Then

$$u'(x)=\frac{S'(x+1)}{S(x+1)}-\frac{S'(x)}{S(x)}<0,$$

since $\frac{S'(x)}{S(x)}$ is decreasing. Thus $u(x)$ and consequently $Q(x)$ are decreasing. Hence for all $x\in(-\infty,\infty)$, we have

$$1=\lim_{x\to\infty}Q(x)<Q(x)<\lim_{x\to-\infty}Q(x)=e,$$

which gives the inequality (17). For $x\in(-\infty,0)$, we have

$$\frac{2e}{1+e}=\lim_{x\to 0}Q(x)<Q(x)<\lim_{x\to-\infty}Q(x)=e,$$

which gives (18). Then for $x\in(0,\infty)$, we have

$$1=\lim_{x\to\infty}Q(x)<Q(x)<\lim_{x\to 0}Q(x)=\frac{2e}{1+e},$$

which gives (19).

Theorem 3.3. *The function $S(x)$ is GG-convex on $(0,1)$. That is, for $x,y\in(0,1)$, the inequality*

$$S(x^{\frac{1}{a}}y^{\frac{1}{b}})\leq[S(x)]^{\frac{1}{a}}[S(y)]^{\frac{1}{b}}, \tag{20}$$

is satisfied, where $a>1$ and $\frac{1}{a}+\frac{1}{b}=1$.

Proof. Recall that a function $f : I \subseteq (0,\infty) \to (0,\infty)$ is GG-convex if and only if $\frac{xf'(x)}{f(x)}$ is increasing for all $x \in I$ (see Corollary 2.5 of [1]). We have $\frac{xS'(x)}{S(x)} = \frac{x}{1+e^x}$ and $\left(\frac{xS'(x)}{S(x)}\right)' = \frac{1+(1-x)e^x}{(1+e^x)^2} > 0$, which concludes the proof.

Remark 3.4. By the classical Young's inequality, the right hand side of (20) gives

$$[S(x)]^{\frac{1}{a}}[S(y)]^{\frac{1}{b}} \leq \frac{S(x)}{a} + \frac{S(y)}{b}. \tag{21}$$

Hence

$$S(x^{\frac{1}{a}}y^{\frac{1}{b}}) \leq \frac{S(x)}{a} + \frac{S(y)}{b}, \tag{22}$$

which shows that $S(x)$ is also GA-convex on $(0,1)$.

Theorem 3.5. *The function $S(x)$ is AH-concave on $(0,\infty)$. That is,*

$$S\left(\frac{x+y}{2}\right) \geq \frac{2S(x)S(y)}{S(x)+S(y)}, \tag{23}$$

for all $x, y \in (0,\infty)$.

Proof. A function $f : I \subseteq (0,\infty) \to (0,\infty)$ is AH-convex if and only if $\frac{f'(x)}{f(x)^2}$ is decreasing for all $x \in I$ (see Corollary 2.5 of [1]). Now we have $\left(\frac{S'(x)}{S(x)^2}\right)' = -\frac{1}{e^x} < 0$, which yields the desired result.

Theorem 3.6. *The function $S(x)$ is logarithmically concave on $(-\infty,\infty)$. That is, for $x, y \in (-\infty,\infty)$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, the inequality*

$$S\left(\frac{x}{a} + \frac{y}{b}\right) \geq [S(x)]^{\frac{1}{a}}[S(y)]^{\frac{1}{b}}, \tag{24}$$

is satisfied.

Proof. Let $\phi(x) = \ln S(x) = x - \ln(1+e^x)$ for all $x, y \in (-\infty,\infty)$. Then $\phi''(x) = -\frac{e^x}{(1+e^x)^2} \leq 0$, which concludes the proof.

Corollary 3.7. *The inequalities*

$$S''(x)S(x) - (S'(x))^2 \leq 0, \quad x \in (-\infty,\infty), \tag{25}$$

$$S(1+u)S(1-u) \leq \left(\frac{e}{1+e}\right)^2, \quad u \in (-\infty,\infty), \tag{26}$$

are satisfied.

Proof. Inequality (25) is a direct consequence of the logarithmic concavity of $S(x)$. By letting $a = b = 2$, $x = 1+u$ and $y = 1-u$ in (24), we obtain (26).

