# INEQUALITIES FOR BETA AND GAMMA FUNCTIONS VIA SOME CLASSICAL AND NEW INTEGRAL INEQUALITIES

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Abstract. In this survey paper we present the natural application of certain integral inequalities such as, Chebychev's inequality for synchronous and asynchronous mappings, Holder's inequality and Gruss' and Ostrowski's inequalities for the celebrated Euler's Beta and Gamma functions. Natural applications dealing with some adaptive quadrature formulae which can be deduced from Ostrowski's inequality are also pointed out.

## 1. Introduction

This survey paper is an attempt to present the natural application of certain integral inequalities such as, Chebychev's inequality for synchronous and asynchronous mappings, Hölder's inequality and Grüss' and Ostrowski's inequalities for the celebrated Euler's Beta and Gamma functions.

In the first section, following the well known book on special functions by Larry C. Andrews, we present some fundamental relations and identities for Gamma and Beta functions which will be used frequently in the sequel.

The second section is devoted to the applications of some classical integral inequalities for the particular cases of Beta and Gamma functions in their integral representations.

The first subsection of this is devoted to the applications of Chebychev's inequality for synchronous and asynchronous mappings for Beta and Gamma functions whilst the second subsection is concerned with some functional properties of these functions which can be easily derived by the use of Hölder's inequality. Applications of Grüss' integral inequality, which provides a more general approach than Chebychev's inequality, are considered in the last subsection.

The third and fourth sections are entirely based on some very recent results on Ostrowski type inequalities developed by Dragomir et al. in [10] - [16]. It is shown that Ostrowski's type inequalities can provide general quadrature formulae of the Riemann type for the Beta function. The remainders of the approximation are analyzed and upper bounded using different techniques developed for general classes of real mappings. Those sections can be also seen themselves as new and powerful tools in Numerical Analysis and the interested reader can use them for other applications besides those considered here.

For a different approach on Theory of Inequalities for Gamma and Beta Functions we recommend the papers [17] - [27].

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## 2. Gamma and Beta Functions

2.1. **Introduction.** In the eighteenth century, L. Euler (1707 - 1783) concerned himself with the problem of interpolating between the numbers

$$n! = \int_0^\infty e^{-t} t^n dt, \quad n = 0, 1, 2, .....$$

with non-integer values of $n$. This problem led Euler, in 1729, to the now famous *Gamma function,* a generalization of the factorial function that gives meaning to $x!$ where $x$ is any positive number.

The notation $\Gamma(x)$ is not due to Euler however, but was introduced in 1809 by A. Legendre (1752 - 1833), who was also responsible for the *Duplication Formula* for the Gamma function.

Nearly 150 years after Euler's discovery of it, the theory concerning the Gamma function was greatly expanded by means of the theory of entire functions developed by K. Weierstrass (1815 - 1897).

The Gamma function has several equivalent definitions, most of which are due to Euler. To begin with, we define [1, p. 51]

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)(x+2)...(x+n)}. \tag{2.1}$$

If $x$ is not zero or a negative integer, it can be shown that the limit (2.1) exists [2, p. 5]. It is apparent, however, that $\Gamma(x)$ cannot be defined at $x = 0, -1, -2, ...$ since the limit becomes infinite for any of these values.

By setting $x = 1$ in (2.1) we see that

$$\Gamma(1) = 1. \tag{2.2}$$

Other values of $\Gamma(x)$ are not so easily obtained, but the substitution of $x + 1$ for $x$ in (2.1) leads to the *Recurrence Formula* [1, p. 23]

$$\Gamma(x+1) = x\Gamma(x). \tag{2.3}$$

Equation (2.3) is the basic functional relation for the Gamma function; it is in the form of a difference equation.

A direct connection between the Gamma function and factorials can be obtained from (2.2) and (2.3)

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, ... \tag{2.4}$$

2.2. **Integral Representation.** The gamma function rarely appears in the form (2.2) in applications. Instead, it most often arises in the evaluation of certain integrals; for example, Euler was able to show that [1, p. 53]

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0. \tag{2.5}$$

This *integral representation* of $\Gamma(x)$ is the most common way in which the Gamma function is now defined. Lastly, we note that (2.5) is an improper integral, due to the infinite limit of integration and because the factor $t^{x-1}$ becomes infinite if

$t = 0$ for values of $x$ in the interval $0 < x < 1$. None the less, the integral (2.5) is *uniformly convergent* for all $a \le x \le b$, where $0 < a \le b < \infty$.

A consequence of the uniform convergence of the defining integral for $\Gamma(x)$ is that we may differentiate the function under the integral sign to obtain [1, p. 54]

$$\Gamma'(x) = \int_0^\infty e^{-t} t^{x-1} \log t \, dt, \quad x > 0 \tag{2.6}$$

and

$$\Gamma''(x) = \int_0^\infty e^{-t} t^{x-1} (\log t)^2 dt, \quad x > 0. \tag{2.7}$$

The integrand in (2.6) is positive over the entire interval of integration and thus it follows that $\Gamma''(x) > 0$, i.e., $\Gamma$ is convex on $(0, \infty)$.

In addition to (2.5), there are a variety of other integral representations of $\Gamma(x)$, most of which can be derived from that one by simple changes of variable [1, p. 57]

$$\Gamma(x) = \int_0^1 \left(\log \frac{1}{u}\right)^{x-1} du, \quad x > 0 \tag{2.8}$$

and

$$\frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)} = \int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \, \sin^{2y-1}\theta \, d\theta, \quad x, y > 0. \tag{2.9}$$

By setting $x = y = \frac{1}{2}$ in (2.9) we deduce the special value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{2.10}$$

2.3. **Other Special Formulae.** A formula involving Gamma functions that is somewhat comparable to the double-angle formulae for trigonometric functions is the *Legendre Duplication Formula* [1, p. 58]

$$2^{2x-1}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = \sqrt{\pi}\Gamma(2x), \quad x > 0. \tag{2.11}$$

An especially important case of (2.11) occurs when $x = n \quad (n = 0, 1, 2, ...)$ [1, p. 55]

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}, \quad n = 0, 1, 2, ... \quad . \tag{2.12}$$

Although it was originally found by Schlömlich in 1844, thirty-two years before Weierstrass' famous work on entire functions, Weierstrass is usually credited with the *infinite product* definition of the Gamma function

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{\frac{-x}{n}} \tag{2.13}$$

where $\gamma$ is the *Euler-Mascheroni constant* defined by

$$\gamma=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k}-\log n\right]=0.577215...... \tag{2.14}$$

An important identity involving the Gamma function and sine function can now be derived by using (2.13) [1, p. 60]. We obtain the identity,

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}\quad(x\text{ non-integer}). \tag{2.15}$$

The following properties of the Gamma function also hold (for example, see [1, p. 63 - p. 65]):

$$\Gamma(x)=s^{x}\int_{0}^{\infty}e^{-st}t^{x-1}dt,\quad x,s>0; \tag{2.16}$$

$$\Gamma(x)=\int_{-\infty}^{\infty}\exp\left(xt-e^{t}\right)dt,\quad x>0; \tag{2.17}$$

$$\Gamma(x)=\int_{1}^{\infty}e^{-t}t^{x-1}d+\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!\,(x+n)},\quad x>0; \tag{2.18}$$

$$\Gamma(x)=(\log b)^{x}\int_{0}^{\infty}t^{x-1}b^{-t}dt,\quad x>0,\ b>1; \tag{2.19}$$

$$\Gamma(x)=\Gamma'(x+1)-x\Gamma'(x),\quad x>0; \tag{2.20}$$

$$\Gamma(x)=\int_{0}^{\infty}e^{-t}(t-x)\,t^{x-1}\log t\ dt,\quad x>0; \tag{2.21}$$

$$\Gamma\left(\frac{1}{2}-n\right)=\frac{(-1)^{n}\,2^{2n-1}(n-1)!\sqrt{\pi}}{(2n-1)!},\quad n=0,1,2,....\ ; \tag{2.22}$$

$$\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}-n\right)=(-1)^{n}\pi,\quad n=0,1,2,...\ ; \tag{2.23}$$

$$\Gamma(3x)=\frac{1}{2\pi}3^{3x-\frac{1}{2}}\Gamma(x)\Gamma\left(x+\frac{1}{3}\right)\Gamma\left(x+\frac{2}{3}\right),\quad x>0; \tag{2.24}$$

$$\left[\Gamma'(x)\right]^{2}\le\Gamma(x)\Gamma''(x),\quad x>0. \tag{2.25}$$

2.4. **Beta Function.** A useful function of two variables is the *Beta function* [1, p. 66] where

$$\beta(x,y):=\int_{0}^{1}t^{x-1}(1-t)^{y-1}dt,\quad x>0,y>0. \tag{2.26}$$

The utility of the Beta function is often overshadowed by that of the Gamma function, partly perhaps because it can be evaluated in terms of the Gamma function. However, since it occurs so frequently in practice, a special designation for it is widely accepted.

It is obvious that the Beta mapping has the symmetry property

$$\beta(x,y)=\beta(y,x) \tag{2.27}$$

and the following connection between the Beta and Gamma functions holds

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad x>0,y>0. \tag{2.28}$$

The following properties of the Beta mapping also hold (see for example [1, p. 68 - p. 70])

$$\beta(x+1,y)+\beta(x,y+1)=\beta(x,y),\quad x,y>0; \tag{2.29}$$

$$\beta(x,y+1)=\frac{y}{x}\beta(x+1,y)=\frac{y}{x+y}\beta(x,y),\quad x,y>0; \tag{2.30}$$

$$\beta(x,x)=2^{1-2x}\beta\left(x,\frac{1}{2}\right),\quad x>0; \tag{2.31}$$

(2.32)

$$\beta(x,y)\beta(x+y,z)\beta(x+y+z,w)=\frac{\Gamma(x)\Gamma(y)\Gamma(z)\Gamma(w)}{\Gamma(x+y+z+w)},\quad x,y,z,w>0;$$

$$\beta\left(\frac{1+p}{2},\frac{1-p}{2}\right)=\pi\sec\left(\frac{p\pi}{2}\right),\quad 0<p<1; \tag{2.33}$$

$$\beta(x,y)=\frac{1}{2}\int_0^1\frac{t^{x-1}+t^{y-1}}{(t+1)^{x+y}}dt=p^x(1+p)^{x+y}\int_0^1\frac{t^{x-1}(1-t)^{y-1}}{(t+p)^{x+y}}dt \tag{2.34}$$

for $x,y,p>0$.

## 3. Inequalities for the Gamma and Beta Functions Via Some Classical Results

3.1. **Inequalities Via Chebychev's Inequality.** The following result is well known in the literature as Chebychev's integral inequality for synchronous (asynchronous) mappings.

**Lemma 1.** *Let $f,g,h:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be so that $h(x)\geq 0$ for $x\in I$ and $h$, $hfg$, $hf$ and $hg$ are integrable on $I$. If $f,g$ are synchronous (asynchronous) on $I$, i.e., we recall it*

$$(f(x)-f(y))(g(x)-g(y))\geq(\leq)\,0\ \ \text{for all } x,y\in I, \tag{3.1}$$

*then we have the inequality*

$$\int_I h(x)\,dx\int_I h(x)f(x)g(x)\,dx\ \geq(\leq)\int_I h(x)f(x)\,dx\int_I h(x)g(x)\,dx. \tag{3.2}$$

A simple proof of this result can be obtained using Korkine's identity [3]

$$\begin{aligned}\int_I h(x)\,dx\int_I h(x)f(x)g(x)\,dx\ -\int_I h(x)f(x)\,dx\int_I h(x)g(x)\,dx \\ =\frac{1}{2}\int_I\int_I h(x)h(y)(f(x)-f(y))(g(x)-g(y))\,dxdy.\end{aligned} \tag{3.3}$$

The following result holds (see also [4]).

