AN IDENTITY OF FINK TYPE FOR THE INTEGRAL OF ANALYTIC COMPLEX FUNCTIONS ON PATHS FROM GENERAL DOMAINS

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Abstract. In this paper we establish an identity of Fink type for approximating the integral of analytic complex functions on paths from general domains. Error bounds for these expansions in terms of p-norms are also provided. Examples for the complex logarithm and the complex exponential are given as well.

1. Introduction

In 1992, [6] A. M. Fink obtained the following identity for a function $f:[a,b]\to\mathbb{R}$ whose $(n-1)$-derivative $f^{(n-1)}$ with $n\geq 1$ is absolutely continuous on $[a,b]$

$$\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt=R_n(x), \tag{1.1}$$

for $x\in[a,b]$, where

$$F_k(x):=\frac{n-k}{k!}\left[\frac{f^{(k-1)}(a)(x-a)^k+(-1)^{k-1}(a-x)^k f^{(k-1)}(a)}{b-a}\right], \tag{1.2}$$

for $k=1,...,n-1$ where $n\geq 2$ and

$$R_n(x):=\frac{1}{n!(b-a)}\times\left[\int_a^x (x-t)^{n-1}(t-a)f^{(n)}(t)\,dt+\int_x^b (x-t)^{n-1}(t-b)f^{(n)}(t)\,dt\right]. \tag{1.3}$$

If $n=1$ the sum $\sum_{k=1}^{n-1}F_k(x)$ is taken to be zero.

In the case $f^{(n)}\in L_\infty[a,b]$, namely

$$\left\|f^{(n)}\right\|_{[a,b],\infty}:=\operatorname*{essup}_{t\in[a,b]}\left|f^{(n)}(t)\right|<\infty,$$

then the following bound for the remainder obtained by Milovanović and Pečarić in 1976, [8] holds

$$|R_n(x)|\leq\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!(b-a)}. \tag{1.4}$$

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In the case of $f^{(n)} \in L_p[a,b]$, $p \geq 1$, namely

$$\left\| f^{(n)} \right\|_{[a,b],p} := \left(\int_a^b \left| f^{(n)}(t) \right|^p dt \right)^{1/p} < \infty,$$

then the following bounds for the remainder obtained by Fink in 1992, [6] hold

$$(1.5)\quad |R_n(x)| \leq \begin{cases} \frac{\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{1/q}}{n!(b-a)} B\left((n-1)q+1, q+1\right) \left\| f^{(n)} \right\|_{[a,b],p} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{(n-1)^{n-1}}{n^n n!(b-a)} \max\left[(x-a)^n, (b-x)^n\right] \left\| f^{(n)} \right\|_{[a,b],1}. \end{cases}$$

For other results connected with Fink's identity, see [1], [2], [3] and [7].

In order to extend these results for the complex integral, we need the following preparations.

Suppose γ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth.* Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$(1.6)\qquad \int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz.$$

We recall also the *triangle inequality* for the complex integral, namely

$$(1.7)\qquad \left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma)$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^{p} |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_{\gamma} |f(z)| |dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$

In the recent paper [4] we obtained the following identity:

Theorem 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$, $z(t) = x$ and $z(b) = w$ where u, $w \in D$. Then we have the equality*

$$\int_{\gamma} f(z)\,dz = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} f^{(k)}(x) \left[(w-x)^{k+1} + (-1)^{k} (x-u)^{k+1} \right] + O_n(x,\gamma), \tag{1.8}$$

where the remainder $O_n(x,\gamma)$ is given by

$$O_n(x,\gamma) := \frac{(-1)^n}{n!} \left[\int_{\gamma_{u,x}} (z-u)^n f^{(n)}(z)\,dz + \int_{\gamma_{x,w}} (z-w)^n f^{(n)}(z)\,dz \right] \tag{1.9}$$

and n is a natural number, $n \geq 1$.

The remainder $O_n(x,\gamma)$ satisfies the following bounds

$$|O_n(x,\gamma)| \leq \frac{1}{n!} \begin{cases} \left\|f^{(n)}\right\|_{\gamma_{u,w},\infty} \left[\int_{\gamma_{u,x}} |z-u|^n |dz| + \int_{\gamma_{x,w}} |z-w|^n |dz| \right], \\ \\ \left\|f^{(n)}\right\|_{\gamma_{u,w},p} \left(\int_{\gamma_{u,x}} |z-u|^{qn} |dz| + \int_{\gamma_{x,w}} |z-w|^{qn} |dz| \right)^{1/q} \\ \text{where } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left\|f^{(n)}\right\|_{\gamma_{u,w},1} \max\left\{ \max_{z \in \gamma_{u,x}} |z-u|^n, \max_{z \in \gamma_{x,w}} |z-w|^n \right\}. \end{cases} \tag{1.10}$$

In this paper we establish an identity of Fink type for approximating the integral of analytic complex functions on paths from general domains. Error bounds for these expansions in terms of p-norms are also provided. Examples for the complex logarithm and the complex exponential are given as well.

2. Representation Results

We start with the following preliminary result that is of interest in itself [4]. For the sake of completeness, we give here a short proof as well.

Lemma 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(z)$, $t \in [a,b]$ with $z(a) = u$, $z(t) = x$ and $z(b) = w$ where u, $w \in D$. Then we have the equality*

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] + \frac{1}{n!}\int_\gamma (x-z)^n f^{(n)}(z)\,dz \tag{2.1}$$

for $n \geq 1$.