Theorem 3.8. *The function $S(x)$ satisfies the following inequalities.*

$$S^2(x+y) \geq S(x)S(y), \quad x, y \in [0,\infty), \tag{27}$$

$$S^2(x+y) \leq S(x)S(y), \quad x, y \in (-\infty,0]. \tag{28}$$

Equality holds if $x = y = 0$.

Proof. Let $x, y \in [0, \infty)$ and recall that $S(x)$ is increasing. Then we have

$$S(x+y) \geq S(x) > 0,$$

$$S(x+y) \geq S(y) > 0,$$

since $x+y \geq x$ and $x+y \geq y$. Now by multiplying these inequalities, we obtain (27). For $x, y \in (-\infty, 0]$, we have

$$0 < S(x+y) \leq S(x),$$

$$0 < S(x+y) \leq S(y),$$

since $x+y \leq x$ and $x+y \leq y$. Then by multiplication, we obtain (28).

Theorem 3.9. *The function $S(x)$ satisfies the following inequalities.*

$$S^2(xy) \leq S(x)S(y), \quad x, y \in (0, 1], \tag{29}$$

$$S^2(xy) \geq S(x)S(y), \quad x, y \in [1, \infty). \tag{30}$$

Equality holds if $x = y = 1$.

Proof. Suppose that $x, y \in (0, 1]$. Then $xy \leq x$ and $xy \leq y$. Since $S(x)$ is increasing, we have

$$0 < S(xy) \leq S(x),$$

$$0 < S(xy) \leq S(y),$$

and by multiplication, we obtain (29). Next suppose that $x, y \in [1, \infty)$. Then $xy \geq x$ and $xy \geq y$ and consequently, we have

$$S(xy) \geq S(x) > 0,$$

$$S(xy) \geq S(y) > 0,$$

which yields (30). This completes the proof.

Theorem 3.10. *The function $S(x)$ is supermultiplicative on $(1, \infty)$. That is*

$$S(xy) > S(x)S(y), \tag{31}$$

for all $x, y \in (1, \infty)$.

Proof. Since $S(z) \in (0, 1)$ for all $z \in (-\infty, \infty)$, then $S^2(z) < S(z)$ for all $z \in (-\infty, \infty)$. Hence

$$S^2(xy) < S(xy),$$

for all $x, y \in (1, \infty)$. This together with (30) yields

$$S(x)S(y) \leq S^2(xy) < S(xy),$$

which completes the proof.

Theorem 3.11. *The function $S(x)$ satisfies the Grumbaum-type inequality*

$$1 + S(z^2) \geq S(x^2) + S(y^2), \tag{32}$$

where $x, y \in (0, \infty)$ and $z^2 = x^2 + y^2$.

Proof. Let $g(x)$ be defined for $x \in (0, \infty)$ as $g(x) = \frac{S(x)-1}{x}$. That is,

$$g(x) = \frac{\frac{e^x}{1+e^x} - 1}{x} = -\frac{1}{x(1+e^x)}.$$

Then

$$g'(x) = \frac{1}{x^2(1+e^x)} + \frac{e^x}{x(1+e^x)} > 0,$$

which implies that $g(x)$ is increasing. Hence by applying Lemma 2.3, we obtain the desired result (32).

In what follows we give some sharp inequalities connecting the sigmoid and the softplus functions.

Theorem 3.12. *The inequalities*

$$\frac{e^x}{1+e^x} < \ln(1+e^x) < \ln 2 - \frac{1}{2} + \frac{e^x}{1+e^x}, \quad x \in (-\infty, 0), \tag{33}$$

$$\ln 2 - \frac{1}{2} + \frac{e^x}{1+e^x} < \ln(1+e^x), \quad x \in (0, \infty), \tag{34}$$

$$\frac{e^x}{1+e^x} < \ln(1+e^x), \quad x \in (-\infty, \infty), \tag{35}$$

are valid.