**Theorem 1.** *Let* $m,n,p,q$ *be positive numbers with the property that*

$$(p-m)(q-n)\leq(\geq)\,0. \tag{3.4}$$

*Then*

$$\beta(p,q)\,\beta(m,n)\geq(\leq)\ \beta(p,n)\,\beta(m,q) \tag{3.5}$$

*and*

$$\Gamma(p+n)\,\Gamma(q+m)\geq(\leq)\ \Gamma(p+q)\,\Gamma(m+n). \tag{3.6}$$

*Proof.* Define the mappings $f,g,h:[0,1]\longrightarrow[0,\infty)$ given by

$$f(x)=x^{p-m},\quad g(x)=(1-x)^{q-n}\ \text{ and }\ h(x)=x^{m-1}(1-x)^{n-1}.$$

Then

$$f'(x)=(p-m)\,x^{p-m-1},\ g'(x)=(n-q)(1-x)^{q-n-1},\quad x\in(0,1).$$

As, by (3.3), $(p-m)(q-n)\leq(\geq)\,0$, then the mappings $f$ and $g$ are synchronous (asynchronous) having the same (opposite) monotonicity on $[0,1]$. Also, $h$ is non-negative on $[0,1]$.
Writing Chebychev's inequality for the above selection of $f$, $g$ and $h$ we get,

$$\begin{aligned}&\int_0^1 x^{m-1}(1-x)^{n-1}dx\int_0^1 x^{m-1}(1-x)^{n-1}x^{p-m}(1-x)^{q-n}dx\\&\geq(\leq)\int_0^1 x^{m-1}(1-x)^{n-1}x^{p-m}dx\int_0^1 x^{m-1}(1-x)^{n-1}(1-x)^{q-n}dx.\end{aligned}$$

That is,

$$\begin{aligned}&\int_0^1 x^{m-1}(1-x)^{n-1}dx\int_0^1 x^{p-1}(1-x)^{q-1}dx\\&\geq(\leq)\int_0^1 x^{p-1}(1-x)^{n-1}dx\int_0^1 x^{m-1}(1-x)^{q-1}dx,\end{aligned}$$

which, via (2.26), is equivalent to (3.5).
Now, using (3.5) and (2.28), we can state

$$\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\cdot\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\geq(\leq)\ \frac{\Gamma(p)\,\Gamma(n)}{\Gamma(p+n)}\cdot\frac{\Gamma(m)\,\Gamma(q)}{\Gamma(m+q)}$$

which is clearly equivalent to (3.6). ∎

The following corollary of Theorem 1 may be noted as well:

**Corollary 1.** *For any* $p,\ m>0$ *we have the inequalities*

$$\beta(m,p)\geq[\beta(p,p)\,\beta(m,m)]^{\frac{1}{2}} \tag{3.7}$$

*and*

$$\Gamma(p+m)\geq[\Gamma(2p)\,\Gamma(2m)]^{\frac{1}{2}}. \tag{3.8}$$

*Proof.* In Theorem 1 set $q = p$ and $n = m$. Then

$$(p-m)(q-n) = (p-m)^2 \geq 0$$

and thus

$$\beta(p,p)\,\beta(m,m) \leq \beta(p,m)\,\beta(m,p) = \beta^2(p,m)$$

and the inequality (3.7) is proved.
The inequality (3.8) follows by (3.7). ∎

The following result employing Chebychev's inequality on an infinite interval holds [4].

**Theorem 2.** *Let $m$, $p$ and $k$ be real numbers with $m, p > 0$ and $p > k > -m$. If*

$$k(p-m-k) \geq (\leq)\, 0, \tag{3.9}$$

*then we have*

$$\Gamma(p)\,\Gamma(m) \geq (\leq)\, \Gamma(p-k)\,\Gamma(m+k) \tag{3.10}$$

*and*

$$\beta(p,m) \geq (\leq)\ \beta(p-k, m+k) \tag{3.11}$$

*respectively.*

*Proof.* Consider the mappings $f, g, h : [0,\infty) \longrightarrow [0,\infty)$ given by

$$f(x) = x^{p-k-m},\ g(x) = x^k,\ h(x) = x^{m-1}e^{-x}.$$

If the condition (3.9) holds, then we can assert that the mappings $f$ and $g$ are synchronous (asynchronous) on $(0,\infty)$ and then, by Chebychev's inequality for $I = [0,\infty)$, we can state

$$\begin{aligned}&\int_0^\infty x^{m-1}e^{-x}dx \int_0^\infty x^{p-k-m}x^k x^{m-1}e^{-x}dx \\ &\geq (\leq) \int_0^\infty x^{p-k-m}x^{m-1}e^{-x}dx \int_0^\infty x^k x^{m-1}e^{-x}dx\end{aligned}$$

i.e.,

$$\int_0^\infty x^{m-1}e^{-x}dx \int_0^\infty x^{p-1}e^{-x}dx \geq (\leq) \int_0^\infty x^{p-k-1}e^{-x}dx \int_0^\infty x^{k+m-1}e^{-x}dx. \tag{3.12}$$

Using the integral representation (2.5), (3.12) provides the desired result (3.10). On the other hand, since

$$\beta(p,m) = \frac{\Gamma(p)\,\Gamma(m)}{\Gamma(p+m)}$$

and

$$\beta(p-k, m+k) = \frac{\Gamma(p-k)\,\Gamma(m+k)}{\Gamma(p+m)}$$

we can easily deduce that (3.11) follows from (3.10). ∎

The following corollary is interesting.

**Corollary 2.** *Let $p > 0$ and $q \in \mathbb{R}$ such that $|q| < p$. Then*

$$\Gamma^2(p) \leq \Gamma(p-q)\Gamma(p+q) \tag{3.13}$$

*and*

$$\beta(p,p) \leq \beta(p-q, p+q). \tag{3.14}$$

*Proof.* Choose in Theorem 2, $m = p$ and $k = q$. Then

$$k(p-m-k) = -q^2 \leq 0$$

and by (3.10) we get

$$\Gamma^2(p) \leq \Gamma(p-q)\Gamma(p+q).$$

The second inequality follows by the relation (2.28). ∎

Let us now consider the following definition [4].

**Definition 1.** *The positive real numbers $a$ and $b$ may be called similarly (oppositely) unitary if*

$$(a-1)(b-1) \geq (\leq) 0. \tag{3.15}$$

**Theorem 3.** *Let $a, b > 0$ and be similarly (oppositely) unitary.*
*Then*

$$\Gamma(a+b) \geq (\leq)\ ab\Gamma(a)\Gamma(b) \tag{3.16}$$

*and*

$$\beta(a,b) \geq (\leq) \frac{1}{ab} \tag{3.17}$$

*respectively.*

*Proof.* Consider the mappings $f, g, h : [0,\infty) \longrightarrow [0,\infty)$ given by

$$f(t) = t^{a-1}, g(t) = t^{b-1} \text{ and } h(t) = te^{-t}.$$

If the condition (3.15) holds, then obviously the mappings $f$ and $g$ are synchronous (asynchronous) on $[0,\infty)$, and by Chebychev's integral inequality we can state that

$$\int_0^\infty te^{-t}dt \int_0^\infty t^{a+b-1}e^{-t}dt \geq (\leq) \int_0^\infty t^a e^{-t}dt \int_0^\infty t^b e^{-t}dt$$

provided $(a-1)(b-1) \geq (\leq) 0$; i.e.,

$$\Gamma(2)\Gamma(a+b) \geq (\leq) \Gamma(a+1)\Gamma(b+1). \tag{3.18}$$

Using the recursive relation (2.3), we have $\Gamma(a+1) = a\Gamma(a)$, $\Gamma(b+1) = b\Gamma(b)$ and $\Gamma(2) = 1$ and thus (3.18) becomes (3.16).
The inequality (3.17) follows by (3.16) via (2.28). ∎

The following corollaries may be noted as well:

**Corollary 3.** *The mapping $\ln\Gamma(x)$ is superadditive for $x > 1$.*

*Proof.* If $a, b \in [1,\infty)$, then, by (3.16),

$$\ln\Gamma(a+b) \geq \ln a + \ln b + \ln\Gamma(a) + \ln\Gamma(b) \geq \ln\Gamma(a) + \ln\Gamma(b)$$

which is the superadditivity of the desired mapping. ∎

**Corollary 4.** *For every $n \in \mathbb{N}$, $n \geq 1$ and $a > 0$, we have the inequality*

$$\Gamma(na) \geq (n-1)! a^{2(n-1)} [\Gamma(a)]^n . \tag{3.19}$$

*Proof.* Using the inequality (3.16) successively, we can state that

$$\Gamma(2a) \geq a^2 \Gamma(a) \Gamma(a)$$

$$\Gamma(3a) \geq 2a^2 \Gamma(2a) \Gamma(a)$$

$$\Gamma(4a) \geq 3a^2 \ \Gamma(3a) \ \Gamma(a)$$

................................

$$\Gamma(na) \geq (n-1) a^2 \ \Gamma[(n-1) a] \ \Gamma(a) .$$

By multiplying these inequalities, we arrive at (3.19) . ∎

**Corollary 5.** *For any $a > 0$, we have*

$$\Gamma(a) \leq \frac{2^{2a-1}}{\sqrt{\pi} a^2} \ \Gamma\left(a + \frac{1}{2}\right) . \tag{3.20}$$

*Proof.* We refer to the identity (2.10) from which we can write

$$2^{2a-1} \Gamma(a) \ \Gamma\left(a + \frac{1}{2}\right) = \sqrt{\pi} \ \Gamma(2a) , \quad a > 0.$$

Since $\Gamma(2a) \geq a^2 \Gamma^2(a)$, we arrive at

$$2^{2a-1} \Gamma(a) \ \Gamma\left(a + \frac{1}{2}\right) \geq \sqrt{\pi} \ a^2 \Gamma^2(a)$$

which is the desired inequality (3.20) . ∎

For a given $m > 0$, consider the mapping $\Gamma_m : [0, \infty) \longrightarrow \mathbb{R}$,

$$\Gamma_m(x) = \frac{\Gamma(x+m)}{\Gamma(m)} .$$

The following result holds.

**Theorem 4.** *The mapping $\Gamma_m(\cdot)$ is supermultiplicative on $[0, \infty)$.*

*Proof.* Consider the mappings $f(t) = t^x$ and $g(t) = t^y$ which are monotonic nondecreasing on $[0, \infty)$ and $h(t) := t^{m-1} e^{-t}$ is non-negative on $[0, \infty)$. Applying Chebychev's inequality for the synchronous mappings $f, g$ and the weight function $h$, we can write

$$\int_0^\infty t^{m-1} e^{-t} dt \int_0^\infty t^{x+y+m-1} e^{-t} dt \geq \int_0^\infty t^{x+m-1} e^{-t} dt \int_0^\infty t^{y+m-1} e^{-t} dt.$$

That is,

$$\Gamma(m) \Gamma(x+y+m) \geq \Gamma(x+m) \Gamma(y+m)$$

which is equivalent to

$$\Gamma_m(x+y) \geq \Gamma_m(x) \Gamma_m(y)$$

and the theorem is proved. ∎

3.2. **Inequalities Via Hölder's Inequality.** Let $I \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$ and assume that $f \in L_p(I)$, $g \in L_q(I)$ $\left(p > 1, \frac{1}{p} + \frac{1}{q} = 1\right)$, i.e.,

$$\int_I |f(s)|^p \, ds, \quad \int_I |g(s)|^q \, ds < \infty.$$

Then $fg \in L_1(I)$ and the following inequality due to Hölder holds

$$\left| \int_I f(s) g(s) \, ds \right| \leq \left( \int_I |f(s)|^p \, ds \right)^{\frac{1}{p}} \left( \int_I |g(s)|^q \, ds \right)^{\frac{1}{q}}. \tag{3.21}$$

For a proof of this classic fact using a Young type inequality

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} x^q, \quad x, y \geq 0, \quad \frac{1}{p} + \frac{1}{q} = 1; \tag{3.22}$$

as well as some related results, see the book [3].

Using Hölder's inequality we point out some functional properties of the mappings Gamma, Beta and Digamma [5].

**Theorem 5.** *Let $a, b \geq 0$ with $a + b = 1$ and $x, y > 0$. Then*

$$\Gamma(ax + by) \leq [\Gamma(x)]^a \; [\Gamma(y)]^b, \tag{3.23}$$

*i.e., the mapping $\Gamma$ is logarithmically convex on $(0, \infty)$.*

*Proof.* We use the following weighted version of Hölder's inequality

$$\left| \int_I f(s) g(s) h(s) \, ds \right| \leq \left( \int_I |f(s)|^p h(s) \, ds \right)^{\frac{1}{p}} \left( \int_I |g(s)|^q h(s) \, ds \right)^{\frac{1}{q}} \tag{3.24}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $h$ is non-negative on $I$ and provided all the other integrals exist and are finite.
Choose

$$f(s) = s^{a(x-1)}, \quad g(s) = s^{b(y-1)} \quad \text{and} \quad h(s) = e^{-s}, \quad s \in (0, \infty)$$

in (3.24) to get (for $I = (0, \infty)$ and $p = \frac{1}{a}$, $q = \frac{1}{b}$)

$$\int_0^\infty s^{a(x-1)} \cdot s^{b(y-1)} e^{-s} ds \leq \left( \int_0^\infty s^{a(y-1) \cdot \frac{1}{a}} e^{-s} ds \right)^a \left( \int_0^\infty s^{b(y-1) \cdot \frac{1}{b}} e^{-s} ds \right)^b$$

which is clearly equivalent to

$$\int_0^\infty s^{ax+by-1} e^{-s} ds \leq \left( \int_0^\infty s^{y-1} e^{-s} ds \right)^a \left( \int_0^\infty s^{y-1} e^{-s} ds \right)^b$$

and the inequality (3.23) is proved. ∎

**Remark 1.** *Consider the mapping $g(x) := \ln \Gamma(x)$, $x \in (0, \infty)$.*
*We have*

$$g'(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{and} \quad g''(x) = \frac{\Gamma''(x) \Gamma(x) - [\Gamma(x)]^2}{\Gamma^2(x)}$$

*for $x \in (0, \infty)$. Using the inequality (2.25) we conclude that $g''(x) \geq 0$ for all $x \in (0, \infty)$ which shows that $\Gamma$ is logarithmically convex on $(0, \infty)$.*

We prove now a similar result for the Beta function [5].