Proof. The proof is by mathematical induction over $n \geq 1$. For $n = 1$, we have to prove that

$$\int_\gamma f(z)\,dz = (x-u) f(u) + (w-x) f(w) + \int_\gamma (x-z) f'(z)\,dz, \tag{2.2}$$

which is straightforward as may be seen by the integration by parts formula applied for the integral

$$\int_\gamma (x-z) f'(z)\,dz.$$

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we wish to show that:

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] + \frac{1}{(n+1)!}\int_\gamma (x-z)^{n+1} f^{(n+1)}(z)\,dz. \tag{2.3}$$

Using the integration by parts rule, we have

$$\begin{aligned} &\frac{1}{(n+1)!}\int_\gamma (x-z)^{n+1} f^{(n+1)}(z)\,dz \\ &= \frac{1}{(n+1)!}\int_\gamma (x-z)^{n+1} \left(f^{(n)}(z)\right)' dz \\ &= \frac{1}{(n+1)!}\left[(x-z)^{n+1} f^{(n)}(z)\Big|_u^w + (n+1)\int_\gamma (x-z)^n f^{(n)}(z)\,dz\right] \\ &= \frac{1}{(n+1)!} \\ &\times \left[(x-w)^{n+1} f^{(n)}(w) - (x-u)^{n+1} f^{(n)}(u) + (n+1)\int_\gamma (x-z)^n f^{(n)}(z)\,dz\right] \\ &= \frac{1}{n!}\int_\gamma (x-z)^n f^{(n)}(z)\,dz \\ &\quad - \frac{1}{(n+1)!}\left[(x-u)^{n+1} f^{(n)}(u) + (-1)^n (w-x)^{n+1} f^{(n)}(w)\right], \end{aligned} \tag{2.4}$$

which gives that

$$\frac{1}{n!}\int_{\gamma}(x-z)^{n}f^{(n)}(z)\,dz \tag{2.5}$$
$$=\frac{1}{(n+1)!}\left[(x-u)^{n+1}f^{(n)}(u)+(-1)^{n}(w-x)^{n+1}f^{(n)}(w)\right]$$
$$+\frac{1}{(n+1)!}\int_{\gamma}(x-z)^{n+1}f^{(n+1)}(z)\,dz.$$

From the induction hypothesis we have

$$\frac{1}{n!}\int_{\gamma}(x-z)^{n}f^{(n)}(z)\,dz \tag{2.6}$$
$$=\int_{\gamma}f(z)\,dz-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1}f^{(k)}(u)+(-1)^{k}(w-x)^{k+1}f^{(k)}(w)\right].$$

By making use of (2.11) and (2.12) we get

$$\int_{\gamma}f(z)\,dz-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1}f^{(k)}(u)+(-1)^{k}(w-x)^{k+1}f^{(k)}(w)\right]$$
$$=\frac{1}{(n+1)!}\left[(x-u)^{n+1}f^{(n)}(u)+(-1)^{n}(w-x)^{n+1}f^{(n)}(w)\right]$$
$$+\frac{1}{(n+1)!}\int_{\gamma}(x-z)^{n+1}f^{(n+1)}(z)\,dz,$$

which is equivalent to (2.3). □

We have the following generalization of Fink identity for the complex integral.

Theorem 2. *Let $f: D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain D and $x\in D$. Suppose $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$, $z(t)=x$ and $z(b)=w$ where $u, w\in D$, $u\neq w$. Define*

$$F_{k}(x):=\frac{n-k}{k!}\left[\frac{f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)}{w-u}\right], \tag{2.7}$$

for $k=1,...,n-1$ where $n\geq 2$.

Then we have the equality

$$\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_{k}(x)\right]-\frac{1}{w-u}\int_{\gamma}f(z)\,dz=R_{n}(x,\gamma), \tag{2.8}$$

where the remainder $R_{n}(x,\gamma)$ is given by

$$R_{n}(x,\gamma):=\frac{1}{n!(w-u)} \tag{2.9}$$
$$\times\left[\int_{\gamma_{u,x}}(x-z)^{n-1}(z-u)f^{(n)}(z)\,dz+\int_{\gamma_{x,w}}(x-z)^{n-1}(z-w)f^{(n)}(z)\,dz\right].$$

For, $n=1$ the identity (2.8) reduces to

$$f(x)-\frac{1}{w-u}\int_{\gamma}f(z)\,dz=R_{1}(x,\gamma), \tag{2.10}$$

where

$$R_1(x,\gamma) := \frac{1}{(w-u)}\left[\int_{\gamma_{u,x}} (z-u) f'(z)\, dz + \int_{\gamma_{x,w}} (z-w) f'(z)\, dz\right]. \tag{2.11}$$

Proof. We prove the identity by induction over n. For $n = 1$, we have to prove the equality (2.10) with the remainder $R_1(x,\gamma)$ given by (2.11).

Integrating by parts, we have:

$$\begin{aligned}
&\int_{\gamma_{u,x}} (z-u) f'(z)\, dz + \int_{\gamma_{x,w}} (z-w) f'(z)\, dz \\
&= (z-u) f(z)\big|_u^x - \int_{\gamma_{u,x}} f(z)\, dz + (z-w) f(z)\big|_x^w - \int_{\gamma_{x,w}} f(z)\, dz \\
&= (x-u) f(x) + (w-x) f(x) - \int_\gamma f(z)\, dz \\
&= (w-u) f(x) - \int_\gamma f(z)\, dz,
\end{aligned}$$

which proves the statement.

