BOUNDS FOR THE DIFFERENCE OF WEIGHTED AND INTEGRAL MEANS OF CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Let f be a convex function on C and $x,\ y\in C$, with $x\neq y$. If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that

$$\frac{1}{\tau}\int_0^\tau g(s)\,ds\le\frac{1}{1-\tau}\int_\tau^1 g(s)\,ds \text{ for all } \tau\in(0,1),$$

then we have the inequalities

$$\begin{aligned}&\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla_+f_x(y-x)\\&\le\int_0^1 p(\tau)f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau\\&\le\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla_-f_y(y-x).\end{aligned}$$

Some applications for norms and semi-inner products are also provided.

1. Introduction

Let X be a real linear space, $x,y\in X$, $x\neq y$ and let $[x,y]:=\{(1-\lambda)x+\lambda y,\ \lambda\in[0,1]\}$ be the *segment* generated by x and y. We consider the function $f:[x,y]\to\mathbb{R}$ and the attached function $\varphi_{(x,y)}:[0,1]\to\mathbb{R}$, $\varphi_{(x,y)}(t):=f[(1-t)x+ty]$, $t\in[0,1]$.

It is well known that f is convex on $[x,y]$ iff $\varphi(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $\varphi'_{\pm(x,y)}(s)=\nabla_\pm f_{(1-s)x+sy}(y-x)$, $s\in[0,1)$,
(ii) $\varphi'_{+(x,y)}(0)=\nabla_+f_x(y-x)$,
(iii) $\varphi'_{-(x,y)}(1)=\nabla_-f_y(y-x)$,

where $\nabla_\pm f_x(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}\nabla_+f_x(y)\ &:=\lim_{h\to0+}\frac{f(x+hy)-f(x)}{h},\\ \nabla_-f_x(y)\ &:=\lim_{k\to0-}\frac{f(x+ky)-f(x)}{k},\quad x,\ y\in X.\end{aligned}$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[x,y]\subset X$:

$$f\left(\frac{x+y}{2}\right)\le\int_0^1 f[(1-t)x+ty]\,dt\le\frac{f(x)+f(y)}{2},\tag{HH}$$

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which easily follows by the classical Hermite-Hadamard inequality for the convex function $\varphi(x,y):[0,1]\to\mathbb{R}$

$$\varphi_{(x,y)}\left(\frac{1}{2}\right)\leq\int_0^1\varphi_{(x,y)}(t)\,dt\leq\frac{\varphi_{(x,y)}(0)+\varphi_{(x,y)}(1)}{2}.$$

For other related results see the monograph on line [8]. For some recent results in linear spaces see [1], [2] and [9]-[12].

In the recent paper [7] we established the following refinements and reverses of Féjer's inequality for functions defined on linear spaces:

Theorem 1. *Let f be an convex function on C and $x,\ y\in C$ with $x\neq y$. If $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$\begin{aligned}(1.1)\qquad 0&\leq\frac{1}{2}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\\&\leq\int_0^1 f((1-t)x+ty)\,p(t)\,dt-f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt\\&\leq\frac{1}{2}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\left(\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\right)\end{aligned}$$

and

$$\begin{aligned}(1.2)\qquad 0&\leq\frac{1}{2}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\&\leq\frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt-\int_0^1 f((1-t)x+ty)\,p(t)\,dt\\&\leq\frac{1}{2}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt.\end{aligned}$$

If we take $p\equiv 1$ in (1.1), then we get

$$\begin{aligned}(1.3)\qquad 0&\leq\frac{1}{8}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\\&\leq\int_0^1 f[(1-t)x+ty]\,dt-f\left(\frac{x+y}{2}\right)\\&\leq\frac{1}{8}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\end{aligned}$$

that was firstly obtained in [4], while from (1.2) we recapture the result obtained in [5]

$$\begin{aligned}(1.4)\qquad 0&\leq\frac{1}{8}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\\&\leq\frac{f(x)+f(y)}{2}-\int_0^1 f[(1-t)x+ty]\,dt\\&\leq\frac{1}{8}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right].\end{aligned}$$

Motivated by the above results, we establish in this paper some upper and lower bounds for the difference

$$\int_0^1 p(\tau)\,f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau$$

where f is a convex function on C and x, $y\in C$, with $x\neq y$ while $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds\leq\frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1).$$

Some applications for norms and semi-inner products are also provided.