**Theorem 6.** *The mapping $\beta$ is logarithmically convex on $(0,\infty)^2$ as a function of two variables.*

*Proof.* Let $(p,q)$, $(m,n) \in (0,\infty)^2$ and $a,b \geq 0$ with $a+b=1$. We have

$$\beta\left[a(p,q)+b(m,n)\right] = \beta(ap+bm, aq+bn)$$

$$= \int_0^1 t^{ap+bm-1}(1-t)^{aq+bn-1}\,dt$$

$$= \int_0^1 t^{a(p-1)+b(m-1)}(1-t)^{a(q-1)+b(n-1)}\,dt$$

$$= \int_0^1 \left[t^{p-1}(1-t)^{q-1}\right]^a \times \left[t^{m-1}(1-t)^{n-1}\right]^b dt.$$

Define the mappings

$$f(t) = \left[t^{p-1}(1-t)^{q-1}\right]^a, \quad t \in (0,1)$$

$$g(t) = \left[t^{m-1}(1-t)^{n-1}\right]^b, \quad t \in (0,1)$$

and choose $p = \frac{1}{a}$, $q = \frac{1}{b}$ $\left(\frac{1}{p}+\frac{1}{q} = a+b = 1, \quad p \geq 1\right)$.

Applying Hölder's inequality for these selections, we get:

$$\int_0^1 \left[t^{p-1}(1-t)^{q-1}\right]^a \left[t^{p-1}(1-t)^{q-1}\right]^b dt$$

$$\leq \left[\int_0^1 t^{p-1}(1-t)^{q-1}\,dt\right]^a \times \left[\int_0^1 t^{m-1}(1-t)^{n-1}\,dt\right]^b$$

i.e.,

$$\beta\left[a(p,q)+b(m,n)\right] \leq \left[\beta(p,q)\right]^a \left[\beta(m,n)\right]^b$$

which is the logarithmic convexity of $\beta$ on $(0,\infty)^2$. ∎

Closely associated with the derivative of the Gamma function is the *logarithmic-derivative function*, or *Digamma function* defined by [1, p. 74]

$$\Psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x \neq 0,-1,-2,\ldots \quad .$$

The function $\Psi(x)$ is also commonly called the *Psi function.*

**Theorem 7.** *The Digamma function is monotonic nondecreasing and concave on* $(0,\infty)$.

*Proof.* As $\Gamma$ is logarithmically convex on $(0,\infty)$, then the derivative of $\ln\Gamma$, which is the Digamma function, is monotonic nondecreasing on $(0,\infty)$.

To prove the concavity of $\Psi$, we use the following known representation of $\Psi$ [6, p. 21].

$$\Psi(x) = \int_0^1 \frac{1-t^{x-1}}{1-t}dt - \gamma, \quad x>0 \tag{3.25}$$

where $\gamma$ is the Euler-Mascheroni constant (see (2.14)).

Now, let $x, y > 0$ and $a, b \geq 0$ with $a + b = 1$. Then

$$\Psi(ax+by)+\gamma = \int_0^1 \frac{1-t^{ax+by-1}}{1-t}dt = \int_0^1 \frac{1-t^{a(x-1)+b(y-1)}}{1-t}dt. \tag{3.26}$$

As the mapping $\mathbb{R} \ni x \longmapsto a^x \in (0,\infty)$ is convex for $a \in (0,1)$, we can state that

$$t^{a(x-1)+b(y-1)} \leq at^{x-1} + bt^{y-1} \tag{3.27}$$

for all $t \in (0,1)$ and $x, y > 0$.

Using (3.27) we can obtain, by integrating over $t \in (0,1)$,

$$\int_0^1 \frac{1-t^{ax+by-1}}{1-t}dt \geq \int_0^1 \frac{1-\left(at^{x-1}+bt^{y-1}\right)}{1-t}dt$$

$$= \int_0^1 \frac{a\left(1-t^{x-1}\right)+b\left(1-t^{y-1}\right)}{1-t}dt = a\int_0^1 \frac{1-t^{x-1}}{1-t}dt + b\int_0^1 \frac{1-t^{y-1}}{1-t}dt$$

$$= a\left[\Psi(x)+\gamma\right] + b\left[\Psi(y)+\gamma\right] = a\Psi(x)+b\Psi(y)+\gamma. \tag{3.28}$$

Now, by (3.26) and (3.28) we deduce

$$\Psi(ax+by) \geq a\Psi(x)+b\Psi(y), \quad x,y>0, \quad a,b\geq 0, \quad a+b=1;$$

i.e., the concavity of $\Psi$. ∎

### 3.3. **Inequalities Via Grüss' Inequality.**

In 1935, G. Grüss established an integral inequality which gives an estimation for the integral of a product in terms of the product of integrals [3, p. 296].

**Lemma 2.** *Let $f$ and $g$ be two functions defined and integrable on $[a,b]$. If*

$$\varphi \leq f(x) \leq \Phi, \ \gamma \leq g(x) \leq \Gamma \text{ for each } x \in [a,b]; \tag{3.29}$$

*where $\varphi, \Phi, \gamma$ and $\Gamma$ are given real constants, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \tag{3.30}$$

*and the constant $\frac{1}{4}$ is the best possible.*

The following application of Grüss' inequality for the Beta mapping holds [7].

**Theorem 8.** *Let $m, n, p$ and $q$ be positive numbers. Then*

$$\left|\beta(m+p+1,n+q+1) - \beta(m+1,n+1)\cdot\beta(p+1,q+1)\right| \leq \frac{1}{4}\frac{p^p q^q}{(p+q)^{p+q}} \cdot \frac{m^m n^n}{(m+n)^{m+n}}. \tag{3.31}$$

*Proof.* Consider the mappings

$$l_{m,n}(x) := x^m(1-x)^n, l_{p,q}(x) := x^p(1-x)^q, \ x \in [0,1].$$

In order to apply Grüss' inequality, we need to find the minima and the maxima of $l_{a,b}$ $(a,b>0)$.

We have

$$\frac{d}{dx}l_{a,b}(x) = ax^{a-1}(1-x)^{b} - bx^{a}(1-x)^{b-1}$$
$$= x^{a-1}(1-x)^{b-1}[a(1-x)-bx]$$
$$= x^{a-1}(1-x)^{b-1}[a-(a+b)x].$$

We observe that the unique solution of $l'_{a,b}(x) = 0$ in $(0,1)$ is $x_0 = \frac{a}{a+b}$ and as $l'_{a,b}(x) > 0$ on $(0,x_0)$ and $l_{a,b}(x) < 0$ on $(x_0,1)$, we conclude that $x_0$ is a point of maximum for $l_{a,b}$ in $(0,1)$. Consequently

$$m_{a,b} := \inf_{x\in[0,1]} l_{a,b}(x) = 0$$

and

$$M_{a,b} := \sup_{x\in[0,1]} l_{a,b}(x) := l_{a,b}\left(\frac{a}{a+b}\right) = \frac{a^{a}b^{b}}{(a+b)^{a+b}}.$$

Now, if we apply Grüss' inequality for the mappings $l_{m,n}$ and $l_{p,q}$, we get

$$\left|\int_0^1 l_{m,n}(x)\cdot l_{p,q}(x)\,dx - \int_0^1 l_{m,,n}(x)\,dx\cdot\int_0^1 l_{p,q}(x)\,dx\right|$$
$$\leq \frac{1}{4}(M_{m,n}-m_{m,n})(M_{p,q}-m_{p,q})$$

which is equivalent to

$$\left|\int_0^1 l_{m+p,n+q}(x)\,dx - \int_0^1 l_{m,,n}(x)\,dx\cdot\int_0^1 l_{p,q}(x)\,dx\right|$$
$$\leq \frac{1}{4}\frac{m^{m}n^{n}}{(m+n)^{m+n}}\cdot\frac{p^{p}q^{q}}{(p+q)^{p+q}}$$

and the inequality (3.31) is obtained. ∎

Another simpler inequality that we can derive via Grüss' inequality is the following.

**Theorem 9.** *Let $p,q>0$. Then we have the inequality*

$$\left|\beta(p+1,q+1) - \frac{1}{(p+1)(q+1)}\right| \leq \frac{1}{4} \tag{3.32}$$

*or, equivalently,*

$$\max\left\{0, \frac{3-pq-p-q}{4(p+1)(q+1)}\right\} \leq \beta(p+1,q+1) \leq \frac{5+pq+p+q}{4(p+1)(q+1)}. \tag{3.33}$$

*Proof.* Consider the mappings

$$f(x) = x^{p},\quad g(x) = (1-x)^{q},\quad x\in[0,1],\quad p,q>0.$$

Then, obviously

$$\inf_{x\in[0,1]} f(x) = \inf_{x\in[0,1]} g(x) = 0;$$

$$\sup_{x\in[0,1]} f(x) = \sup_{x\in[0,1]} g(x) = 1;$$

and

$$\int_0^1 f(x)\,dx = \frac{1}{p+1}, \quad \int_0^1 g(x)\,dx = \frac{1}{q+1}.$$

Using Grüss' inequality we get (3.32). Algebraic computations will show that (3.32) is equivalent to (3.33). ∎

**Remark 2.** *Taking into account that* $\beta(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}$, *the inequality* (3.32) *is equivalent to*

$$\left|\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} - \frac{1}{(p+1)(q+1)}\right| \leq \frac{1}{4},$$

*i.e.,*

$$\begin{aligned} &|(p+1)\Gamma(p+1)\cdot(q+1)\Gamma(q+1) - \Gamma(p+q+2)| \\ &\leq \frac{1}{4}(p+1)(q+1)\Gamma(p+q+2) \end{aligned}$$

*and as* $(p+1)\Gamma(p+1) = \Gamma(p+2)$, $(q+1)\Gamma(q+1) = \Gamma(q+2)$, *we get*

$$|\Gamma(p+q+2) - \Gamma(p+2)\Gamma(q+2)| \leq \frac{1}{4}(p+1)(q+1)\Gamma(p+q+2). \tag{3.34}$$

Grüss' inequality has a weighted version as follows.

**Lemma 3.** *Let* $f, g$ *be as in Lemma 2 and* $h : [a,b] \longrightarrow [0,\infty)$ *such that* $\int_a^b h(x)\,dx > 0$. *Then*

$$\begin{aligned} &\left|\frac{1}{\int_a^b h(x)\,dx}\int_a^b f(x)g(x)h(x)\,dx\right. \\ &\left.- \frac{1}{\int_a^b h(x)\,dx}\int_a^b f(x)h(x)\,dx \cdot \frac{1}{\int_a^b h(x)\,dx}\int_a^b g(x)h(x)\,dx\right| \\ \leq\ &\frac{1}{4}(\Gamma-\gamma)(\Phi-\varphi) \end{aligned} \tag{3.35}$$

*The constant* $\frac{1}{4}$ *is best.*

For a proof of this fact which is similar to the classical one, see the recent paper [8].

Using Lemma 3, we can state the following proposition generalizing Theorem 8.

**Proposition 1.** *Let* $m, n, p, q > 0$ *and* $r, s > -1$. *Then we have*

$$\begin{aligned} &|\beta(r+1,s+1)\beta(m+p+r+1,n+q+s+1) \\ &-\beta(m+r+1,n+s+1)\beta(p+r+1,q+s+1)| \\ &\leq \frac{1}{4}\frac{m^m n^n}{(m+n)^{m+n}}\cdot\frac{p^p q^q}{(p+q)^{p+q}}\beta^2(r+1,s+1). \end{aligned} \tag{3.36}$$

The proof follows by the inequality (3.35) by choosing

$$h(x) = l_{r,s}(x), \;\; f(x) = l_{m,n}(x) \quad \text{and} \quad g(x) = l_{p,q}(x), \quad x \in (0,1).$$

Now, applying the same inequality, but for the mappings

$$h(x) = l_{r,s}(x), \;\; f(x) = x^p \quad \text{and} \quad g(x) = (1-x)^q, \quad x \in (0,1),$$

we deduce the following proposition generalizing Theorem 9.

**Proposition 2.** *Let* $p, q > 0$ *and* $r, s > -1$. *Then*

(3.37)

$$\begin{aligned} &|\beta(r+1,s+1)\,\beta(p+r+1,q+s+1) - \beta(p+r+1,s+1)\,\beta(r+1,q+s+1)| \\ &\leq \frac{1}{4}\beta^2(r+1,s+1). \end{aligned}$$

The weighted version of Grüss' inequality allows us to obtain inequalities directly for the Gamma mapping.

**Theorem 10.** *Let* $\alpha, \beta, \gamma > 0$. *Then*

(3.38)

$$\left|\frac{1}{3^{\alpha+\beta+\gamma+1}}\Gamma(\alpha+\beta+\gamma+1)\,\Gamma(\gamma+1) - \frac{1}{2^{\alpha+\beta+2\gamma+2}}\Gamma(\alpha+\gamma+1)\,\Gamma(\beta+\gamma+1)\right|$$

$$\leq \frac{1}{4}\cdot\frac{\alpha^\alpha}{e^\alpha}\cdot\frac{\beta^\beta}{e^\beta}\Gamma^2(\gamma+1).$$

*Proof.* Consider the mapping $f_\alpha(t) = t^\alpha e^{-t}$ defined on $(0,\infty)$. Then

$$f_\alpha'(t) = \alpha t^{\alpha-1}e^{-t} - t^\alpha e^{-t} = e^{-t}t^{\alpha-1}(\alpha - t)$$

which shows that $f_\alpha$ is increasing on $(0,\alpha)$ and decreasing on $(0,\infty)$ and the maximum value is $f_\alpha(\alpha) = \frac{\alpha^\alpha}{e^\alpha}$.