Assume that the representation (2.8) holds for "n" and let us prove it for "$n+1$". That is, we have to prove the equality

$$\begin{aligned}
&\frac{1}{n+1} \\
&\times\left[f(x) + \sum_{k=1}^{n} \frac{n+1-k}{k!}\left[\frac{f^{(k-1)}(u)(x-u)^k + (-1)^{k-1}(w-x)^k f^{(k-1)}(w)}{w-u}\right]\right] \\
&\qquad - \frac{1}{w-u}\int_\gamma f(z)\, dz \\
&\qquad = \frac{1}{(n+1)!\,(w-u)} \\
&\times\left[\int_{\gamma_{u,x}} (x-z)^n (z-u) f^{(n+1)}(z)\, dz + \int_{\gamma_{x,w}} (x-z)^n (z-w) f^{(n+1)}(z)\, dz\right].
\end{aligned} \tag{2.12}$$

Using the integration by parts, we have

$$\begin{aligned}
&\int_{\gamma_{u,x}} (x-z)^n (z-u) f^{(n+1)}(z)\, dz \\
&\qquad = \int_{\gamma_{u,x}} (x-z)^n (z-u) \left(f^{(n)}(z)\right)' dz \\
&= (x-z)^n (z-u) f^{(n)}(z)\Big|_u^x - \int_{\gamma_{u,x}} \left((x-z)^n (z-u)\right)' f^{(n)}(z)\, dz \\
&= -\int_{\gamma_{u,x}} \left[-n(x-z)^{n-1}(z-u) + (x-z)^n\right] f^{(n)}(z)\, dz \\
&= n\int_{\gamma_{u,x}} (x-z)^{n-1}(z-u) f^{(n)}(z)\, dz - \int_{\gamma_{u,x}} (x-z)^n f^{(n)}(z)\, dz
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}(2.14)\quad &\int_{\gamma_{x,w}} (x-z)^n (z-w) f^{(n+1)}(z)\,dz \\ &= \int_{\gamma_{x,w}} (x-z)^n (z-w) \left(f^{(n)}(z)\right)' dz \\ &= (x-z)^n (z-w) f^{(n)}(z)\Big|_x^w - \int_{\gamma_{x,w}} \left((x-z)^n (z-w)\right)' f^{(n)}(z)\,dz \\ &= -\int_{\gamma_{x,w}} \left[-n(x-z)^{n-1}(z-w) + (x-z)^n\right] f^{(n)}(z)\,dz \\ &= n\int_{\gamma_{x,w}} (x-z)^{n-1}(z-w) f^{(n)}(z)\,dz - \int_{\gamma_{x,w}} (x-z)^n f^{(n)}(z)\,dz.\end{aligned}$$

If we add these two equalities, we get

$$\begin{aligned}(2.15)\quad &\int_{\gamma_{u,x}} (x-z)^n (z-u) f^{(n+1)}(z)\,dz + \int_{\gamma_{x,w}} (x-z)^n (z-w) f^{(n+1)}(z)\,dz \\ &= n\left[\int_{\gamma_{u,x}} (x-z)^{n-1}(z-u) f^{(n)}(z)\,dz + \int_{\gamma_{x,w}} (x-z)^{n-1}(z-w) f^{(n)}(z)\,dz\right] \\ &\quad - \int_{\gamma} (x-z)^n f^{(n)}(z)\,dz.\end{aligned}$$

By dividing with $(n+1)!\,(w-u)$ in (2.15) we get

$$\begin{aligned}R_{n+1}(x,\gamma) &:= \frac{1}{(n+1)!\,(w-u)} \\ &\times \left[\int_{\gamma_{u,x}} (x-z)^n (z-u) f^{(n+1)}(z)\,dz + \int_{\gamma_{x,w}} (x-z)^n (z-w) f^{(n+1)}(z)\,dz\right] \\ &= \frac{n}{(n+1)!\,(w-u)} \\ &\times \left[\int_{\gamma_{u,x}} (x-z)^{n-1}(z-u) f^{(n)}(z)\,dz + \int_{\gamma_{x,w}} (x-z)^{n-1}(z-w) f^{(n)}(z)\,dz\right] \\ &\quad - \frac{1}{(n+1)!\,(w-u)} \int_{\gamma} (x-z)^n f^{(n)}(z)\,dz \\ &= \frac{n}{n+1} R_n(x,\gamma) - \frac{1}{(n+1)!\,(w-u)} \int_{\gamma} (x-z)^n f^{(n)}(z)\,dz.\end{aligned}$$

Using the representation (2.9) for $R_n(x,\gamma)$, which is assumed to be true by the induction hypothesis, we get

$$
\begin{aligned}
(2.16)\quad & R_{n+1}(x,\gamma) \\
&= \frac{n}{n+1}\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}\frac{n-k}{k!}\left[\frac{f^{(k-1)}(u)(x-u)^k+(-1)^{k-1}(w-x)^k f^{(k-1)}(w)}{w-u}\right]\right] \\
&\qquad - \frac{n}{n+1}\frac{1}{w-u}\int_\gamma f(z)\,dz \\
&\quad - \frac{1}{(n+1)!(w-u)}\left[\int_{\gamma_{u,x}}(x-z)^n f^{(n)}(z)\,dz+\int_{\gamma_{x,w}}(x-z)^n f^{(n)}(z)\,dz\right] \\
&= \frac{1}{n+1} \\
&\times\left[f(x)+\sum_{k=1}^{n-1}\frac{n-k}{k!(w-u)}\left[f^{(k-1)}(u)(x-u)^k+(-1)^{k-1}(w-x)^k f^{(k-1)}(w)\right]\right] \\
&\qquad - \frac{n}{n+1}\frac{1}{w-u}\int_\gamma f(z)\,dz - \frac{1}{(n+1)!(w-u)}\int_\gamma (x-z)^n f^{(n)}(z)\,dz.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\sum_{k=1}^{n}\frac{n+1-k}{k!}\left[f^{(k-1)}(u)(x-u)^k+(-1)^{k-1}(w-x)^k f^{(k-1)}(w)\right] \\
&= \sum_{k=1}^{n-1}\frac{n+1-k}{k!}\left[f^{(k-1)}(u)(x-u)^k+(-1)^{k-1}(w-x)^k f^{(k-1)}(w)\right] \\
&\quad + \frac{n+1-n}{n!}\left[f^{(n-1)}(u)(x-u)^n+(-1)^{n-1}(w-x)^n f^{(n-1)}(w)\right] \\
&= \sum_{k=1}^{n-1}\frac{n-k}{k!}\left[f^{(k-1)}(u)(x-u)^k+(-1)^{k-1}(w-x)^k f^{(k-1)}(w)\right] \\
&\quad + \sum_{k=1}^{n-1}\frac{1}{k!}\left[f^{(k-1)}(u)(x-u)^k+(-1)^{k-1}(w-x)^k f^{(k-1)}(w)\right] \\
&\quad + \frac{1}{n!}\left[f^{(n-1)}(u)(x-u)^n+(-1)^{n-1}(w-x)^n f^{(n-1)}(w)\right] \\
&= \sum_{k=1}^{n-1}\frac{n-k}{k!}\left[f^{(k-1)}(u)(x-u)^k+(-1)^{k-1}(w-x)^k f^{(k-1)}(w)\right] \\
&\qquad + \sum_{k=1}^{n}\frac{1}{k!}\left[f^{(k-1)}(u)(x-u)^k+(-1)^{k-1}(w-x)^k f^{(k-1)}(w)\right],
\end{aligned}
$$