Proof. Let $F(x) = \ln(1+e^x) - \frac{e^x}{1+e^x}$ for all $x \in (-\infty, \infty)$. Then

$$F'(x) = \frac{e^x}{1+e^x}\left(1 - \frac{1}{1+e^x}\right) = \left(\frac{e^x}{1+e^x}\right)^2 > 0.$$

Thus $F(x)$ is increasing for all $x \in (-\infty, \infty)$. Then for $x \in (-\infty, 0)$, we have

$$0 = \lim_{x\to-\infty} F(x) < F(x) < \lim_{x\to 0} F(x) = \ln 2 - \frac{1}{2},$$

which gives inequality (33). For $x \in (0, \infty)$, we have

$$\ln 2 - \frac{1}{2} = \lim_{x\to 0} F(x) < F(x) < \lim_{x\to\infty} F(x) = \infty,$$

which gives inequality (34). Finally, for $x \in (-\infty, \infty)$, we have

$$0 = \lim_{x\to-\infty} F(x) < F(x) < \lim_{x\to\infty} F(x) = \infty,$$

which gives inequality (35). This completes the proof.

Lemma 3.13. *The inequality*

$$e^x - \ln(1+e^x) > 0 \tag{36}$$

holds for all $x \in (-\infty, \infty)$.

Proof. Let $T(x) = e^x - \ln(1+e^x)$ for all $x \in (-\infty, \infty)$. Then

$$T'(x) = e^x \left(1 - \frac{1}{1+e^x}\right) = \frac{e^{2x}}{1+e^x} > 0,$$

which means that $T(x)$ is increasing. Then we have

$$T(x) > \lim_{x\to-\infty} T(x) = \lim_{x\to-\infty} [e^x - \ln(1+e^x)] = 0,$$

which gives inequality (36).

Theorem 3.14. *Let* $f(x) = (1+e^x)^{\frac{1}{e^x}}$ *and* $g(x) = (1+e^x)^{1+\frac{1}{e^x}}$ *for all* $x \in (-\infty, \infty)$. *Then* $f(x)$ *is decreasing and* $g(x)$ *is increasing. Consequently the inequalities*

$$(\ln 2)e^x < \ln(1+e^x) < e^x, \quad x \in (-\infty, 0), \tag{37}$$

$$\frac{e^x}{1+e^x} < \ln(1+e^x) < (2\ln 2)\frac{e^x}{1+e^x}, \quad x \in (-\infty, 0), \tag{38}$$

$$\frac{e^x}{1+e^x} < \ln(1+e^x) < e^x, \quad x \in (-\infty, \infty), \tag{39}$$

are satisfied.

Proof. Let $K(x) = \ln f(x) = \frac{\ln(1+e^x)}{e^x}$ and $L(x) = \ln g(x) = \frac{1+e^x}{e^x}\ln(1+e^x)$ for all $x \in (-\infty, \infty)$. Then

$$\begin{aligned} K'(x) &= \frac{1}{1+e^x} - \frac{\ln(1+e^x)}{e^x} \\ &= \frac{1}{e^x}\left[\frac{e^x}{1+e^x} - \ln(1+e^x)\right] < 0, \end{aligned}$$

which follows from (35). Hence $K(x)$ is decreasing and consequently, $f(x)$ is also decreasing. Also, we have

$$\begin{aligned} L'(x) &= 1 - \frac{\ln(1+e^x)}{e^x} \\ &= \frac{1}{e^x}[e^x - \ln(1+e^x)] > 0 \end{aligned}$$

which follows from (36). Thus $L(x)$ is increasing and consequently, $g(x)$ is also increasing. Moreover, we have

$$K(0) = \ln 2, \tag{40}$$

$$\lim_{x\to-\infty} K(x) = \lim_{x\to-\infty} \frac{\ln(1+e^x)}{e^x} = \lim_{x\to-\infty} \frac{1}{1+e^x} = 1, \tag{41}$$

$$\lim_{x\to\infty} K(x) = \lim_{x\to\infty} \frac{1}{1+e^x} = 0, \tag{42}$$

$$L(0) = 2\ln 2, \tag{43}$$

$$\lim_{x\to-\infty} L(x) = \lim_{x\to-\infty} \frac{\ln(1+e^x)}{\left(\frac{e^x}{1+e^x}\right)} = \lim_{x\to-\infty} (1+e^x) = 1, \tag{44}$$

$$\lim_{x\to\infty} L(x) = \lim_{x\to\infty} \frac{\ln(1+e^x)}{\left(\frac{e^x}{1+e^x}\right)} = \infty. \tag{45}$$