Using (3.35), we can state that

$$\begin{aligned} &\left|\int_0^x f_\alpha(t)\,f_\beta(t)\,f_\gamma(t)\,dt \cdot \int_0^x f_\gamma(t)\,dt - \int_0^x f_\alpha(t)\,f_\gamma(t)\,dt \cdot \int_0^x f_\beta(t)\,f_\gamma(t)\,dt\right| \\ &\leq \frac{1}{4}\left(\max_{t\in[0,x]} f_\alpha(t) - \min_{t\in[0,x]} f_\alpha(t)\right)\left(\max_{t\in[0,x]} f_\beta(t) - \min_{t\in[0,x]} f_\beta(t)\right)\left(\int_0^x f_\gamma(t)\,dt\right)^2 \end{aligned}$$

for all $x > 0$, which is equivalent to

$$\begin{aligned} &\left|\int_0^x t^{\alpha+\beta+\gamma}e^{-3t}dt \cdot \int_0^x e^\gamma e^{-t}dt - \int_0^x t^{\alpha+\gamma}e^{-2t}dt \cdot \int_0^x t^{\beta+\gamma}e^{-2t}dt\right| \\ &\leq \frac{1}{4}\;\frac{\alpha^\alpha}{e^\alpha}\cdot\frac{\beta^\beta}{e^\beta}\left(\int_0^x e^\gamma e^{-t}dt\right)^2 \end{aligned}$$

for all $x > 0$.

As the involved integrals are convergent on $[0,\infty)$, we get

$$\begin{aligned} &\left|\int_0^\infty t^{\alpha+\beta+\gamma}e^{-3t}dt \cdot \int_0^\infty e^\gamma e^{-t}dt - \int_0^\infty t^{\alpha+\gamma}e^{-2t}dt \cdot \int_0^\infty t^{\beta+\gamma}e^{-2t}dt\right| \\ &\leq \frac{1}{4}\;\frac{\alpha^\alpha}{e^\alpha}\cdot\frac{\beta^\beta}{e^\beta}\left(\int_0^\infty e^\gamma e^{-t}dt\right)^2. \end{aligned} \tag{3.39}$$

Now, using the change of variable $u = 3t$, we get

$$\int_0^\infty t^{\alpha+\beta+\gamma} e^{-3t} dt = \frac{1}{3} \int_0^\infty \left(\frac{u}{3}\right)^{\alpha+\beta+\gamma} e^{-u} du = \frac{1}{3^{\alpha+\beta+\gamma+1}} \Gamma(\alpha+\beta+\gamma+1)$$

and, similarly,

$$\int_0^\infty t^{\alpha+\gamma} e^{-2t} dt = \frac{1}{2^{\alpha+\gamma+1}} \Gamma(\alpha+\gamma+1)$$

and

$$\int_0^\infty t^{\beta+\gamma} e^{-2t} dt = \frac{1}{2^{\beta+\gamma+1}} \Gamma(\beta+\gamma+1)$$

and then, by $(3.39)$, we deduce the desired inequality $(3.38)$. ∎

## 4. Inequalities for the Gamma and Beta Functions Via Some New Results

### 4.1. Inequalities Via Ostrowski's Inequality for Lipschitzian Mappings.

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality (see for example, [9, p. 469]).

**Theorem 11.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$, whose derivative is bounded on $(a,b)$ and let $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \tag{4.1}$$

*for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.*

The following generalization of (4.1) has been done in [10].

**Theorem 12.** *Let $u : [a,b] \longrightarrow \mathbb{R}$ be a L-lipschitzian mapping on $[a,b]$, i.e.,*

$$|u(x) - u(y)| \le L |x-y| \quad \text{for all } x, y \in [a,b].$$

*Then we have the inequality*

$$\left| \int_a^b u(t)\, dt - u(x)(b-a) \right| \le L(b-a)^2 \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \tag{4.2}$$

*for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.*

*Proof.* Using the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_a^x (t-a)\, du(t) = u(x)(x-a) - \int_a^x u(t)\, dt$$

and

$$\int_x^b (t-b)\, du(t) = u(x)(b-x) - \int_x^b u(t)\, dt.$$

If we add the above two equalities, we get

$$u(x)(b-a)-\int_a^b u(t)\,dt=\int_a^x (t-a)\,du(t)+\int_x^b (t-b)\,du(t). \tag{4.3}$$

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < .... < x_{n-1}^{(n)} < x_n^{(n)} = d$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \longrightarrow \infty$, where $\nu(\Delta_n) := \max_{i\in\{0,\dots,n-1\}}\left(x_{i+1}^{(n)} - x_i^{(n)}\right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$. If $p : [c,d] \longrightarrow \mathbb{R}$ is Riemann integrable on $[c,d]$ and $v : [c,d] \longrightarrow \mathbb{R}$ is L-Lipschitzian on $[a,b]$, then

(4.4)

$$\begin{aligned}\left|\int_c^d p(x)\,dv(x)\right| &= \left|\lim_{\nu(\Delta n)\longrightarrow 0}\sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right]\right| \\ &\le \lim_{\nu(\Delta n)\longrightarrow 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)}-x_i^{(n)}\right)\left|\frac{v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)}{x_{i+1}^{(n)}-x_i^{(n)}}\right| \\ &\le L\lim_{\nu(\Delta n)\longrightarrow 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)}-x_i^{(n)}\right) \\ &= L\int_c^d |p(x)|\,dx.\end{aligned}$$

Applying the inequality (4.4) on $[a,x]$ and $[x,b]$ successively, we get

(4.5)

$$\begin{aligned}\left|\int_a^x (t-a)\,du(t)+\int_x^b (t-b)\,du(t)\right| &\le \left|\int_a^x (t-a)\,du(t)\right|+\left|\int_x^b (t-b)\,du(t)\right| \\ &\le L\left[\int_a^x |t-a|\,dt+\int_x^b |t-b|\,dt\right] \\ &= \frac{L}{2}\left[(x-a)^2+(b-x)^2\right] \\ &= \frac{L}{2}(b-a)^2\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right],\end{aligned}$$

and then, by (4.5), via the identity (4.3), we get the desired inequality (4.2). To prove the sharpness of the constant $\frac{1}{4}$, assume that the inequality (4.2) holds with a constant $C > 0$, i.e.,

$$\left|\int_a^b u(t)\,dt-u(x)(b-a)\right| \le L(b-a)^2\left[C+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{4.6}$$

for all $x \in [a,b]$.
Consider the mapping $f : [a,b] \longrightarrow \mathbb{R}$, $f(x) = x$ in (4.6). Then

$$\left|x-\frac{a+b}{2}\right| \le \left[C+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)$$

for all $x \in [a,b]$, and then for $x = a$, we get

$$\frac{b-a}{2} \leq \left(C + \frac{1}{4}\right)(b-a)$$

which implies that $C \geq \frac{1}{4}$, and the theorem is completely proved. ∎

The best inequality we can get from (4.2) is the following one.

**Corollary 6.** *Let $u : [a,b] \longrightarrow \mathbb{R}$ be as above. Then we have the inequality:*

$$\left|\int_a^b u(t)\,dt - u\left(\frac{a+b}{2}\right)(b-a)\right| \leq \frac{1}{4}L(b-a)^2. \tag{4.7}$$

The previous results are useful in the estimation of the remainder for a general quadrature formula of the Riemann type for L-lipschitzian mappings as follows: Let $I_n : a = x_0 < x_1 < .... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,1,...,n-1)$ a sequence of intermediate points for $I_n$. Construct the Riemann sums

$$R_n(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i) h_i$$

where $h_i := x_{i+1} - x_i$ $(i = 0,1,...,n-1)$.
We now have the following quadrature formula.

**Theorem 13.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be an L-lipschitzian mapping on $[a,b]$ and $I_n$, $\xi_i$, $(i = 0,1,...,n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)\,dx = R_n(f, I_n, \xi) + W_n(f, I_n, \xi) \tag{4.8}$$

*where the remainder satisfies the estimate*

$$\begin{aligned} |W_n(f, I_n, \xi)| &\leq L\left[\frac{1}{4}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right] \\ &\leq \frac{1}{2}L\sum_{i=0}^{n-1} h_i^2 \end{aligned} \tag{4.9}$$

*for all $\xi_i$ $(i = 0,1,...,n-1)$ as above.*
*The constant $\frac{1}{4}$ is sharp.*

*Proof.* Apply Theorem 12 on the interval $[x_i, x_{i+1}]$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i)h_i\right| \leq L\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right].$$

Summing over $i$ from 0 to $n-1$ and using the generalized triangle inequality, we get

$$\begin{aligned} |W_n(f, I_n, \xi)| &\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i)h_i\right| \\ &\leq L\sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]. \end{aligned}$$

Now, as

$$\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \leq \frac{1}{4}h_i^2$$

for all $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$, the second part of (4.9) is also proved. ∎

Note that, the best estimation we can obtain from (4.9) is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$, obtaining the following midpoint formula.

**Corollary 7.** *Let $f, I_n$ be as above. Then we have the midpoint rule.*

$$\int_a^b f(x)\,dx = M_n(f, I_n) + S_n(f, I_n)$$

*where*

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

*and the remainder $S_n(f, I_n)$ satisfies the estimation.*

$$|S_n(f, I_n)| \leq \frac{1}{4} L \sum_{i=0}^{n-1} h_i^2.$$

**Remark 3.** *If we assume that $f : [a,b] \longrightarrow \mathbb{R}$ is differentiable on $(a,b)$ and whose derivative $f'$ is bounded on $(a,b)$, we can put instead of $L$ the infinity norm $\|f'\|_\infty$ obtaining the estimation due to Dragomir-Wang from* [11].

We are able now to state and prove our results for the Beta mapping.

**Theorem 14.** *Let $p, q > 2$ and $x \in [0,1]$. Then we have the inequality*

$$\begin{aligned} &\left|\beta(p,q) - x^{p-1}(1-x)^{q-1}\right| \\ &\leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2\right]. \end{aligned} \tag{4.10}$$

*Proof.* Reconsider the mapping $l_{a,b} : (0,1) \longrightarrow \mathbb{R}$, $l_{a,b}(x) = x^a (1-x)^b$. For $p, q > 1$, we get:

$$l'_{p-1,q-1}(t) = l_{p-2,q-2}(t)\left[(p-1) - (p+q-2)t\right], \quad t \in (0,1).$$

If $t \in \left(0, \frac{p-1}{p+q-2}\right)$, then $l'_{p-1,q-1}(t) > 0$. Otherwise, if $t \in \left(\frac{p-1}{p+q-2}, 1\right)$, then $l'_{p-1,q-1}(t) < 0$, which shows that for $t_0 = \frac{p-1}{p+q-2}$, we have a maximum for $l_{p-1,q-1}$ and

$$\sup_{t\in(0,1)} l_{p-1,q-1}(t) = l_{p-1,q-1}(t_0) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}, \quad p, q > 1.$$

Consequently

$$\begin{aligned} \left|l'_{p-1,q-1}(t)\right| &\leq |l_{p-2,q-2}(t)| \max_{t\in[0,1]} |(p-1) - (p+q-2)t| \\ &\leq \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \max\{p-1, q-1\} \end{aligned}$$

for all $t \in [0,1]$, and then

$$\left\| l'_{p-1,q-1}(t) \right\|_{\infty} \leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}, \quad p,q > 2. \tag{4.11}$$

Applying now the inequality (4.2) for $f(x) = l_{p-1,q-1}(x)$, $x \in [0,1]$ and using the bound (4.11), we derive the desired inequality (4.10). ∎

The best inequality we can get from (4.10) is the following.

**Corollary 8.** *Let $p,q > 2$. Then we have the inequality:*

$$\left| \beta(p,q) - \frac{1}{2^{p+q-2}} \right| \leq \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}. \tag{4.12}$$

The following approximation formula for the Beta mapping holds.