which implies that

$$\sum_{k=1}^{n-1}\frac{n-k}{k!\,(w-u)}\left[f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)\right]$$
$$=\sum_{k=1}^{n}\frac{n+1-k}{k!\,(w-u)}\left[f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)\right]$$
$$-\sum_{k=1}^{n}\frac{1}{k!\,(w-u)}\left[f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)\right].$$

Therefore, by (2.16) we get

$$R_{n+1}(x,\gamma)=\frac{1}{n+1}\tag{2.17}$$
$$\times\left[f(x)+\sum_{k=1}^{n}\frac{n+1-k}{k!}\left[\frac{f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)}{w-u}\right]\right]$$
$$-\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{k!\,(w-u)}\left[f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)\right].$$
$$-\frac{n}{n+1}\frac{1}{w-u}\int_{\gamma}f(z)\,dz-\frac{1}{(n+1)!\,(w-u)}\int_{\gamma}(x-z)^{n}f^{(n)}(z)\,dz$$

$$=\frac{1}{n+1}$$
$$\times\left[f(x)+\sum_{k=1}^{n}\frac{n+1-k}{k!}\left[\frac{f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)}{w-u}\right]\right]$$
$$-\frac{1}{w-u}\int_{\gamma}f(z)\,dz$$
$$-\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{k!\,(w-u)}\left[f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)\right]$$
$$+\frac{1}{(n+1)(w-u)}\int_{\gamma}f(z)\,dz-\frac{1}{(n+1)!\,(w-u)}\int_{\gamma}(x-z)^{n}f^{(n)}(z)\,dz.$$

We must prove now that

$$-\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{k!\,(w-u)}\left[f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)\right]$$
$$+\frac{1}{(n+1)(w-u)}\int_{\gamma}f(z)\,dz$$
$$-\frac{1}{(n+1)!\,(w-u)}\left[\int_{\gamma_{u,x}}(x-z)^{n}f^{(n)}(z)\,dz+\int_{\gamma_{x,w}}(x-z)^{n}f^{(n)}(z)\,dz\right]=0,$$

namely

$$\int_{\gamma} f(z)\,dz=\sum_{k=1}^{n}\frac{1}{k!}\left[f^{(k-1)}(u)(x-u)^{k}+(-1)^{k-1}(w-x)^{k}f^{(k-1)}(w)\right]+\frac{1}{n!}\int_{\gamma}(x-z)^{n}f^{(n)}(z)\,dz.$$

This however follows by Lemma 1. □

We have following trapezoid type representation:

Corollary 1. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}\frac{1}{n}&\left[\frac{f(u)+f(w)}{2}+\sum_{k=1}^{n-1}\frac{n-k}{k!}\frac{f^{(k-1)}(u)+(-1)^{k-1}f^{(k-1)}(w)}{2}(w-u)^{k-1}\right]-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\\&=\frac{1}{2n!(w-u)}\int_{\gamma_{u,w}}(z-u)(w-z)\left[(w-z)^{n-2}+(-1)^{n}(z-u)^{n-2}\right]f^{(n)}(z)\,dz\\&=:T_{n}(\gamma)\end{aligned}\tag{2.18}$$

for $n\geq 2$.

For $n=1$, *we have*

$$\frac{f(u)+f(w)}{2}-\frac{1}{w-u}\int_{\gamma}f(z)\,dz=\frac{1}{w-u}\int_{\gamma}\left(z-\frac{u+w}{2}\right)f'(z)\,dz.\tag{2.19}$$

Proof. We have

$$F_{k}(u)=(-1)^{k-1}\frac{n-k}{k!}(w-u)^{k-1}f^{(k-1)}(w)$$

and

$$F_{k}(w)=\frac{n-k}{k!}(w-u)^{k-1}f^{(k-1)}(u)$$

for $k=1,...,n-1$ where $n\geq 2$.