2. Main Results

We start to the following identity that is of interest in itself as well:

Lemma 1. *Let f be a convex function on C and x, $y\in C$, with $x\neq y$. If $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function, then we have the equality*

$$\begin{aligned}\int_0^1 g(\tau)\varphi_{(x,y)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(x,y)}(\tau)\,d\tau\\=\int_0^1\tau(1-\tau)\left(\frac{\int_\tau^1 g(s)\,ds}{1-\tau}-\frac{\int_0^\tau g(s)\,ds}{\tau}\right)\varphi'_{(x,y)}(\tau)\,d\tau.\end{aligned}\tag{2.1}$$

Proof. Integrating by parts in the Lebesgue integral, we have

$$\begin{aligned}&\int_0^\tau t\varphi'_{(x,y)}(t)\,dt+\int_\tau^1(t-1)\varphi'_{(x,y)}(t)\,dt\\&=\tau\varphi_{(x,y)}(\tau)-\int_0^\tau\varphi_{(x,y)}(t)\,dt-(\tau-1)\varphi_{(x,y)}(\tau)-\int_\tau^1\varphi_{(x,y)}(t)\,dt\\&=\varphi_{(x,y)}(\tau)-\int_0^1\varphi_{(x,y)}(t)\,dt\end{aligned}$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}&\int_0^1 g(\tau)\varphi_{(x,y)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(x,y)}(t)\,dt\\&=\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\tau+\int_0^1 g(\tau)\left(\int_\tau^1(t-1)\varphi'_{(x,y)}(t)\,dt\right)d\tau.\end{aligned}\tag{2.2}$$

Using integration by parts, we derive

$$\begin{aligned}&\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\tau\\&=\int_0^1\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)\\&=\left(\int_0^\tau g(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)\Bigg|_0^1\\&-\int_0^1\left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau\end{aligned}\tag{2.3}$$

$$
\begin{aligned}
&= \left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 t\varphi'_{(x,y)}(t)\,dt\right) \\
&\quad - \int_0^1 \left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)d\tau \\
&= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right) d\left(\int_0^\tau g(s)\,ds\right) \\
&= \left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)\left(\int_0^\tau g(s)\,ds\right)\Bigg|_0^1 \\
&\quad + \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau,
\end{aligned} \tag{2.4}
$$

which proves the identity

$$
\begin{aligned}
&\int_0^1 g(\tau)\,\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&\quad + \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau.
\end{aligned} \tag{2.5}
$$

Now, observe that

$$
\begin{aligned}
&\int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau + \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \tau\left(\int_\tau^1 g(s)\,ds\right)\varphi'_{(x,y)}(\tau)\,d\tau - \int_0^1 (1-\tau)\left(\int_0^\tau g(s)\,ds\right)\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 g(s)\,ds}{1-\tau} - \frac{\int_0^\tau g(s)\,ds}{t}\right)\varphi'_{(x,y)}(\tau)\,d\tau
\end{aligned}
$$

and by (2.5) we obtain the desired equality (2.1). □

We have the following result:

Theorem 2. *Let f be a convex function on C and $x,\ y \in C$, with $x \neq y$. If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that*

$$
\frac{1}{\tau}\int_0^\tau p(s)\,ds \leq \frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau \in (0,1), \tag{2.6}
$$

then we have the inequalities

$$
\begin{aligned}
&\left[\int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla_+ f_x(y-x) \\
&\le \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \\
&\le \left[\int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla_- f_y(y-x).
\end{aligned} \tag{2.7}
$$

Proof. By the properties of the convex function $\varphi_{(x,y)}$ from the above section, we have that

$$
\nabla_- f_y(y-x) \ge \varphi'_{(x,y)}(\tau) \ge \nabla_+ f_x(y-x) \tag{2.8}
$$

for all $\tau \in (0,1)$.

Since

$$
\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t} \ge 0
$$

for all $\tau \in (0,1)$, hence

$$
\begin{aligned}
&\tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\nabla_- f_y(y-x) \\
&\ge \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\varphi'_{(x,y)}(\tau) \\
&\ge \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\nabla_+ f_x(y-x)
\end{aligned}
$$

for all $\tau \in (0,1)$.

