ON A BOJANIĆ–STANOJEVIĆ TYPE INEQUALITY
AND ITS APPLICATIONS

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Abstract. An extension of the Bojanić–Stanojević type inequality [1] is made, considering
the r-th derivative of the Dirichlet’s kernel \( D_k^{(r)} \) instead of \( D_k \).
Namely, the following inequality is proved:

\[
\left\| \sum_{k=1}^{n} \alpha_k D_k^{(r)}(x) \right\|_1 \leq M_p n^{r+1} \left( \frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^p \right)^{1/p},
\]

where \( \| \cdot \|_1 \) is the \( L^1 \)-norm, \( \{ \alpha_k \} \) is a sequence of real numbers, \( 1 < p \leq 2, \ r = 0, 1, 2, \ldots \) and \( M_p \)
is an absolute constant dependents only on \( p \). As an application of this inequality, it is shown that
the class \( F_{pr} \) is a subclass of \( BV \cap C_r \), where \( F_{pr} \) is the extension of the Fomin’s class, \( C_r \) is the
extension of the Garrett–Stanojević class [7] and \( BV \) is the class of all null sequences of bounded
variation.

1. Introduction.

Sidon [5] proved the inequality named after him in 1939 year. It is an upper estimate
for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed
in terms of the coefficients. Since the estimate has many applications for instance in \( L^1 \)-
convergence problems and summation methods with respects to trigonometric series, newer
and newer improvements of the original inequality has been proved by several authors.

Fomin [3] applying the linear method for summing of Fourier series has given another
proof of this inequality. Thus the inequality is called as Sidon-Fomin’s inequality. Also,
S. A. Telyakovskii in [6] has given an elegant proof of Sidon-Fomin’s inequality.

Lemma 1 (Sidon-Fomin). Let \( \{ \alpha_k \}_{k=0}^{n} \) be a sequence of real numbers such that
\( |\alpha_k| \leq 1 \) for all \( k \). Then there exists a positive constant \( M \) such that for any \( n \geq 1, \)

\[
\left\| \sum_{k=0}^{n} \alpha_k D_k(x) \right\|_1 \leq M(n + 1).
\]  

(1.1)

In [8] we extended this result and we given two different proof of the following lemma.

Lemma 2 [8]. Let \( \{ \alpha_j \}_{j=0}^{k} \) be a sequence of real numbers such that \( |\alpha_k| \leq 1 \) for all \( k \). Then there exists a positive constant \( M > 0 \), such that for any \( n \geq 1, \)

\[
\left\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \right\|_1 \leq M(n + 1)^{r+1}.
\]  

(1.2)

But Bojanić and Stanojević [1] proved the following more general inequality of (1.1).

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\( L^1 \)-convergence cosine series.
Lemma 3 [1]. Let \( \{\alpha_i\}_{i=0}^n \) be a sequence of real numbers. Then for any \( 1 < p \leq 2 \) and \( n \geq 1 \)
\[
\left\| \sum_{k=0}^{n} \alpha_k D_k(x) \right\|_1 \leq M_p(n + 1) \left( \frac{1}{n+1} \sum_{k=1}^{n} |\alpha_k|^p \right)^{1/p},
\]
where the constant \( M_p \) dependents only on \( p \).

We note that this estimate is essentially contained (case \( p = 2 \)) in Fomin [3]. Sidon-Fomin’s inequality is a special case of the Bojanić-Stanojević inequality, i.e. it can easily be deduced from Lemma 3.

It is easy to see that Bojanić-Stanojević inequality is not valid for \( p = 1 \). Indeed, if \( \alpha_n = 1 \) and \( \alpha_k = 0 \) \((k \neq n, k \in \mathbb{N})\) then the left side is of order \( \frac{\log n}{n} \) while the right side is of order \( \frac{1}{n} \) as \( n \to \infty \).

For the proof of our new results we need the following lemma.

Lemma 4 [9]. If \( T_n(x) \) is a trigonometrical polynomial of order \( n \), then
\[
\| T_n^{(r)} \| \leq n^r \| T_n \|.
\]
This is S. Bernstein’s inequality in the \( L^1(0, \pi) \)-metric (see [9], Vol.2, p.11).

2. Main result

Now we will prove a counterpart of inequality (1.3) in the case where the \( r \)-th derivate of the Dirichlet’s kernel \( D_k^{(r)} \) is used instead of \( D_k \).

Lemma 5. Let \( \{\alpha_k\}_{k=1}^n \) be a sequence of real numbers. Then for any \( 1 < p \leq 2 \) and \( r = 0, 1, 2, \ldots, n \in \mathbb{N} \) the following inequality holds:
\[
\left\| \sum_{k=1}^{n} \alpha_k D_k^{(r)}(x) \right\|_1 \leq M_p n^{r+1} \left( \frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^p \right)^{1/p},
\]
where the constant \( M_p \) dependents only on \( p \).