From (2.8) we have

$$\frac{1}{n}\left[f(u)+\sum_{k=1}^{n-1}F_{k}(u)\right]-\frac{1}{w-u}\int_{\gamma}f(z)\,dz=R_{n}(u,\gamma)=\frac{1}{n!(w-u)}\int_{\gamma_{u,w}}(u-z)^{n-1}(z-w)f^{(n)}(z)\,dz\tag{2.20}$$

and

$$\frac{1}{n}\left[f(w)+\sum_{k=1}^{n-1}F_{k}(w)\right]-\frac{1}{w-u}\int_{\gamma}f(z)\,dz=R_{n}(w,\gamma)=\frac{1}{n!(w-u)}\int_{\gamma_{u,w}}(w-z)^{n-1}(z-u)f^{(n)}(z)\,dz.\tag{2.21}$$

If we add the equalities (2.20) and (2.21) and divide by 2, then we get

$$\begin{aligned}
&\frac{1}{n}\left[\frac{f(u)+f(w)}{2}+\sum_{k=1}^{n-1}\frac{F_k(u)+F_k(w)}{2}\right]-\frac{1}{w-u}\int_\gamma f(z)\,dz \\
&=\frac{1}{2n!(w-u)}\int_{\gamma_{u,w}}\left[(u-z)^{n-1}(z-w)+(w-z)^{n-1}(z-u)\right]f^{(n)}(z)\,dz \\
&=\frac{1}{2n!(w-u)}\int_{\gamma_{u,w}}\left[(u-z)^{n-1}(z-w)+(w-z)^{n-1}(z-u)\right]f^{(n)}(z)\,dz \\
&=\frac{1}{2n!(w-u)}\int_{\gamma_{u,w}}\left[(-1)^n(z-u)^{n-1}(w-z)+(w-z)^{n-1}(z-u)\right]f^{(n)}(z)\,dz \\
&=\frac{1}{2n!(w-u)}\int_{\gamma_{u,w}}(z-u)(w-z)\left[(w-z)^{n-2}+(-1)^n(z-u)^{n-2}\right]f^{(n)}(z)\,dz,
\end{aligned}$$

which proves (2.18). □

Remark 1. *If the function f is of real variable and defined on the interval $[a,b]$ then from (2.18) we obtain the following trapezoid identity obtained by Dragomir & Sofo in* [5]

$$\begin{aligned}
(2.22)\quad &\frac{1}{n}\left[\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n-1}\frac{n-k}{k!}\frac{f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)}{2}(b-a)^{k-1}\right] \\
&\qquad -\frac{1}{b-a}\int_a^b f(t)\,dt \\
&=\frac{1}{2n!(b-a)}\int_a^b(t-a)(b-t)\left[(b-t)^{n-2}+(-1)^n(t-a)^{n-2}\right]f^{(n)}(t)\,dt
\end{aligned}$$

for $n\geq 2$.

If $n=1$, then we have

$$(2.23)\qquad \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)\,dt.$$

It is natural to consider the case of linear path γ, namely the path parametrized by $z(s):=(1-s)u+sw$, $s\in[0,1]$ that join the distinct complex numbers u, $w\in D$. If $x=(1-t)u+tw$ for some $t\in[0,1]$, then

$$\begin{aligned}
(2.24)\quad &F_k((1-s)u+tw) \\
&=\frac{n-k}{k!}\left[\frac{f^{(k-1)}(u)t^k(w-u)^k+(-1)^{k-1}(1-t)^k(w-u)^kf^{(k-1)}(w)}{w-u}\right] \\
&=\frac{n-k}{k!}\left[f^{(k-1)}(u)t^k+(-1)^{k-1}(1-t)^kf^{(k-1)}(w)\right](w-u)^{k-1}
\end{aligned}$$

for $k=1,...,n-1$ where $n\geq 2$,

$$\frac{1}{w-u}\int_\gamma f(z)\,dz=\int_\gamma f((1-s)u+sw)\,ds,$$

$$R_n\left((1-s)u+tw,\gamma\right)=\frac{1}{n!}(w-u)^n \times\left[\int_0^t (t-s)^{n-1} sf^{(n)}\left((1-s)u+sw\right)ds - \int_t^1 (t-s)^{n-1}(1-s) f^{(n)}\left((1-s)u+sw\right)ds\right] \tag{2.25}$$

and the equality (2.8) becomes

$$\frac{1}{n}f\left((1-t)u+tw\right) + \frac{1}{n}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left[f^{(k-1)}(u)t^k+(-1)^{k-1}(1-t)^k f^{(k-1)}(w)\right](w-u)^{k-1} - \int_\gamma f\left((1-s)u+sw\right)ds$$
$$= \frac{1}{n!}(w-u)^n\left[\int_0^t (t-s)^{n-1} sf^{(n)}\left((1-s)u+sw\right)ds - \int_t^1 (t-s)^{n-1}(1-s) f^{(n)}\left((1-s)u+sw\right)ds\right]. \tag{2.26}$$

3. Error Bounds

We have the following error bounds:

Theorem 3. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a)=u$, $z(t)=x$ and $z(b)=w$ where u, $w \in D$, $u \neq w$. Then we have the representation (2.8) and the remainder $R_n(x,\gamma)$ satisfies the bounds*

$$|R_n(x,\gamma)| \le \frac{1}{n!\,|w-u|} \times\left[\int_{\gamma_{u,x}} |x-z|^{n-1}|z-u|\left|f^{(n)}(z)\right||dz| + \int_{\gamma_{x,w}} |x-z|^{n-1}|z-w|\left|f^{(n)}(z)\right||dz|\right] \tag{3.1}$$

$$\le \frac{1}{n!\,|w-u|}\begin{cases} \int_{\gamma_{u,x}} |x-z|^{n-1}|z-u||dz|\left\|f^{(n)}\right\|_{\gamma_{u,x},\infty} \\ \left(\int_{\gamma_{u,x}} |x-z|^{(n-1)q}|z-u|^q|dz|\right)^{1/q}\left\|f^{(n)}\right\|_{\gamma_{u,x},p} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \max_{z\in\gamma_{u,x}}\left\{|x-z|^{n-1}|z-u|\right\}\left\|f^{(n)}\right\|_{\gamma_{u,x},1} \end{cases}$$