Since $K(x)$ is decreasing and $L(x)$ is increasing, we obtain the following. For $x \in (-\infty, 0)$, we have

$$\ln 2 = K(0) < K(x) < \lim_{x\to-\infty} K(x) = 1,$$

which gives inequality (37). Also, for $x \in (-\infty, 0)$, we have

$$1 = \lim_{x\to-\infty} L(x) < L(x) < L(0) = 2\ln 2,$$

which gives inequality (38). Furthermore, for $x \in (-\infty, \infty)$, we have

$$0 = \lim_{x\to\infty} K(x) < K(x) < \lim_{x\to-\infty} K(x) = 1,$$

which gives

$$\ln(1+e^x) < e^x. \tag{46}$$

Also, we have

$$1 = \lim_{x\to-\infty} L(x) < L(x) < \lim_{x\to\infty} L(x) = \infty,$$

which gives

$$\frac{e^x}{1+e^x} < \ln(1+e^x). \tag{47}$$

Then by combining (46) and (47), we obtain (38).

Theorem 3.15. *Let Ω be defined for $x \in (-\infty, 0)$ by*

$$\Omega(x) = \frac{e^x \ln(1+e^x)}{e^x - \ln(1+e^x)}.$$

Then $\Omega(x)$ is increasing and consequently, the inequality

$$0 < \frac{e^x \ln(1+e^x)}{e^x - \ln(1+e^x)} < \frac{\ln 2}{1-\ln 2}, \tag{48}$$

is satisfied.

Proof. To begin with, we have

$$\lim_{x\to0} \Omega(x) = \frac{\ln 2}{1-\ln 2},$$

and

$$\begin{aligned}
\lim_{x\to-\infty} \Omega(x) &= \lim_{x\to-\infty} \frac{e^x \ln(1+e^x)}{e^x - \ln(1+e^x)} \\
&= \lim_{x\to-\infty} \frac{\ln(1+e^x) - \frac{e^x}{1+e^x}}{\frac{e^x}{1+e^x}} \\
&= \lim_{x\to-\infty} \frac{\frac{e^x}{1+e^x} - \frac{e^x}{(1+e^x)^2}}{\frac{e^x}{(1+e^x)^2}} \\
&= \lim_{x\to-\infty} e^x \\
&= 0.
\end{aligned}$$

Next, let $f(x) = e^x \ln(1+e^x)$ and $g(x) = e^x - \ln(1+e^x)$. Then $f(-\infty) = \lim_{x\to-\infty} f(x) = 0$ and $g(-\infty) = \lim_{x\to-\infty} g(x) = 0$. Also,

$$f'(x) = e^x \left[\ln(1+e^x) + \frac{e^x}{1+e^x}\right],$$

and

$$g'(x) = e^x \left[1 - \frac{1}{1+e^x}\right] = e^x \frac{e^x}{1+e^x}.$$

Then

$$\frac{f'(x)}{g'(x)} = \frac{\ln(1+e^x) + \frac{e^x}{1+e^x}}{\frac{e^x}{1+e^x}} = \frac{\ln(1+e^x)}{\left(\frac{e^x}{1+e^x}\right)} - 1 = \frac{1+e^x}{e^x}\ln(1+e^x) - 1,$$

which implies that

$$\left(\frac{f'(x)}{g'(x)}\right)' = \left(\frac{1+e^x}{e^x}\ln(1+e^x)\right)' = \frac{1}{e^x}[e^x - \ln(1+e^x)] > 0.$$

Thus $\frac{f'(x)}{g'(x)}$ is increasing. Hence in view of Lemma 2.4, we conclude that $\frac{f(x)}{g(x)} = \Omega(x)$ is increasing. Then for $x \in (-\infty, 0)$ we have

$$0 = \lim_{x\to-\infty} \Omega(x) < \Omega(x) < \lim_{x\to 0} \Omega(x) = \frac{\ln 2}{1-\ln 2},$$

which yields inequality (48).

Remark 3.16. Let $\lambda = \frac{\ln 2}{1-\ln 2}$. Then inequality (48) can be written as

$$\ln(1+e^x) < \frac{\lambda e^x}{\lambda + e^x}, \tag{49}$$

for all $x \in (-\infty, 0)$.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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