Proof. Without loss of generality, we may assume that \( n \) is of the form \( n = 2^m - 1 \) with some \( m \geq 1 \). Let \( j \geq 1 \). Applying first Bernstein’s inequality, then Bojanić-Stanojević inequality, yields
\[
\left\| \sum_{k=2^j-1}^{2^j-1} \alpha_k D_k^{(r)}(x) \right\| \leq (2^{j-1})^r \left\| \sum_{k=2^j-1}^{2^j-1} \alpha_k D_k \right\| \leq (2^{j-1})^r M_p n^{r+1} \left( \frac{1}{n} \sum_{k=2^j-1}^{2^j-1} |\alpha_k|^p \right)^{1/p}.
\]
Continuing by making use of the triangle inequality, then Holder’s inequality with the exponents \( p \) and \( q, \frac{1}{p} + \frac{1}{q} = 1 \), we get:
\[
\left\| \sum_{k=1}^{2^m-1} \alpha_k D_k^{(r)}(x) \right\| \leq \sum_{j=1}^{m} \left\| \sum_{k=2^j-1}^{2^j-1} \alpha_k D_k^{(r)}(x) \right\| \leq \sum_{j=1}^{m} (2^{j-1})^r M_p n^{r+1} \left( \frac{1}{n} \sum_{k=2^j-1}^{2^j-1} |\alpha_k|^p \right)^{1/p}.
\]
\[
M_p \left( \sum_{j=1}^{m} (2^{j-1})^r q 2^{(j-1)(1-1/p)q} \right)^{1/q} \left( \sum_{j=1}^{m} \sum_{k=2^{j-1}}^{2^j-1} |\alpha_k|^p \right)^{1/p} \leq M_p (2^{m-1})^r \left( \sum_{j=1}^{m} 2^{j-1} \right)^{1/q} \left( \sum_{k=1}^{2^m-1} |\alpha_k|^p \right)^{1/p} \leq M_p (2^{m-1})^r (2^m - 1) \left( \sum_{k=1}^{2^{m-1}} |\alpha_k|^p \right)^{1/p} = M_p n^{r+1} \left( \frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^p \right)^{1/p}.
\]

It is easy to see that the inequality (1.2) is a special case of the inequality (2.1), i.e. it can easily be deduced from Lemma 5.

3. Application

The problem of $L^1$-convergence, via Fourier coefficient, consists of finding the properties of Fourier coefficients such that the necessary and sufficient condition for $\|S_n - f\| = o(1)$, $n \to \infty$ is given in the form $a_n \log n = o(1)$, $n \to \infty$. Here $S_n$ is the partial sums of the cosine series

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos nx.
\]

Sidon-Telyakovskii class $S$ [6] is a classical example for which condition $a_n \log n = o(1)$, $n \to \infty$ is equivalent to $\|S_n - f\| = o(1)$, $n \to \infty$. Later Fomin [2] have extended the Sidon-Telyakovskii class. He defined a class $F_p$, $p > 1$ of Fourier coefficients as follows: a sequence $\{a_k\}$ belongs to $F_p$, $p > 1$ if $a_k \to 0$ as $k \to \infty$ and

\[
\sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=k}^{\infty} |\Delta a_i|^p \right)^{1/p} < \infty.
\]  

(3.1)

We note that Fomin [2] has given an equivalent form of the condition (3.1). Namely, he proved that $\{a_n\} \in F_p$, $p > 1$ iff $\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} < \infty$, where

\[
\Delta_s^{(p)} = \left\{ \frac{1}{2^{s-1}} \sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p \right\}^{1/p}.
\]

Let $BV$ denote the class of null sequence $\{a_n\}$ of bounded variation, i.e. $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$.

The class $C$ was defined by Garrett and Stanojević [4] as follows: a null sequence of real numbers satisfy the condition $C$ if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ independent of $n$, such that

\[
\int_{0}^{\delta} \sum_{k=n}^{\infty} |\Delta a_k D_k(x)| \, dx < \varepsilon, \quad \text{for every } n.
\]
On the other hand, Stanojević [10] proved the following inclusion between the classes $F_p$, $C$ and $BV$.

**Theorem 1** [10]. For all $1 < p \leq 2$ the following inclusion holds $F_p \subset BV \cap C$.

In [7] we defined an extension $C_r$, $r = 0, 1, 2, \ldots$, of the Garrett-Stanojević class. Namely, a null sequence $\{a_k\}$ belongs to the class $C_r$, $r = 0, 1, 2, \ldots$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$
\int_0^\delta \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) < \varepsilon, \quad \text{for all } n.
$$

When $r = 0$, we denote $C_r = C$.

Denote by $I_m$ the dyadic interval $[2^{m-1}, 2^m)$, for $m \geq 1$. A null sequence $\{a_n\}$ belongs to the class $F_{pr}$, $p > 1$, $r = 0, 1, 2, \ldots$ if

$$
\sum_{m=1}^\infty 2^{m(1/q+r)} \left( \sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} < \infty, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.
$$

It is obvious that $F_{pr} \subset F_p$. For $r = 0$, we obtain the Fomin’s class $F_p$.

**Theorem 2.** For all $1 < p \leq 2$ and $r = 0, 1, 2, \ldots$ the following inclusion holds $F_{pr} \subset BV \cap C_r$.

**Proof.** By theorem 1, it is clear that $F_{pr} \subset BV$. It suffices to show that

$$
\left\| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) \right\| = o(1), \quad n \to \infty.
$$

Since

$$
\sum_{m=1}^\infty 2^{m(1/q+r)} \left( \sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} = 2 \sum_{m=1}^\infty \left\{ 2^{(m-1)((r+1)p-1)} \sum_{k \in I_m} |\Delta a_k|^p \right\}^{1/p},
$$

we have:

$$
\sum_{k=1}^\infty k^{(r+1)p-1} |\Delta a_k|^p < \infty.
$$

Applying the Lemma 6, we obtain

$$
\left\| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) \right\| \leq M_p \left( \sum_{k=n}^\infty k^{(r+1)p-1} |\Delta a_k|^p \right) = o(1), \quad n \to \infty.
$$

**References**


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