$$+\frac{1}{n!\,|w-u|}\begin{cases}\int_{\gamma_{x,w}}|x-z|^{n-1}|z-w|\,|dz|\,\left\|f^{(n)}\right\|_{\gamma_{x,w},\infty}\\[2ex] \left(\int_{\gamma_{x,w}}|x-z|^{(n-1)q}|z-w|^{q}\,|dz|\right)^{1/q}\left\|f^{(n)}\right\|_{\gamma_{x,w},p}\\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1\\[2ex] \max_{z\in\gamma_{x,w}}\left\{|x-z|^{n-1}|z-w|\right\}\left\|f^{(n)}\right\|_{\gamma_{x,w},1}\end{cases}$$

$$\leq\frac{1}{n!\,(|w-u|)}$$

$$\times\begin{cases}\left[\int_{\gamma_{u,x}}|x-z|^{n-1}|z-u|\,|dz|+\int_{\gamma_{x,w}}|x-z|^{n-1}|z-w|\,|dz|\right]\\ \times\left\|f^{(n)}\right\|_{\gamma_{u,w},\infty}\\[2ex] \left[\int_{\gamma_{u,x}}|x-z|^{(n-1)q}|z-u|^{q}\,|dz|+\int_{\gamma_{x,w}}|x-z|^{(n-1)q}|z-w|^{q}\,|dz|\right]^{1/q}\\ \times\left\|f^{(n)}\right\|_{\gamma_{u,w},p}\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1\\[2ex] \max\left\{\max_{z\in\gamma_{u,x}}\left\{|x-z|^{n-1}|z-u|\right\},\max_{z\in\gamma_{x,w}}\left\{|x-z|^{n-1}|z-w|\right\}\right\}\\ \times\left\|f^{(n)}\right\|_{\gamma_{u,w},1}.\end{cases}$$

Proof. By the equality (2.9) we have

$$|R_n(x,\gamma)|\leq\frac{1}{n!\,|w-u|}\tag{3.2}$$

$$\times\left[\left|\int_{\gamma_{u,x}}(x-z)^{n-1}(z-u)\,f^{(n)}(z)\,dz\right|+\left|\int_{\gamma_{x,w}}(x-z)^{n-1}(z-w)\,f^{(n)}(z)\,dz\right|\right]$$

$$\leq\frac{1}{n!\,|w-u|}$$

$$\times\left[\int_{\gamma_{u,x}}|x-z|^{n-1}|z-u|\left|f^{(n)}(z)\right||dz|+\int_{\gamma_{x,w}}|x-z|^{n-1}|z-w|\left|f^{(n)}(z)\right||dz|\right]$$

$$=:A,$$

which proves the first inequality in (3.1).

Using Hölder's integral inequality we have

$$\int_{\gamma_{u,x}}|x-z|^{n-1}|z-u|\left|f^{(n)}(z)\right||dz|$$

$$\leq\begin{cases}\int_{\gamma_{u,x}}|x-z|^{n-1}|z-u|\,|dz|\,\left\|f^{(n)}\right\|_{\gamma_{u,x},\infty}\\[2ex] \left(\int_{\gamma_{u,x}}|x-z|^{(n-1)q}|z-u|^{q}\,|dz|\right)^{1/q}\left\|f^{(n)}\right\|_{\gamma_{u,x},p}\\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1\\[2ex] \max_{z\in\gamma_{u,x}}\left\{|x-z|^{n-1}|z-u|\right\}\left\|f^{(n)}\right\|_{\gamma_{u,x},1}\end{cases}$$

and

$$\int_{\gamma_{x,w}} |x-z|^{n-1} |z-w| \left| f^{(n)}(z) \right| |dz|$$
$$\leq \begin{cases} \int_{\gamma_{x,w}} |x-z|^{n-1} |z-w| |dz| \left\| f^{(n)} \right\|_{\gamma_{x,w},\infty} \\ \\ \left(\int_{\gamma_{x,w}} |x-z|^{(n-1)q} |z-w|^q |dz| \right)^{1/q} \left\| f^{(n)} \right\|_{\gamma_{x,w},p} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \max_{z\in\gamma_{x,w}} \left\{ |x-z|^{n-1} |z-w| \right\} \left\| f^{(n)} \right\|_{\gamma_{x,w},1}, \end{cases}$$

which proves the the second inequality in (3.1).

We also have

$$\int_{\gamma_{u,x}} |x-z|^{n-1} |z-u| |dz| \left\| f^{(n)} \right\|_{\gamma_{u,x},\infty} + \int_{\gamma_{x,w}} |x-z|^{n-1} |z-w| |dz| \left\| f^{(n)} \right\|_{\gamma_{x,w},\infty}$$
$$\leq \max\left\{ \left\| f^{(n)} \right\|_{\gamma_{u,x},\infty}, \left\| f^{(n)} \right\|_{\gamma_{x,w},\infty} \right\}$$
$$\times \left[\int_{\gamma_{u,x}} |x-z|^{n-1} |z-u| |dz| + \int_{\gamma_{x,w}} |x-z|^{n-1} |z-w| |dz| \right]$$
$$= \left[\int_{\gamma_{u,x}} |x-z|^{n-1} |z-u| |dz| + \int_{\gamma_{x,w}} |x-z|^{n-1} |z-w| |dz| \right] \left\| f^{(n)} \right\|_{\gamma_{u,w},\infty},$$