**Theorem 15.** *Let $I_n : 0 = x_0 < x_1 < .... < x_{n-1} < x_n = 1$ be a division of the interval $[0,1]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ a sequence of intermediate points for $I_n$ and $p,q > 2$. Then we have the formula*

$$\beta(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1} (1-\xi_i)^{q-1} h_i + T_n(p,q)$$

*where the remainder $T_n(p,q)$ satisfies the estimation*

$$\begin{aligned} |T_n(p,q)| &\leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \\ &\quad \times \left[ \frac{1}{4} \sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1} \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \\ &\leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2. \end{aligned}$$

*In particular, if we choose for the above*

$$\xi_i = \frac{x_i + x_{i+1}}{2}, \quad (i = 0, 1, ..., n-1);$$

*then we get the approximation*

$$\beta(p,q) = \frac{1}{2^{p+q-2}} \sum_{i=0}^{n-1} (x_i + x_{i+1})^{p-1} (2 - x_i - x_{i+1})^{q-1} + V_n(p,q),$$

*where*

$$|V_n(p,q)| \leq \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2.$$

4.2. **Some Inequalities Via Ostrowski's Inequality for Mappings of Bounded Variation.** The following inequality for mappings of bounded variation [15] holds:

**Theorem 16.** *Let $u : [a,b] \longrightarrow \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then for all $x \in [a,b]$, we have*

$$\left| \int_a^b u(t)\,dt - u(x)(b-a) \right| \leq \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (u) \tag{4.13}$$

*where* $\bigvee_a^b (u)$ *denotes the total variation of* $u$.
*The constant* $\frac{1}{2}$ *is the best possible.*

*Proof.* Using the integration by parts formula for Riemann-Stieltjes integral, we have (see also the proof of the Theorem 11 ) that

$$u(x)(b-a) - \int_a^b u(t)\,dt = \int_a^b (t-a)\,du(t) + \int_a^b (t-b)\,du(t) \tag{4.14}$$

for all $x \in [a,b]$.
Now, assume that $\Delta_n : c = x_0^{(n)} < x_1^{(n)} < .... < x_{n-1}^{(n)} < x_n^{(n)} = d$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \longrightarrow \infty$, where $\nu(\Delta_n) := \max_{i\in\{0,\dots,n-1\}} \left(x_{i+1}^{(n)} - x_i^{(n)}\right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$. If $p : [c,d] \longrightarrow \mathbb{R}$ is continuous on $[c,d]$ and $v : [c,d] \longrightarrow \mathbb{R}$ is of bounded variation on $[a,b]$, then

$$\begin{aligned}
\left|\int_c^d p(x)\,dv(x)\right| &= \left|\lim_{\nu(\Delta n)\longrightarrow 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right]\right| \\
&\le \lim_{\nu(\Delta n)\longrightarrow 0} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right| \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right| \\
&\le \sup_{x\in[c,d]} |p(x)| \sup_{\Delta_n} \sum_{i=0}^{n-1} \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right| \\
&= \sup_{x\in[c,d]} |p(x)| \bigvee_c^d (v).
\end{aligned} \tag{4.15}$$

Applying (4.15), we have successively

$$\left|\int_a^x (t-a)\,du(t)\right| \le (x-a)\bigvee_a^x (u)$$

and

$$\left|\int_x^b (t-b)\,du(t)\right| \le (b-x)\bigvee_x^b (u)$$

and then

$$\begin{aligned}
\left|\int_a^x (t-a)\,du(t) + \int_x^b (t-b)\,du(t)\right| &\le \left|\int_a^x (t-a)\,du(t)\right| + \left|\int_x^b (t-b)\,du(t)\right| \\
&\le (x-a)\bigvee_a^x (u) + (b-x)\bigvee_x^b (u) \\
&\le \max\{x-a, b-x\}\left[\bigvee_a^x (u) + \bigvee_x^b (u)\right] \\
&= \max\{x-a, b-x\}\bigvee_a^b (u)
\end{aligned}$$

$$= \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u).$$

Using the identity (4.14), we get the desired inequality $(4.13)$.
Now, assume that the inequality (4.13) holds with a constant $C > 0$, i.e.,

$$\left|\int_a^b u(t)\,dt - u(x)(b-a)\right| \le \left[C(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u), \tag{4.16}$$

for all $x \in [a,b]$.

Consider the mapping $u : [a,b] \longrightarrow \mathbb{R}$ given by

$$u(x) = \begin{cases} 0 \text{ if } & x \in [a,b] \setminus \left\{\frac{a+b}{2}\right\} \\ 1 \text{ if } & x = \frac{a+b}{2} \end{cases}$$

in $(4.16)$. Note that $u$ is of bounded variation on $[a,b]$ and

$$\bigvee_a^b (u) = 2, \quad \int_a^b u(t)\,dt = 0$$

and for $x = \frac{a+b}{2}$ we get by (4.16) that $1 \le 2C$ which implies $C \ge \frac{1}{2}$ and the theorem is completely proved. ∎

The following corollaries hold.

**Corollary 9.** *Let $u : [a,b] \longrightarrow \mathbb{R}$ be a L-lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b u(t)\,dt - u(x)(b-a)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] |f(b) - f(a)| \tag{4.17}$$

*for all $x \in [a,b]$.*

The case of Lipschitzian mappings is embodied in the following corollary.

**Corollary 10.** *Let $u : [a,b] \longrightarrow \mathbb{R}$ be a L-lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b u(t)\,dt - u(x)(b-a)\right| \le L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] (b-a), \tag{4.18}$$

*for all $x \in [a,b]$.*

The following particular case can be more useful in practice.

**Corollary 11.** *If $u : [a,b] \longrightarrow \mathbb{R}$ is continuous and differentiable on $(a,b)$, $u'$ is continuous on $(a,b)$ and $\|u'\|_1 := \int_a^b |u'(t)|\,dt < \infty$, then*

$$\left|\int_a^b u(t)\,dt - u(x)(b-a)\right| \le L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \|u'\|_1 \tag{4.19}$$

*for all $x \in [a,b]$.*

**Remark 4.** *The best inequality we can obtain from $(4.13)$ is that one for $x = \frac{a+b}{2}$, obtaining the inequality*

$$\left|\int_a^b u(t)\,dt - u\left(\frac{a+b}{2}\right)(b-a)\right| \le \frac{1}{2}(b-a) \bigvee_a^b (u). \tag{4.20}$$

Now, consider the Riemann sums

$$R_n(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i) h_i$$

where $I_n : a = x_0 < x_1 < .... < x_{n-1} < x_n = b$ is a division of the interval $[a, b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ is a sequence of intermediate points for $I_n$, $h_i := x_{i+1} - x_i$ $(i = 0, 1, ..., n-1)$.

We have the following quadrature formula.

**Theorem 17.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a mapping of bounded variation on $[a, b]$ and $I_n$, $\xi_i$ $(i = 0, 1, ..., n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)\,dx = R_n(f, I_n, \xi) + W_n(f, I_n, \xi) \tag{4.21}$$

*where the remainder satisfies the estimate*

$$\begin{aligned} |W_n(f, I_n, \xi)| &\le \sup_{i=0,1,\dots,n-1} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (f) \\ &\le \left[\frac{1}{2}\nu(h) + \sup_{i=0,1,\dots,n-1} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (f) \\ &\le \nu(h) \bigvee_a^b (f); \end{aligned} \tag{4.22}$$

*for all* $\xi_i$ $(i = 0, 1, ..., n-1)$ *as above, where* $\nu(h) := \max_{i=0,1,\dots,n-1}\{h_i\}$*. The constant* $\frac{1}{2}$ *is sharp.*

*Proof.* Apply Theorem 16 in the interval $[x_i, x_{i+1}]$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i) h_i\right| \le \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}} (f). \tag{4.23}$$

Summing over $i$ from 0 to $n-1$ and using the generalized triangle inequality, we get

$$\begin{aligned} |W_n(f, I_n, \xi)| &\le \sum_{i=0}^{n-1} \left|\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i) h_i\right| \\ &\le \sum_{i=0}^{n-1} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}} (f) \\ &\le \sup_{i=0,1,\dots,n-1} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) \\ &= \sup_{i=0,1,\dots,n-1} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (f). \end{aligned}$$

The second inequality follows by the properties if $\sup(\cdot)$.
Now, as

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2} h_i$$

for all $\xi_i \in [x_i, x_{i+1}] \quad (i = 0, 1, ..., n-1)$, the last part of (4.22) is also proved. ∎

Note that the best estimation we can get from (4.22) is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$ obtaining the following midpoint quadrature formula.

**Corollary 12.** *Let $f, I_n$ be as above. Then we have the midpoint rule*

$$\int_a^b f(x)\,dx = M_n(f, I_n) + S_n(f, I_n)$$

*where*

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

*and the remainder $S_n(f, I_n)$ satisfies the estimation*

$$|S_n(f, I_n)| \leq \frac{1}{2} \nu(h) \bigvee_a^b (f).$$

We are able now to apply the above results for Euler's Beta function.

**Theorem 18.** *Let $p, q > 1$ and $x \in [0, 1]$. Then we have the inequality:*

$$\left|\beta(p, q) - x^{p-1}(1-x)^{q-1}\right| \tag{4.24}$$

$$\leq \max\{p-1, q-1\} \beta(p-1, q-1) \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right].$$

*Proof.* Consider the mapping $l_{p-1,q-1}(t) = t^{p-1}(1-x)^{q-1}$, $t \in [0, 1]$.
We have for $p, q > 1$ that

$$l'_{p-1,q-1}(t) = l_{p-2,q-2}(t)[p - 1 - (p + q - 2)t]$$

and, as

$$|p - 1 - (p + q - 2)t| \leq \max\{p - 1, q - 1\}$$

for all $t \in [0, 1]$, then

$$\begin{aligned} \left\|l'_{p-1,q-1}\right\|_1 &= \int_0^1 l_{p-2,q-2}(t)\,|p - 1 - (p + q - 2)t|\,dt \\ &\leq \max\{p - 1, q - 1\} \left\|l_{p-2,q-2}\right\|_1 \\ &= \max\{p - 1, q - 1\} \beta(p - 1, q - 1), \quad p, q > 1. \end{aligned}$$

Now, applying Theorem 16 for $u(t) = l_{p-1,q-1}$, we deduce

$$\left|\int_0^1 l_{p-1,q-1}(t)\,dt - x^{p-1}(1-x)^{q-1}\right| \leq \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right] \bigvee_0^1 (l_{p-1,q-1})$$

$$\leq \max\{p-1,q-1\}\,\beta(p-1,q-1)\left[\frac{1}{2}+\left|x-\frac{1}{2}\right|\right]$$

for all $x \in [0,1]$, and the theorem is proved. ∎

The best inequality that we can get from (4.24) is embodied in the following corollary.

**Corollary 13.** *Let $p, q > 1$. Then we have the inequality*

$$\left|\beta(p,q)-\frac{1}{2^{p+q-2}}\right| \leq \frac{1}{2}\max\{p-1,q-1\}\,\beta(p-1,q-1). \tag{4.25}$$

Now, if we apply Theorem 16 for the mapping $l_{p-1,q-1}$, we get the following approximation of the Beta function in terms of Riemann sums.

**Theorem 19.** *Let $I_n : a = x_0 < x_1 < .... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,1,...,n-1)$ a sequence of intermediate points for $I_n$, and $p, q > 1$. Then we have the formula*

$$\beta(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1}(1-\xi_i)^{q-1} h_i + T_n(p,q) \tag{4.26}$$

*where the remainder $T_n(p,q)$ satisfies the estimation*

$$\begin{aligned}
&|T_n(p,q)| \\
&\leq \max\{p-1,q-1\}\left[\frac{1}{2}\nu(h)+\sup_{i=0,1,\ldots,n-1}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\beta(p-1,q-1) \\
&\qquad \leq \max\{p-1,q-1\}\,\nu(h)\,\beta(p-1,q-1).
\end{aligned}$$

*In particular, if we choose above $\xi_i = \frac{x_i+x_{i+1}}{2}$ $(i = 0,1,...,n-1)$, then we get the approximation*

$$\beta(p,q) = \frac{1}{2^{p+q-2}}\sum_{i=0}^{n-1}(x_i+x_{i+1})^{p-1}(2-x_i-x_{i+1})^{q-1} + V_n(p,q)$$

*where*

$$|V_n(p,q)| \leq \frac{1}{2}\max\{p-1,q-1\}\,\nu(h)\,\beta(p-1,q-1).$$

4.3. **Inequalities Via Ostrowski's Inequality for Absolutely Continuous Mappings Whose Derivatives Belong to $L_p$-Spaces.** The following theorem concerning Ostrowski's inequality for absolutely continuous mappings whose derivatives belong to $L_p$-spaces holds (see also [12]).

**Theorem 20.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be an absolutely continuous mapping for which $f' \in L_p[a,b]$, $p > 1$. Then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f'\|_p \\
&\leq \frac{1}{(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\|f'\|_p
\end{aligned} \tag{4.27}$$

*for all* $x \in [a,b]$, *where*

$$\|f'\|_p := \left( \int_a^b |f'(t)|^p \, dt \right)^{\frac{1}{p}}. \tag{4.28}$$

*Proof.* Integrating by parts, we have

$$\int_a^x (t-a) f'(t) \, dt = (x-a) f(x) - \int_a^x f(t) \, dt$$

and

$$\int_x^b (t-b) f'(t) \, dt = (b-x) f(x) - \int_x^b f(t) \, dt.$$

If we add the above two equalities, we get

$$\int_a^x (t-a) f'(t) \, dt + \int_x^b (t-b) f'(t) \, dt = (b-a) f(x) - \int_a^b f(t) \, dt.$$

From this we obtain

$$f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt = \frac{1}{b-a} \int_a^b p(x,t) f'(t) \, dt \tag{4.29}$$

where

$$p(x,t) := \begin{cases} t-a \text{ if } t \in [a,x] \\ t-b \text{ if } t \in (x,b] \end{cases}, \quad (t,x) \in [a,b]^2.$$

Now, using Hölder's integral inequality, we have

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \\ \leq & \frac{1}{b-a} \int_a^b |p(x,t)| \, |f'(t)| \, dt \\ \leq & \frac{1}{b-a} \left( \int_a^b |p(x,t)|^q \, dt \right)^{\frac{1}{q}} \left( \int_a^b |f'(t)|^p \, dt \right)^{\frac{1}{p}}. \end{aligned} \tag{4.30}$$

A simple calculation shows that

$$\begin{aligned} \int_a^b |p(x,t)|^q \, dt &= \int_a^x |t-a|^q \, dt + \int_x^b |t-b|^q \, dt \\ &= \int_a^x (t-a)^q \, dt + \int_x^b (t-b)^q \, dt \\ &= \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \\ &= \frac{1}{q+1} \left[ \left( \frac{x-a}{b-a} \right)^{q+1} + \left( \frac{b-x}{b-a} \right)^{q+1} \right] (b-a)^{q+1}. \end{aligned}$$

Now, using the inequality (4.30), we have

$$\begin{aligned}
& \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\
\leq\ & \frac{1}{b-a} \left[ \frac{1}{q+1} \left[ \left( \frac{x-a}{b-a} \right)^{q+1} + \left( \frac{b-x}{b-a} \right)^{q+1} \right] (b-a)^{q+1} \right]^{\frac{1}{q}} \|f'\|_p \\
=\ & \frac{1}{(q+1)^{\frac{1}{q}}} \left[ \left( \frac{x-a}{b-a} \right)^{q+1} + \left( \frac{b-x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_p
\end{aligned}$$

and the first inequality in (4.27) is proved.
Now, for $s \geq 1$ and $\alpha < \beta$, consider the mapping $h : [\alpha, \beta] \longrightarrow \mathbb{R}$ defined by $h(x) := (x-\alpha)^s + (\beta - x)^s$. Observe that

$$h'(x) = s\left[ (x-\alpha)^{s-1} - (\beta - x)^{s-1} \right]$$

and so $h'(x) < 0$ on $\left[\alpha, \frac{\alpha+\beta}{2}\right)$ and $h'(x) > 0$ on $\left(\frac{\alpha+\beta}{2}, \beta\right]$.
Therefore, we have

$$\inf_{x \in [\alpha,\beta]} h(x) = h\left( \frac{\alpha+\beta}{2} \right) = \frac{(\beta-\alpha)^s}{2^{s-1}}$$

and

$$\sup_{x \in [\alpha,\beta]} h(x) = h(\alpha) = h(\beta) = (\beta - \alpha)^s .$$

Consequently, we have

$$(b-x)^{q+1} + (x-a)^{q+1} \leq (b-a)^{q+1}, \quad x \in [\alpha, \beta]$$

and the last part of (4.27) is thus proved. ∎

The best inequality we can get from (4.27) is embodied in the following corollary.