By taking the integral in this inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)d\tau\nabla_- f_y(y-x) \\
&\ge \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\varphi'_{(x,y)}(\tau)\,d\tau \\
&\ge \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)d\tau\nabla_+ f_x(y-x).
\end{aligned} \tag{2.9}
$$

By the identity (2.1) we also have

$$
\begin{aligned}
&\int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)d\tau \\
&= \int_0^1 g(\tau)\tau d\tau - \int_0^1 g(\tau)\,d\tau\int_0^1 \tau d\tau = \int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau
\end{aligned}
$$

and by employing Lemma 1 and the inequality (2.9) we obtain (2.7). □

Corollary 1. *Let f be a convex function on C and x, $y \in C$, with $x \ne y$. If $p : [0,1] \to \mathbb{R}$ is a monotonic nondecreasing function, then we have the inequalities (2.7).*

Proof. If $p:[0,1]\to\mathbb{R}$ is a monotonic nondecreasing function, then

$$\frac{1}{x}\int_0^x p(s)\,ds \le p(x) \le \frac{1}{1-x}\int_x^1 p(s)\,ds$$

for $x\in(0,1)$. Then by applying Theorem 2 we get the desired result. □

If $p:[0,1]\to\mathbb{R}$ is asymmetric, namely

$$p(1-t) = -p(t) \text{ for all } t\in[0,1]$$

and Lebesgue integrable, then $\int_0^1 p(s)\,ds = 0$. If $\tau\in[0,1]$ then $\int_0^\tau p(s)\,ds + \int_\tau^1 p(s)\,ds = 0$, which implies that $\int_\tau^1 p(s)\,ds = -\int_0^\tau p(s)\,ds$.

Corollary 2. *Let f be a convex function on C and $x,\ y\in C$, with $x\neq y$. If $p:[0,1]\to\mathbb{R}$ is an asymmetric Lebesgue integrable function such that*

$$\int_0^\tau p(s)\,ds \le 0 \text{ for all } \tau\in[0,1], \tag{2.10}$$

or, equivalently,

$$0 \le \int_\tau^1 p(s)\,ds \text{ for all } \tau\in[0,1], \tag{2.11}$$

then we have the inequalities

$$\begin{aligned}\int_0^1 \tau p(\tau)\,d\tau\nabla_+ f_x(y-x) &\le \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau \\ &\le \int_0^1 \tau p(\tau)\,d\tau\nabla_- f_y(y-x).\end{aligned} \tag{2.12}$$

Proof. The condition

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds \le \frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1)$$

is equivalent to

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds \le -\frac{1}{1-\tau}\int_0^\tau p(s)\,ds$$

namely

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds + \frac{1}{1-\tau}\int_0^\tau p(s)\,ds \le 0,$$

which is equivalent to (2.10).

By utilising (2.7) we derive the desired result (2.12). □

If $q:[0,1]\to\mathbb{R}$ is integrable, then the function $p(s) = q(s) - q(1-s)$ is asymmetric. By the condition (2.10) we have

$$\int_0^\tau [q(s) - q(1-s)]\,ds \le 0$$

namely

$$\int_0^\tau q(s)\,ds \le \int_0^\tau q(1-s)\,ds,\ \ \tau\in[0,1]. \tag{2.13}$$

If we put $u = 1 - s$, then

$$\int_0^\tau q(1-s)\,ds = \int_{1-\tau}^1 q(s)\,ds$$

and we obtain

$$\int_0^\tau q(s)\,ds \le \int_{1-\tau}^1 q(s)\,ds,\ \ \tau \in [0,1]. \tag{2.14}$$

We also have

$$\begin{aligned}\int_0^1 \tau p(\tau)\,d\tau &= \int_0^1 s\left[q(s) - q(1-s)\right] ds \\ &= \int_0^1 sq(s)\,ds - \int_0^1 (1-s)\,q(s)\,ds \\ &= \int_0^1 [2s-1]\,q(s)\,ds = 2\int_0^1 \left(s - \frac{1}{2}\right) q(s)\,ds\end{aligned}$$

and, for an integrable function $f : [0,1] \to \mathbb{R}$ we have

$$\begin{aligned}\int_0^1 p(s)\,f(s)\,ds &= \int_0^1 \left[q(s) - q(1-s)\right] f(s)\,ds \\ &= \int_0^1 q(s)\,f(s)\,ds - \int_0^1 q(1-s)\,f(s)\,ds \\ &= \int_0^1 q(s)\,f(s)\,ds - \int_0^1 q(s)\,f(1-s)\,ds \\ &= \int_0^1 q(s)\left[f(s) - f(1-s)\right] ds.\end{aligned}$$

We can state:

Corollary 3. *Let f be a convex function on C and $x,\ y \in C$, with $x \neq y$. If $q : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that (2.13) holds, then we have the inequalities*

$$\begin{aligned}&\int_0^1 \left(\tau - \frac{1}{2}\right) q(\tau)\,d\tau \nabla_+ f_x(y-x) \\ &\le \frac{1}{2}\int_0^1 q(\tau)\left[f((1-\tau)\,x + \tau y) - f(\tau x + (1-\tau)\,y)\right] d\tau \\ &\le \int_0^1 \left(\tau - \frac{1}{2}\right) q(\tau)\,d\tau \nabla_- f_y(y-x).\end{aligned} \tag{2.15}$$