$$\left(\int_{\gamma_{u,x}} |x-z|^{(n-1)q} |z-u|^q |dz| \right)^{1/q} \left\| f^{(n)} \right\|_{\gamma_{u,x},p}$$
$$+ \left(\int_{\gamma_{x,w}} |x-z|^{(n-1)q} |z-w|^q |dz| \right)^{1/q} \left\| f^{(n)} \right\|_{\gamma_{x,w},p}$$

$$\leq \left[\int_{\gamma_{u,x}} |x-z|^{(n-1)q} |z-u|^q |dz| + \int_{\gamma_{x,w}} |x-z|^{(n-1)q} |z-w|^q |dz| \right]^{1/q}$$
$$\times \left[\left\| f^{(n)} \right\|_{\gamma_{u,x},p}^p + \left\| f^{(n)} \right\|_{\gamma_{x,w},p}^p \right]^{1/p}$$
$$= \left[\int_{\gamma_{u,x}} |x-z|^{(n-1)q} |z-u|^q |dz| + \int_{\gamma_{x,w}} |x-z|^{(n-1)q} |z-w|^q |dz| \right]^{1/q} \left\| f^{(n)} \right\|_{\gamma_{u,w},p}$$

and

$$\max_{z\in\gamma_{u,x}} \left\{ |x-z|^{n-1} |z-u| \right\} \left\| f^{(n)} \right\|_{\gamma_{u,x},1} + \max_{z\in\gamma_{x,w}} \left\{ |x-z|^{n-1} |z-w| \right\} \left\| f^{(n)} \right\|_{\gamma_{x,w},1}$$
$$\leq \max\left\{ \max_{z\in\gamma_{u,x}} \left\{ |x-z|^{n-1} |z-u| \right\}, \max_{z\in\gamma_{x,w}} \left\{ |x-z|^{n-1} |z-w| \right\} \right\} \left\| f^{(n)} \right\|_{\gamma_{u,w},1},$$

which proves the last part of 3.1). □

We have the following error bounds:

Theorem 4. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$, $z(t) = x$ and $z(b) = w$ where u, $w \in D$, $u \neq w$. Then we have the representation (2.18) and the remainder $T_n(\gamma)$ satisfies the bounds*

$$\begin{aligned}
|T_n(\gamma)| &\leq \frac{1}{2n!\,|w-u|} \\
&\quad \times \int_{\gamma_{u,w}} |z-u|\,|w-z| \left|(w-z)^{n-2} + (-1)^n (z-u)^{n-2}\right| \left|f^{(n)}(z)\right| |dz| \\
&\leq \frac{\left\|f^{(n)}\right\|_{\gamma_{u,w},\infty}}{2n!\,|w-u|} \int_{\gamma_{u,w}} |z-u|\,|w-z| \left|(w-z)^{n-2} + (-1)^n (z-u)^{n-2}\right| |dz| \\
&\leq \frac{\left\|f^{(n)}\right\|_{\gamma_{u,w},\infty}}{2n!\,|w-u|} \int_{\gamma_{u,w}} |z-u|\,|w-z| \left|(w-z)^{n-2} + (-1)^n (z-u)^{n-2}\right| |dz| \\
&\qquad \geq \frac{\left\|f^{(n)}\right\|_{\gamma_{u,w},\infty}}{2n!\,|w-u|} \int_{\gamma_{u,w}} |z-u|\,|w-z| \left[|w-z|^{n-2} + |z-u|^{n-2}\right] |dz|,
\end{aligned}$$

provided that $\left\|f^{(n)}\right\|_{\gamma_{u,w},\infty} < \infty$.

The proof follows by the identity (2.18) by taking the modulus and using the integral properties.

4. Examples for Logarithm and Exponential

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C} \setminus \{x + iy : x \leq 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \geq 1,\ z \in \mathbb{C}_\ell.$$

Suppose $\gamma \subset \mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in \mathbb{C}_\ell$, $u \neq w$. Then

$$\begin{aligned}
\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz &= \int_{\gamma_{u,w}} \operatorname{Log}(z)\,dz = \\
&= z \operatorname{Log}(z)\big|_u^w - \int_{\gamma_{u,w}} (\operatorname{Log}(z))' z\,dz \\
&= w \operatorname{Log}(w) - u \operatorname{Log}(u) - \int_{\gamma_{u,w}} dz \\
&= w \operatorname{Log}(w) - u \operatorname{Log}(u) - (w-u),
\end{aligned}$$

where u, $w \in \mathbb{C}_\ell$.

Define

$$F_k(x) := \frac{n-k}{k(k-1)} \left[\frac{(-1)^k \frac{(x-u)^k}{u^{k-1}} - \frac{(w-x)^k}{u^{k-1}}}{w-u}\right], \tag{4.1}$$

for $k=1,...,n-1$ where $n\geq 2$.

Then we have the equality

$$\frac{1}{n}\left[\operatorname{Log}(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{w\operatorname{Log}(w)-u\operatorname{Log}(u)-(w-u)}{w-u}=R_n(x,\gamma), \tag{4.2}$$

where the remainder $R_n(x,\gamma)$ is given by

$$R_n(x,\gamma):=\frac{(-1)^{n-1}}{n(w-u)}\times\left[\int_{\gamma_{u,x}}\frac{(x-z)^{n-1}(z-u)}{z^n}dz+\int_{\gamma_{x,w}}\frac{(x-z)^{n-1}(z-w)}{z^n}dz\right]. \tag{4.3}$$

If $d_\gamma:=\inf_{z\in\gamma}|z|$ is positive and finite, then from (4.3) we get the inequality

$$|R_n(x,\gamma)|\leq\frac{1}{n|w-u|d_\gamma^n}\left[\int_{\gamma_{u,x}}|x-z|^{n-1}|z-u||dz|+\int_{\gamma_{x,w}}|x-z|^{n-1}|z-w||dz|\right]. \tag{4.4}$$

Consider the function $f(z)=\frac{1}{z}$, $z\in\mathbb{C}\backslash\{0\}$. Then

$$f^{(k)}(z)=\frac{(-1)^k k!}{z^{k+1}}\text{ for }k\geq 0,\ z\in\mathbb{C}\backslash\{0\}$$

and suppose $\gamma\subset\mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in\mathbb{C}_\ell$, $u\neq w$. Then

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}}f(z)\,dz=\int_{\gamma_{u,w}}\frac{dz}{z}=\operatorname{Log}(w)-\operatorname{Log}(u)$$

for u, $w\in\mathbb{C}_\ell$.