**Corollary 14.** *Under the above assumptions for $f$, we have*

$$\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{1}{2} \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_p . \tag{4.31}$$

We now consider the application of (4.27) to some numerical quadrature rules.

**Theorem 21.** *Let $f$ be as in Theorem 20. Then for any partition $I_n : a = x_0 < x_1 < .... < x_{n-1} < x_n = b$ of $[a,b]$ and any intermediate point vector $\xi = (\xi_0, \xi_1, ..., \xi_{n-1})$ satisfying $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$, we have*

$$\int_a^b f(x)\,dx = A_R(f, I_n, \xi) + R_R(f, I_n, \xi) \tag{4.32}$$

*where $A_R$ denotes the quadrature rules of the Riemann type defined by*

$$A_R(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i)\,h_i, \quad h_i := x_{i+1} - x_i,$$

*and the remainder satisfies the estimate*

$$|R_R(f,I_n,\xi)| \leq \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{n-1}\left[(\xi_i-x_i)^{q+1}+(x_{i+1}-\xi_i)^{q+1}\right]\right)^{\frac{1}{q}} \leq \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left(\sum_{i=0}^{n-1}h_i^{q+1}\right)^{\frac{1}{q}} \tag{4.33}$$

*where* $h_i := x_{i+1}-x_i \quad (i=0,1,...,n-1)$.

*Proof.* Apply Theorem 20 on the intervals $[x_i,x_{i+1}] \quad (i=0,1,...,n-1)$ to get

$$\left|f(\xi_i)h_i-\frac{1}{b-a}\int_{x_i}^{x_{i+1}}f(t)\,dt\right|$$
$$\leq \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{\xi_i-x_i}{h_i}\right)^{q+1}+\left(\frac{x_{i+1}-\xi_i}{h_i}\right)^{q+1}\right]^{\frac{1}{q}}h_i^{1+\frac{1}{q}}\times\left(\int_{x_i}^{x_{i+1}}|f'(t)|^p\,dt\right)^{\frac{1}{p}}$$

for all $i\in\{0,1,...,n-1\}$.
Summing over $i$ from 0 to $n-1$, using the generalized triangle inequality and Hölder's discrete inequality, we get

$$|R_R(f,I_n,\xi)| \leq \sum_{i=0}^{n-1}\left|f(\xi_i)h_i-\int_{x_i}^{x_{i+1}}f(t)\,dt\right|$$

$$\begin{aligned}
&\leq \frac{1}{(q+1)^{\frac{1}{q}}}\sum_{i=0}^{n-1}\left[(\xi_i-x_i)^{q+1}+(x_{i+1}-\xi_i)^{q+1}\right]^{\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}}|f'(t)|^p\,dt\right)^{\frac{1}{p}}\\
&\leq \frac{1}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{n-1}\left(\left[(\xi_i-x_i)^{q+1}+(x_{i+1}-\xi_i)^{q+1}\right]^{\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}\\
&\quad\times\left[\sum_{i=0}^{n-1}\left[\left(\int_{x_i}^{x_{i+1}}|f'(t)|^p\,dt\right)^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}}\\
&= \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\sum_{i=0}^{n-1}\left[(\xi_i-x_i)^{q+1}+(x_{i+1}-\xi_i)^{q+1}\right]
\end{aligned}$$

and the first inequality in (4.33) is proved.
The second inequality follows from the fact that $(\xi_i-x_i)^{q+1}+(x_{i+1}-\xi_i)^{q+1}\leq h_i^{q+1}$ for all $i\in\{0,1,...,n-1\}$, and the theorem is thus proved. ∎

The best quadrature formula we can get from the above general result is that one for which $\xi_i := \frac{x_i+x_{i+1}}{2}$, $i=0,1,...,n-1$, obtaining the following corollary.

**Corollary 15.** *Let* $f$ *and* $I_n$ *be as in the above theorem. Then*

$$\int_a^b f(x)\,dx = A_M(f,I_n)+R_M(f,I_n) \tag{4.34}$$

*where $A_M$ is the midpoint quadrature rule, i.e.,*

$$A_M(f, I_n) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

*and the remainder $R_M$ satisfies the estimation:*

$$|R_M(f, I_n)| \leq \frac{1}{2} \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}. \tag{4.35}$$

We are able to now apply the above results for Euler's Beta mapping.

**Theorem 22.** *Let $s > 1$, $p, q > 2 - \frac{1}{s} > 1$. Then we have the inequality*

$$\begin{aligned} & \left|\beta(p,q) - x^{p-1}(1-x)^{q-1}\right| \\ \leq\ & \frac{1}{(l+1)^{\frac{1}{l}}} \left[x^{l+1} + (1-x)^{l+1}\right]^{\frac{1}{l}} \max\{p-1, q-1\} \\ & \times \left[\beta(s(p-2)+1, s(q-2)+1)\right]^{\frac{1}{s}} \end{aligned} \tag{4.36}$$

*provided $\frac{1}{s} + \frac{1}{l} = 1$.*

*Proof.* We apply Theorem 20 for the mapping $f(t) = t^{p-1}(1-t)^{q-1} = l_{p-1,q-1}(t)$, $t \in [0,1]$ to get

$$\begin{aligned} & |\beta(p,q) - l_{p-1,q-1}(x)| \\ \leq\ & \frac{1}{(l+1)^{\frac{1}{l}}} \left[x^{l+1} + (1-x)^{l+1}\right]^{\frac{1}{l}} \left\|l'_{p-1,q-1}\right\|_s, \quad x \in [0,1] \end{aligned} \tag{4.37}$$

where $s > 1$ and $\frac{1}{s} + \frac{1}{l} = 1$.
However, as in the proof of Theorem 18,

$$l'_{p-1,q-1}(t) = l_{p-2,q-2}(t)\left[p - 1 - (p+q-2)t\right]$$

and then

$$\begin{aligned} \left\|l'_{p-1,q-1}\right\|_s &= \left(\int_0^1 l^s_{p-2,q-2}(t) \left|p-1-(p+q-2)\right|^s ds\right)^{\frac{1}{s}} \\ &= \left(\int_0^1 t^{s(p-2)}(1-t)^{s(q-2)} \left|p-1-(p+q-2)\right|^s ds\right)^{\frac{1}{s}} \\ &\leq \max\{p-1, q-1\}\left[\beta(s(p-2)+1, s(q-2)+1)\right]^{\frac{1}{s}}. \end{aligned}$$

Using (4.37) we deduce (4.36). ∎

We can state now the following result concerning the approximation of the Beta function in terms of Riemann sums.

**Theorem 23.** *Let $s > 1$, $p, q > 2 - \frac{1}{s} > 1$. If $I_n : 0 = x_0 < x_1 < .... < x_{n-1} < x_n = 1$ is a division of $[0,1]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ a sequence of intermediate points for $I_n$, then we have the formula*

$$\beta(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1}(1-\xi_i)^{q-1} h_i + T_n(p,q) \tag{4.38}$$

*where the remainder $T_n(p,q)$ satisfies the estimate*

$$\begin{aligned}|T_n(p,q)| &\leq \frac{\max\{p-1,q-1\}}{(l+1)^{\frac{1}{l}}}\left[\beta\left(s(p-2)+1,s(q-2)+1\right)\right]^{\frac{1}{s}} \\ &\quad\times\left(\sum_{i=0}^{n-1}\left[(\xi_i-x_i)^{l+1}+(x_{i+1}-\xi_i)^{l+1}\right]\right)^{\frac{1}{l}} \\ &\leq \frac{\max\{p-1,q-1\}}{(l+1)^{\frac{1}{l}}}\left[\beta\left(s(p-2)+1,s(q-2)+1\right)\right]^{\frac{1}{s}}\left(\sum_{i=0}^{n-1}h_i^{l+1}\right)^{\frac{1}{l}}\end{aligned}$$

*where $h_i := x_{i+1}-x_i \quad (i=0,1,...,n-1)$ and $\frac{1}{s}+\frac{1}{l}=1$.*
*The proof follows by Theorem 21 applied for the mapping $f(t)=t^{p-1}(1-t)^{q-1}$, $t\in[0,1]$, and we omit the details.*

4.4. **An Ostrowski Type Inequality for Monotonic Mappings.** The following result of the Ostrowski type holds [13].

**Theorem 24.** *Let $u:[a,b]\longrightarrow\mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then for all $x\in[a,b]$, we have the inequality*

$$\begin{aligned}&\left|u(x)-\frac{1}{b-a}\int_a^b u(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left\{[2x-(a+b)]\,u(x)+\int_a^b sgn(t-x)\,u(t)\,dt\right\} \\ &\leq \frac{1}{b-a}\left[(x-a)(u(x)-u(a))+(b-x)(u(b)-u(x))\right] \\ &\leq \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right](u(b)-u(a)).\end{aligned} \tag{4.39}$$

*The inequalities in (4.39) are sharp and the constant $\frac{1}{2}$ is the best possible.*

*Proof.* Using the integration by parts formula for Riemann-Stieltjes integral (4.14), we have the identity

$$u(x)-\frac{1}{b-a}\int_a^b u(t)\,dt=\frac{1}{b-a}\int_a^b p(x,t)\,du(t) \tag{4.40}$$

where

$$p(x,t):=\begin{cases} t-a \text{ if } t\in[a,x] \\ t-b \text{ if } t\in(x,b]\end{cases}.$$

Now, assume that $\Delta_n : a=x_0^{(n)}<x_1^{(n)}<....<x_{n-1}^{(n)}<x_n^{(n)}=b$ is a sequence of divisions with $\nu(\Delta_n)\to 0$ as $n\longrightarrow\infty$, where $\nu(\Delta_n):=\max_{i\in\{0,...,n-1\}}\left(x_{i+1}^{(n)}-x_i^{(n)}\right)$ and $\xi_i^{(n)}\in\left[x_i^{(n)},x_{i+1}^{(n)}\right]$.

If $p$ is Riemann-Stieltjes integrable by rapport of $v$, and $v$ is monotonic nondecreasing on $[a,b]$, then

$$\begin{aligned}(4.41)\quad \left|\int_a^b p(x)\,dv(x)\right| &= \left|\lim_{\nu(\Delta n)\longrightarrow 0}\sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right]\right| \\ &\le \lim_{\nu(\Delta n)\longrightarrow 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left|v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right| \\ &\le \lim_{\nu(\Delta n)\longrightarrow 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right) \\ &= \int_a^b |p(x)|\,dv(x).\end{aligned}$$

Using the above inequality, we can state that

$$\left|\int_a^b p(x,t)\,du(t)\right| \le \int_a^b |p(x,t)|\,du(t). \tag{4.42}$$

Now, let observe that

$$\begin{aligned}\int_a^b |p(x,t)|\,du(t) &= \int_a^x |t-a|\,du(t)+\int_x^b |t-b|\,du(t) \\ &= \int_a^x (t-a)\,du(t)+\int_x^b (b-t)\,du(t) \\ &= (t-a)\,u(t)]_a^x-\int_a^x u(t)\,dt-(b-t)\,u(t)]_x^b+\int_x^b u(t)\,dt \\ &= [2x-(a+b)]\,u(x)-\int_a^x u(t)\,dt+\int_x^b u(t)\,dt \\ &= [2x-(a+b)]\,u(x)+\int_a^b sgn(t-x)\,u(t)\,dt.\end{aligned}$$

Using the inequality $(4.42)$ and the identity $(4.40)$, we get the first part of $(4.39)$.