3. Examples for Norms

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x, y\rangle_s := \nabla_+ f_{0,y}(x) = \lim_{t\to 0+} \frac{\|y+tx\|^2 - \|y\|^2}{2t}$;

(v) $\langle x, y\rangle_i := \nabla_- f_{0,y}(x) = \lim_{s\to 0-} \frac{\|y+sx\|^2 - \|y\|^2}{2s}$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2] or [6]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x, y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x, y\rangle_p\right| \leq \|x\| \|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x, x\rangle_p + \langle y, x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x, y\rangle_p = -\langle x, y\rangle_q$ for all x, $y \in X$;
(va) $\langle x + y, z\rangle_p \leq \|x\| \|z\| + \langle y, z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot, \cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X\setminus\{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \leq \langle y, x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

The function $f_r(x) = \|x\|^r$ ($x \in X$ and $1 \leq r < \infty$) is also convex. Therefore, the following limits, which are related to the superior (inferior) semi-inner products,

$$\begin{aligned}\nabla_{\pm} f_{r,y}(x) &:= \lim_{t\to 0\pm} \frac{\|y + tx\|^r - \|y\|^r}{t} \\ &= r\|y\|^{r-1} \lim_{t\to 0\pm} \frac{\|y + tx\| - \|y\|}{t} = r\|y\|^{r-2} \langle x, y\rangle_{s(i)}\end{aligned}$$

exist for all x, $y \in X$ whenever $r \geq 2$; otherwise, they exist for any $x \in X$ and nonzero $y \in X$. In particular, if $r = 1$, then the following limits

$$\nabla_{\pm} f_{1,y}(x) := \lim_{t\to 0\pm} \frac{\|y + tx\| - \|y\|}{t} = \frac{\langle x, y\rangle_{s(i)}}{\|y\|}$$

exist for x, $y \in X$ and $y \neq 0$.

If $p : [0, 1] \to \mathbb{R}$ is a Lebesgue integrable function such that

$$\frac{1}{\tau}\int_0^{\tau} p(s)\, ds \leq \frac{1}{1-\tau}\int_{\tau}^1 p(s)\, ds \text{ for all } \tau \in (0, 1),$$

then we have the inequalities

$$\begin{aligned}&r\left[\int_0^1 \tau p(\tau)\, d\tau - \frac{1}{2}\int_0^1 p(\tau)\, d\tau\right] \|x\|^{r-2} \langle y - x, x\rangle_s \\ &\leq \int_0^1 p(\tau) f((1-\tau)x + \tau y)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)x + \tau y)\, d\tau \\ &\leq r\left[\int_0^1 \tau p(\tau)\, d\tau - \frac{1}{2}\int_0^1 p(\tau)\, d\tau\right] \|y\|^{r-2} \langle y - x, y\rangle_s .\end{aligned} \tag{3.1}$$

If $r \geq 2$, then the inequality (3.1) holds for all x, $y \in X$. If $r \in [1, 2)$, then the inequality (3.1) holds for all x, $y \in X$ with x, $y \neq 0$.

For $r=2$ we get

$$(3.2)\qquad 2\left[\int_0^1 \tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\langle y-x,x\rangle_s$$
$$\leq \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau$$
$$\leq 2\left[\int_0^1 \tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\langle y-x,y\rangle_s$$

for all x, $y\in X$.

4. Examples for Functions of Several Variables

Now, let $\Omega\subset\mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. If $F:\Omega\to\mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c}\in\Omega$ we have

$$\nabla F_{\bar{c}}(\bar{y})=\sum_{i=1}^n\frac{\partial F(\bar{c})}{\partial x_i}\cdot y_i,\ \ \bar{y}=(y_1,...,y_n)\in\mathbb{R}^n,$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i $(i=1,...,n)$.

If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds\leq\frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1),$$

then we have the inequalities

$$(4.1)\qquad \left[\int_0^1 \tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\sum_{i=1}^n\frac{\partial F(\bar{a})}{\partial x_i}(b_i-a_i)$$
$$\leq \int_0^1 p(\tau) f\left((1-\tau)\bar{a}+\tau\bar{b}\right)d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f\left((1-\tau)\bar{a}+\tau\bar{b}\right)d\tau$$
$$\leq \left[\int_0^1 \tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\sum_{i=1}^n\frac{\partial F(\bar{b})}{\partial x_i}(b_i-a_i)$$

for all $\bar{a}$, $\bar{b}\in\Omega$.

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