Define for $x\in\gamma$

$$L_k(x):=\frac{n-k}{k}\left[\frac{\frac{(w-x)^k}{w^k}+(-1)^{k-1}\frac{(x-u)^k}{u^k}}{w-u}\right], \tag{4.5}$$

for $k=1,...,n-1$ where $n\geq 2$.

Then we have the equality

$$\frac{1}{n}\left[\frac{1}{x}+\sum_{k=1}^{n-1}L_k(x)\right]-\frac{\operatorname{Log}(w)-\operatorname{Log}(u)}{w-u}=L_n(x,\gamma), \tag{4.6}$$

where the remainder $L_n(x,\gamma)$ is given by

$$L_n(x,\gamma):=\frac{(-1)^n}{(w-u)}\times\left[\int_{\gamma_{u,x}}\frac{(x-z)^{n-1}(z-u)}{z^{n+1}}dz+\int_{\gamma_{x,w}}\frac{(x-z)^{n-1}(z-w)}{z^{n+1}}dz\right]. \tag{4.7}$$

If d_γ defined above is positive and finite, then from (4.3) we get the inequality

$$|L_n(x,\gamma)| \leq \frac{1}{|w-u|\, d_\gamma^{n+1}} \left[\int_{\gamma_{u,x}} |x-z|^{n-1} |z-u| \, |dz| + \int_{\gamma_{x,w}} |x-z|^{n-1} |z-w| \, |dz| \right] \tag{4.8}$$

Consider the function $f(z) = \exp(z)$, $z \in \mathbb{C}$. Then

$$f^{(k)}(z) = \exp(z) \text{ for } k \geq 0, \ z \in \mathbb{C}$$

and suppose $\gamma \subset \mathbb{C}$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in \mathbb{C}$, $u \neq w$. Then

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz = \int_{\gamma_{u,w}} \exp(z)\, dz = \exp(w) - \exp(u).$$

Define for $x \in \gamma$

$$E_k(x) := \frac{n-k}{k!} \left[\frac{(x-u)^k \exp u + (-1)^{k-1} (w-x)^k \exp w}{w-u} \right], \tag{4.9}$$

for $k = 1, ..., n-1$ where $n \geq 2$.

Then we have the equality

$$\frac{1}{n} \left[\exp(x) + \sum_{k=1}^{n-1} E_k(x) \right] - \frac{\exp(w) - \exp(u)}{w-u} = E_n(x,\gamma), \tag{4.10}$$

where the remainder $E_n(x,\gamma)$ is given by

$$E_n(x,\gamma) := \frac{1}{n!\,(w-u)} \times \left[\int_{\gamma_{u,x}} (x-z)^{n-1} (z-u) \exp z dz + \int_{\gamma_{x,w}} (x-z)^{n-1} (z-w) \exp z dz \right]. \tag{4.11}$$

Since $|\exp z| = \exp(\operatorname{Re} z)$ and if assume that for $\gamma \subset \mathbb{C}$ we have

$$M_\gamma := \sup_{z \in \gamma} [\exp(\operatorname{Re} z)] < \infty,$$

then by (4.11) we get the inequality

$$|E_n(x,\gamma)| \leq \frac{M_\gamma}{n!\,|w-u|} \left[\int_{\gamma_{u,x}} |x-z|^{n-1} |z-u| \, |dz| + \int_{\gamma_{x,w}} |x-z|^{n-1} |z-w| \, |dz| \right]. \tag{4.12}$$

References

[1] G. A. Anastassiou, Multivariate Fink type identity and multivariate Ostrowski, comparison of means and Grüss type inequalities. *Math. Comput. Modelling* **46** (2007), no. 3-4, 351–374.

[2] A. Aglić Aljinović, J. Pečarić and A. Vukelić, The extension of Montgomery identity via Fink identity with applications. *J. Inequal. Appl.* 2005, no. **1**, 67–80.

[3] S. Ivelić Bradanović, N. Latif and J. Pečarić, Generalizations of Sherman's inequality via Fink's identity and Green's function. Reprinted in *Ukrainian Math. J.* **70** (2019), no. 8, 1192–1204. Ukraïn. Mat. Zh. 70 (2018), no. 8, 1033–1043

[4] S. S. Dragomir, New trapezoid type rules for approximating the integral of analytic complex functions on paths from general domains, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 5. [http://rgmia.org/papers/v22/v22a05.pdf].

[5] S. S. Dragomir and A. Sofo, Trapezoidal type inequalities for n-time differentiable functions, *J. Appl. Math. Comput.* **28** (2008): 367–379.

[6] A. M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.* **42** (117) (1992), no. 2, 289–310.

[7] S. Kovač, J. Pečarić and A. Vukelić, Weighted generalization of the trapezoidal rule via Fink identity. *Aust. J. Math. Anal. Appl.* **4** (2007), no. 1, Art. 8, 12 pp.

[8] G. V. Milovanović and J. E. Pečarić, On generalization of the inequality of A. Ostrowski and some related applications, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* (1976), no. **544–576**, 155–158.

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