We know that

$$\int_a^b sgn(t-x)\,u(t)\,dt = -\int_a^x u(t)\,dt+\int_x^b u(t)\,dt.$$

As $u$ is monotonic nondecreasing on $[a,b]$, we can state that

$$\int_a^x u(t)\,dt \ge (x-a)\,u(a)$$

and

$$\int_x^b u(t)\,dt \le (b-x)\,u(b)$$

and then

$$\int_a^b sgn(t-x)\,u(t)\,dt \le (b-x)\,u(b)-(x-a)\,u(a).$$

Consequently, we can state that

$$\begin{aligned} & [2x-(a+b)]\,u(x)+\int_a^b sgn\,(t-x)\,u(t)\,dt \\ \leq\; & [2x-(a+b)]\,u(x)+(b-x)\,u(b)-(x-a)\,u(a) \\ =\; & (b-x)\,(u(b)-u(x))+(x-a)\,(u(x)-u(a)) \end{aligned}$$

and the second part of (4.39) is proved.
Finally, let us observe that

$$\begin{aligned} & (b-x)\,(u(b)-u(x))+(x-a)\,(u(x)-u(a)) \\ \leq\; & \max\{b-x,x-a\}\,[u(b)-u(x)+u(x)-u(a)] \\ =\; & \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](u(b)-u(a)) \end{aligned}$$

and the inequality (4.39) is proved.
Assume that (4.39) holds with a constant $C>0$ instead of $\frac{1}{2}$, i.e.,

$$\begin{aligned} & \left|u(x)-\frac{1}{b-a}\int_a^b u(t)\,dt\right| \\ \leq\; & \frac{1}{b-a}\left\{[2x-(a+b)]\,u(x)+\int_a^b sgn\,(t-x)\,u(t)\,dt\right\} \\ \leq\; & \frac{1}{b-a}\,[(x-a)\,(u(x)-u(a))+(b-x)\,(u(b)-u(x))] \\ \leq\; & \left[C+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right](u(b)-u(a)). \end{aligned} \tag{4.43}$$

Consider the mapping $u_0:[a,b]\longrightarrow\mathbb{R}$ given by

$$u_0(x):=\begin{cases} -1 \text{ if } x=a \\ 0 \text{ if } x\in(a,b] \end{cases}.$$

Putting in (4.43) $u=u_0$ and $x=a$, we get

$$\begin{aligned} & \left|u(x)-\frac{1}{b-a}\int_a^b u(t)\,dt\right| \\ =\; & \frac{1}{b-a}\left\{[2x-(a+b)]\,u(x)+\int_a^b sgn\,(t-x)\,u(t)\,dt\right\} \\ =\; & \frac{1}{b-a}\,[(x-a)\,(u(x)-u(a))+(b-x)\,(u(b)-u(x))]=1 \\ \leq\; & \left[C+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right](u(b)-u(a))=C+\frac{1}{2}, \end{aligned}$$

which proves the sharpness of the first two inequalities and the fact that $C$ should not be less than $\frac{1}{2}$. ∎

The following corollaries are interesting.

**Corollary 16.** *Let $u$ be as above. Then we have the midpoint inequality*

$$\left|u\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b u(t)\,dt\right| \leq \frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt \leq \frac{1}{2}\left[u(b)-u(a)\right]. \tag{4.44}$$

Also, the following "trapezoid inequality" for monotonic nondecreasing mappings holds.

**Corollary 17.** *Under the above assumption, we have*

$$\left|\frac{u(b)+u(a)}{2}-\frac{1}{b-a}\int_a^b u(t)\,dt\right| \leq \frac{1}{2}\left[u(b)-u(a)\right]. \tag{4.45}$$

*Proof.* Let us choose in Theorem 24, $x=a$ and $x=b$ to obtain

$$\left|u(a)-\frac{1}{b-a}\int_a^b u(t)\,dt\right| \leq \frac{1}{b-a}\left[-(b-a)\,u(a)+\int_a^b u(t)\,dt\right]$$

and

$$\left|u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right| \leq \frac{1}{b-a}\left[(b-a)\,u(b)+\int_a^b u(t)\,dt\right].$$

Summing the above inequalities, using the triangle inequality and dividing by 2, we get the desired inequality (4.45). ∎

## 5. Inequalities of The Ostrowski Type in Probability Theory and Applications for the Beta Function

5.1. **An inequality of Ostrowski's Type for Cumulative Distribution Functions and Applications for the Beta Function.** Let $X$ be a random variable taking values in the finite interval $[a,b]$, with the cumulative distribution function $F(x)=\Pr(X\leq x)$.

The following result of Ostrowski type holds [14].

**Theorem 25.** *Let $X$ and $F$ be as above. Then*

$$\begin{aligned} &\left|\Pr(X\leq x)-\frac{b-E(X)}{b-a}\right| \\ \leq\ & \frac{1}{b-a}\left[\left[2x-(a+b)\right]\Pr(X\leq x)+\int_a^b sgn(t-x)\,F(t)\,dt\right] \\ \leq\ & \frac{1}{b-a}\left[(b-x)\Pr(X\geq x)+(x-a)\Pr(X\leq x)\right] \\ \leq\ & \frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a} \end{aligned} \tag{5.1}$$

*for all $x\in[a,b]$. All the inequalities in (5.1) are sharp and the constant $\frac{1}{2}$ is the best possible.*

*Proof.* We know, by Theorem 24, that for a monotonic nondecreasing mapping $u : [a,b] \longrightarrow \mathbb{R}$, we have the inequality

$$\begin{aligned} (5.2) \qquad & \left| u(x) - \frac{1}{b-a} \int_a^b u(t)\,dt \right| \\ \leq & \frac{1}{b-a} \left\{ [2x-(a+b)]\,u(x) + \int_a^b sgn(t-x)\,u(t)\,dt \right\} \\ \leq & \frac{1}{b-a} \left[ (x-a)(u(x)-u(a)) + (b-x)(u(b)-u(x)) \right] \\ \leq & \frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} (u(b)-u(a)) \end{aligned}$$

for all $x \in [a,b]$.

Apply (5.2) for the monotonic nondecreasing mapping $u(x) = F(x)$ and take into account that $F(a) = 0$, $F(b) = 1$, to get

$$\begin{aligned} (5.3) \qquad & \left| F(x) - \frac{1}{b-a} \int_a^b F(t)\,dt \right| \\ \leq & \frac{1}{b-a} \left[ [2x-(a+b)]\,F(x) + \int_a^b sgn(t-x)\,F(t)\,dt \right] \\ \leq & \frac{1}{b-a} \left[ (x-a)F(x) + (b-x)(1-F(x)) \right] \\ \leq & \frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a}. \end{aligned}$$

However, by the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned} E(X) \quad : &= \int_a^b t dF(t) = tF(t)\big|_a^b - \int_a^b F(t)\,dt \\ &= bF(b) - aF(a) - \int_a^b F(t)\,dt = b - \int_a^b F(t)\,dt, \end{aligned}$$

and

$$1 - F(x) = \Pr(X \geq x).$$

Then, by (5.3), we get the desired inequality (5.1).

To prove the sharpness of the inequalities in (5.1), we choose the random variable $X$ such that $F : [0,1] \longrightarrow \mathbb{R}$

$$F(x) := \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x \in (0,1]. \end{cases}$$

We omit the details. ∎

**Remark 5.** *Taking into account the fact that*

$$\Pr(X \geq x) = 1 - \Pr(X \leq x)$$

*then, from* $(5.1)$, *we get the equivalent inequality*

$$\begin{aligned}(5.4)\qquad & \left|\Pr(X\geq x)-\frac{E(X)-a}{b-a}\right| \\ \leq & \frac{1}{b-a}\left\{[2x-(a+b)]\Pr(X\leq x)+\int_a^b sgn(t-x)F(t)\,dt\right\} \\ \leq & \frac{1}{b-a}[(b-x)\Pr(X\geq x)+(x-a)\Pr(X\leq x)] \\ \leq & \frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\end{aligned}$$

*for all* $x\in[a,b]$.

The following particular inequality can also be interesting

$$(5.5)\qquad \left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{b-E(X)}{b-a}\leq\right|\int_a^b sgn\left(t-\frac{a+b}{2}\right)F(t)\,dt\leq\frac{1}{2}$$

and

$$(5.6)\qquad \left|\Pr\left(X\geq\frac{a+b}{2}\right)-\frac{E(X)-a}{b-a}\leq\right|\int_a^b sgn\left(t-\frac{a+b}{2}\right)F(t)\,dt\leq\frac{1}{2}.$$

The following corollary may be useful in practice.

**Corollary 18.** *Under the above assumptions, we have*

$$(5.7)\qquad \frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]\leq\Pr\left(X\leq\frac{a+b}{2}\right)\leq\frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]+1.$$

*Proof.* From the inequality $(5.1)$, we get

$$-\frac{1}{2}+\frac{b-E(X)}{b-a}\leq\Pr\left(X\leq\frac{a+b}{2}\right)\leq\frac{1}{2}+\frac{b-E(X)}{b-a}.$$

But

$$\begin{aligned}-\frac{1}{2}+\frac{b-E(X)}{b-a} &= \frac{-b+a+2b-2E(X)}{2(b-a)} \\ &= \frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}+\frac{b-E(X)}{b-a} &= 1+\frac{b-E(X)}{b-a}-\frac{1}{2} \\ &= 1+\frac{2b-2E(X)-b+a}{2(b-a)} \\ &= 1+\frac{1}{b-a}\left[\frac{a+b}{2}-E(X)\right]\end{aligned}$$

and the inequality $(5.7)$ is thus proved. ∎

**Remark 6.** *a) Let* $1\geq\varepsilon\geq 0$, *and assume that*

$$(5.8)\qquad E(X)\geq\frac{a+b}{2}+(1-\varepsilon)(b-a),$$

*then*

$$\Pr\left(X \geq \frac{a+b}{2}\right) \leq \varepsilon. \tag{5.9}$$

*Indeed, if* (5.8) *holds, then by the right-hand side of* (5.9), *we get*

$$\begin{aligned} \Pr\left(X \leq \frac{a+b}{2}\right) &\leq \frac{1}{b-a}\left[\frac{a+b}{2} - E(X)\right] + 1 \\ &\leq \frac{(\varepsilon - 1)(b-a)}{b-a} + 1 = \varepsilon. \end{aligned}$$

*b) Also, if*

$$E(X) \leq \frac{a+b}{2} - \varepsilon(b-a), \tag{5.10}$$

*then, by the right-hand side of* (5.7),

$$\begin{aligned} \Pr\left(X \leq \frac{a+b}{2}\right) &\geq \left[\frac{a+b}{2} - E(X)\right] \cdot \frac{1}{b-a} \\ &\geq \frac{\varepsilon(b-a)}{b-a} = \varepsilon. \end{aligned}$$

*That is,*

$$\Pr\left(X \leq \frac{a+b}{2}\right) \geq \varepsilon \quad \varepsilon \in (0,1). \tag{5.11}$$

The following corollary is also interesting.

**Corollary 19.** *Under the above assumptions of Theorem 25, we have the inequality*

$$\frac{1}{b-x}\int_a^b \left[\frac{1 + sgn(t-x)}{2}\right] F(t)\,dt \geq \Pr(X \geq x) \tag{5.12}$$

$$\geq \frac{1}{x-a}\int_a^b \left[\frac{1 - sgn(t-x)}{2}\right] F(t)\,dt$$

*for all* $x \in [a,b]$.

*Proof.* From the equality (5.2), we have

$$\begin{aligned} &\Pr(X \leq x) - \frac{b - E(X)}{b-a} \\ \leq\ & \frac{1}{b-a}\left[[2x - (a+b)]\Pr(X \leq x) + \int_a^b sgn(t-x)\,F(t)\,dt\right] \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(b-a)\Pr(X \leq x) - [2x - (a+b)]\Pr(X \leq x) \\ \leq\ & b - E(X) + \int_a^b sgn(t-x)\,F(t)\,dt. \end{aligned}$$

That is,

$$2(b-x)\Pr(X \leq x) \leq b - E(X) + \int_a^b sgn(t-x)\,F(t)\,dt.$$

Since

$$b - E(X) = \int_a^b F(t)\,dt,$$

then from the above inequality, we deduce the first part of (5.12).

The second part of (5.12) follows by a similar argument from the inequality

$$\begin{aligned}
& \Pr(X \le x) - \frac{b - E(X)}{b-a} \\
\ge\ & -\frac{1}{b-a}\left[[2x - (a+b)]\Pr(X \le x) + \int_a^b sgn(t-x)\,\mathrm{F}(t)\,dt\right]
\end{aligned}$$

and we omit the details. ∎

**Remark 7.** *If we put* $x = \frac{a+b}{2}$ *in* $(5.12)$, *then we get*

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b \left[1 + sgn\left(t - \frac{a+b}{2}\right)\right] F(t)\,dt \\
\ge\ & \Pr\left(X \ge \frac{a+b}{2}\right) \\
\ge\ & \frac{1}{b-a}\int_a^b \left[1 - sgn\left(t - \frac{a+b}{2}\right)\right] F(t)\,dt.
\end{aligned} \tag{5.13}$$

We are able now to give some applications for a Beta random variable.

A Beta random variable $X$ with parameters $(p,q)$ has the probability density function

$$f(x;p,q) := \frac{x^{p-1}(1-x)^{q-1}}{\beta(p,q)};\ 0 < x < 1$$

where $\Omega = \{(p,q) : p,q > 0\}$ and $\beta(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$.
Let us compute the expected value of $X$.
We have

$$\begin{aligned}
E(X) &= \frac{1}{\beta(p,q)}\int_0^1 x \cdot x^{p-1}(1-x)^{q-1}\,dx \\
&= \frac{\beta(p+1,q)}{\beta(p,q)} = \frac{p}{p+q}.
\end{aligned}$$

The following result holds.

**Theorem 26.** *Let* $X$ *be a Beta random variable with parameters* $(p,q) \in \Omega$. *Then we have the inequalities*

$$\left|\Pr(X \le x) - \frac{q}{p+q}\right| \le \frac{1}{2} + \left|x - \frac{1}{2}\right|$$

*and*

$$\left|\Pr(X \ge x) - \frac{p}{p+q}\right| \le \frac{1}{2} + \left|x - \frac{1}{2}\right|$$

*for all* $x \in [0,1]$ *and, particularly,*

$$\left|\Pr\left(X \le \frac{1}{2}\right) - \frac{q}{p+q}\right| \le \frac{1}{2}$$

*and*

$$\left|\Pr\left(X \geq \frac{1}{2}\right) - \frac{p}{p+q}\right| \leq \frac{1}{2}.$$

The proof follows by Theorem 25 applied to the Beta random variable $X$.

5.2. **An Ostrowski Type Inequality For a Probability Density Function $\mathbf{f} \in \mathbf{L}_p\,[a,b]$.** The following theorem holds

**Theorem 27.** *Let $X$ be a random variable with the probability density function $f : [a,b] \subset \mathbf{R} \to \mathbf{R}_+$ and with cumulative distribution function $F(x) = \Pr(X \leq x)$. If $f \in L_p[a,b]$, $p > 1$, then we have the inequality:*

$$\left|\Pr(X \leq x) - \frac{b - E(X)}{b-a}\right| \tag{5.14}$$

$$\leq \frac{q}{q+1}\,\|f\|_p\,(b-a)^{\frac{1}{q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}} + \left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right]$$

$$\leq \frac{q}{q+1}\,\|f\|_p\,(b-a)^{\frac{1}{q}}$$

*for all $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

*Proof.* By Hölder's integral inequality we have

$$|F(x) - F(y)| = \left|\int_x^y f(t)\,dt\right| \tag{5.15}$$

$$\leq \left|\int_x^y dt\right|^{\frac{1}{q}}\left|\int_x^y |f(t)|^p\,dt\right|^{\frac{1}{p}} \leq |x-y|^{\frac{1}{q}}\,\|f\|_p$$

for all $x, y \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\|f\|_p := \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}$$

is the usual $p-$norm on $L_p[a,b]$.

The inequality (5.15) shows in fact that the mapping $F(\cdot)$ is of $r - H-$Hölder type, i.e.,

$$|F(x) - F(y)| \leq H\,|x-y|^r, \qquad (\forall)\, x, y \in [a,b] \tag{5.16}$$

with $0 < H = \|f\|_p$ and $r = \frac{1}{q} \in (0,1)$.

Integrating the inequality (5.15) over $y \in [a,b]$, we get successively

$$\left|F(x) - \frac{1}{b-a}\int_a^b F(y)\,dy\right| \tag{5.17}$$

$$
\begin{aligned}
&\leq \frac{1}{b-a}\int_a^b |F(x)-F(y)|\,dy \leq \frac{1}{b-a}\|f\|_p \int_a^b |x-y|^{\frac{1}{q}}\,dy \\
&= \frac{1}{b-a}\|f\|_p\left[\int_a^x (x-y)^{\frac{1}{q}}\,dy + \int_x^b (y-x)^{\frac{1}{q}}\,dy\right] \\
&= \frac{1}{b-a}\|f\|_p\left[\frac{(x-a)^{\frac{1}{q}+1}}{\frac{1}{q}+1} + \frac{(b-x)^{\frac{1}{q}+1}}{\frac{1}{q}+1}\right] \\
&= \frac{q}{q+1}\cdot\frac{1}{b-a}\|f\|_p\left[(x-a)^{\frac{1}{q}+1} + (b-x)^{\frac{1}{q}+1}\right] \\
&= \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1}{q}+1} + \left(\frac{b-x}{b-a}\right)^{\frac{1}{q}+1}\right]
\end{aligned}
$$

for all $x \in [a,b]$.

It is well known that

$$E(X) = b - \int_a^b F(t)\,dt$$

then, by $(5.17)$, we get the first inequality in $(5.14)$.

For the second inequality, we observe that

$$\left(\frac{x-a}{b-a}\right)^{\frac{1}{q}+1} + \left(\frac{b-x}{b-a}\right)^{\frac{1}{q}+1} \leq 1, \qquad (\forall)\, x \in [a,b]$$

and the theorem is completely proved. ∎

**Remark 8.** *The inequality* (5.14) *is equivalent to*

$$
\begin{aligned}
&\left|\Pr(X \geq x) - \frac{E(X)-a}{b-a}\right| \qquad (5.18)\\
&\leq \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}} + \left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right] \\
&\leq \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}, \qquad (\forall)\, x \in [a,b].
\end{aligned}
$$

**Corollary 20.** *Under the above assumptions, we have the double inequality*

$$b - \frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}} \leq E(X) \leq a + \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}+1}. \tag{5.19}$$

*Proof.* We know that $a \leq E(X) \leq b$.

Now, choose in $(5.14)$ $x = a$ to get

$$\left|\frac{b-E(X)}{b-a}\right| \leq \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}$$

i.e.,

$$b - E(X) \leq \frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}}$$

which is equivalent to the first inequality in $(5.19)$.

Also, choosing $x = b$ in $(5.14)$, we get

$$\left|1 - \frac{b - E(X)}{b-a}\right| \le \frac{q}{q+1} \|f\|_p (b-a)^{\frac{1}{q}}$$

i.e.,

$$E(X) - a \le \frac{q}{q+1} \|f\|_p (b-a)^{\frac{1}{q}+1}$$

which is equivalent to the second inequality in $(5.19)$. ∎

**Remark 9.** *We know that by Hölder's integral inequality*

$$1 = \int_a^b f(t)\,dt \le (b-a)^{\frac{1}{q}} \|f\|_p$$

*which gives*

$$\|f\|_p \ge \frac{1}{(b-a)^{\frac{1}{q}}}.$$

*Now, if we assume that $\|f\|_p$ is not too large, i.e.,*

$$\|f\|_p \le \frac{q+1}{q} \cdot \frac{1}{(b-a)^{\frac{1}{q}}} \tag{5.20}$$

*then we get*

$$a + \frac{q}{q+1} \|f\|_p (b-a)^{\frac{1}{q}+1} \le b$$

*and*

$$b - \frac{q}{q+1} \|f\|_p (b-a)^{1+\frac{1}{q}} \ge a$$

*which shows that the inequality* (5.19) *is a tighter inequality than $a \le E(X) \le b$ when* (5.20) *holds.*

Another equivalent inequality to (5.19) which can be more useful in practice is the following one:

**Corollary 21.** *With the above assumptions, we have the inequality:*

$$\left|E(X) - \frac{a+b}{2}\right| \le (b-a)\left[\frac{q}{q+1} \|f\|_p (b-a)^{\frac{1}{q}} - \frac{1}{2}\right]. \tag{5.21}$$

*Proof.* From the inequality (5.19) we have:

$$b - \frac{a+b}{2} - \frac{q}{q+1} \|f\|_p (b-a)^{1+\frac{1}{q}} \le E(X) - \frac{a+b}{2}$$

$$\le a - \frac{a+b}{2} + \frac{q}{q+1} \|f\|_p (b-a)^{1+\frac{1}{q}}.$$

That is,

$$\frac{b-a}{2} - \frac{q}{q+1} \|f\|_p (b-a)^{1+\frac{1}{q}} \le E(X) - \frac{a+b}{2}$$

$$\le -\frac{b-a}{2} + \frac{q}{q+1} \|f\|_p (b-a)^{1+\frac{1}{q}}$$

which is equivalent to

$$\begin{aligned}\left|E(X)-\frac{a+b}{2}\right| &\leq \frac{q}{q+1}\|f\|_p(b-a)^{1+\frac{1}{q}}-\frac{b-a}{2} \\ &= (b-a)\left[\frac{q}{q+1}\|f\|_p(b-a)^{\frac{1}{q}}-\frac{1}{2}\right]\end{aligned}$$

and the inequality (5.21) is proved. ∎

This corollary provides the possibility of finding a sufficient condition in terms of $\|f\|_p$ $(p>1)$ for the expectation $E(X)$ to be close to the mean value $\frac{a+b}{2}$.

**Corollary 22.** *Let $X$ and $f$ be as above and $\varepsilon>0$. If*

$$\|f\|_p \leq \frac{q+1}{2q}\cdot\frac{1}{(b-a)^{\frac{1}{q}}}+\frac{\varepsilon(q+1)}{q(b-a)^{1+\frac{1}{q}}}$$

*then*

$$\left|E(X)-\frac{a+b}{2}\right|\leq\varepsilon.$$

The proof is similar, and we omit the details.

The following corollary of Theorem 27 also holds:

**Corollary 23.** *Let $X$ and $f$ be as above. Then we have the inequality:*

$$\begin{aligned}&\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}\right| \\ &\leq \frac{q}{2^{\frac{1}{q}}(q+1)}\|f\|_p(b-a)^{\frac{1}{q}}+\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right|.\end{aligned}$$

*Proof.* If we choose in (5.14) $x=\frac{a+b}{2}$, we get

$$\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{b-E(X)}{b-a}\right|\leq\frac{q}{2^{\frac{1}{q}}(q+1)}\|f\|_p(b-a)^{\frac{1}{q}}$$

which is clearly equivalent to:

$$\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|\leq\frac{q}{2^{\frac{1}{q}}(q+1)}\|f\|_p(b-a)^{\frac{1}{q}}.$$

Now, using the triangle inequality, we get

$$\begin{aligned}&\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}\right| \\ &=\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)-\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right| \\ &\leq\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|+\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right|\end{aligned}$$

$$\leq \frac{q}{2^{\frac{1}{q}}(q+1)} \|f\|_p (b-a)^{\frac{1}{q}} + + \frac{1}{b-a} \left| E(X) - \frac{a+b}{2} \right|$$

and the corollary is proved. ∎

Finally, the following result also holds:

**Corollary 24.** *With the above assumptions, we have:*

$$\left| E(X) - \frac{a+b}{2} \right|$$

$$\leq \frac{q}{2^{\frac{1}{q}}(q+1)} \|f\|_p (b-a)^{1+\frac{1}{q}} + (b-a) \left| \Pr\left( X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right|.$$

The proof is similar and we omit the details.

A Beta Random Variable $X$ with parameters $(s,t) \in \Omega$ has the probability density function

$$f(x;s,t) := \frac{x^{s-1}(1-x)^{t-1}}{\beta(s,t)}; \quad 0 < x < 1$$

where

$$\Omega := \{(s,t) : s,t > 0\}$$

and

$$\beta(s,t) := \int_0^1 \tau^{s-1}(1-\tau)^{t-1} d\tau.$$

We observe that, for $p > 1$,

$$\begin{aligned}
\|f(\cdot;s,t)\|_p &= \frac{1}{\beta(s,t)} \left( \int_0^1 \tau^{p(s-1)} (1-\tau)^{p(t-1)} d\tau \right)^{\frac{1}{p}} \\
&= \frac{1}{\beta(s,t)} \left( \int_0^1 \tau^{p(s-1)+1-1} (1-\tau)^{p(t-1)+1-1} d\tau \right)^{\frac{1}{p}} \\
&= \frac{1}{\beta(s,t)} \left[ \beta(p(s-1)+1, p(t-1)+1) \right]^{\frac{1}{p}}
\end{aligned}$$

provided

$$p(s-1)+1, \quad p(t-1)+1 > 0$$

i.e.,

$$s > 1 - \frac{1}{p} \text{ and } t > 1 - \frac{1}{p}.$$

Now, using Theorem 27, we can state the following proposition:

**Proposition 3.** *Let $p > 1$ and $X$ be a Beta random variable with the parameters $(s, t)$, $s > 1 - \frac{1}{p}, t > 1 - \frac{1}{p}$. Then we have the inequality:*

$$\left| \Pr(X \leq x) - \frac{t}{s+t} \right| \tag{5.22}$$
$$\leq \frac{q}{q+1} \frac{\left[ x^{\frac{1+q}{q}} + (1-x)^{\frac{1+q}{q}} \right] [\beta(p(s-1)+1, p(t-1)+1)]^{\frac{1}{p}}}{\beta(s,t)}$$

*for all $x \in [0,1]$.*
*Particularly, we have*

$$\left| \Pr\left( X \leq \frac{1}{2} \right) - \frac{t}{s+t} \right| \leq \frac{q}{2^{\frac{1}{q}}(q+1)} \frac{[\beta(p(s-1)+1, p(t-1)+1)]^{\frac{1}{p}}}{\beta(s,t)}.$$

The proof follows by Theorem 27 choosing $f(x) = f(x; s, t)$, $x \in [0,1]$ and taking into account that $E(X) = \frac{s}{s+t}$.

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