SOME ELEMENTARY INEQUALITIES FOR THE EXPECTATION
AND VARIANCE OF A RANDOM VARIABLE WHOSE PDF IS
DEFINED ON A FINITE INTERVAL

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Abstract. Some elementary inequalities for the expectation and variance of
a continuous random variable whose pdf is defined on a finite interval are
obtained using some standard and recent results from the theory of inequalities.

1. Introduction

Let $X$ be a continuous random variable having the probability density function
$f$ defined on a finite interval $[a, b]$.
By definition

$$E(X) := \int_a^b t f(t) \, dt$$

the expectation of $X$, and

$$\sigma^2(X) := \int_a^b (t - E(X))^2 f(t) \, dt$$

the variance of $X$.

Using some tools from the theory of inequalities, namely Hölder’s inequality,
pre-Grüss inequality, pre-Chebychev inequality, Taylor’s formula with the integral
remainder, we point out some elementary inequalities for the expectation and variance.

2. The Results

Theorem 1. Let $X$ be a continuous random variable defined on $[a, b]$ having p.d.f.,
f. Then
(i) we have the inequality

$$0 \leq \sigma(X) \leq [b - E(X)]^{\frac{1}{2}} [E(X) - a]^{\frac{1}{2}} \leq \frac{1}{2} (b - a)$$

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and

\begin{equation}
0 \leq [b - E(X)] [E(X) - a] - \sigma^2(X)
\end{equation}

\begin{equation}
\begin{aligned}
&\leq \begin{cases}
\frac{(b-a)^3}{6} \|f\|_\infty \\
[B(q+1,q+1)]^{\frac{1}{q}} (b-a)^{2+\frac{1}{q}} \|f\|_p
\end{cases}
\end{aligned}
\end{equation}

where \( B(\cdot, \cdot) \) is Euler’s Beta function.

(ii) If \( m \leq f \leq M \) a.e. on \([a, b]\), then

\begin{equation}
\frac{m(b-a)^3}{6} \leq [b - E(X)] [E(X) - a] - \sigma^2(X) \leq \frac{M(b-a)^3}{6}
\end{equation}

and

\begin{equation}
\left| [b - E(X)] [E(X) - a] - \sigma^2(X) - \frac{(b-a)^2}{6} \right| \leq \frac{\sqrt{5} (b-a)^3 (M-m)}{60}.
\end{equation}

Proof. Note that:

\begin{equation}
\int_a^b (b-t) (t-a) f(t) \, dt
= \int_a^b [(b - E(X)) + (E(X) - t)] [(E(X) - a) + (t - E(X))] f(t) \, dt
= (b - E(X)) (E(X) - a) \int_a^b f(t) \, dt + (E(X) - a) \int_a^b (E(X) - t) f(t) \, dt
+ (b - E(X)) \int_a^b (t - E(X)) f(t) \, dt - \int_a^b (t - E(X))^2 f(t) \, dt
= [b - E(X)] [E(X) - a] - \sigma^2(X)
\end{equation}

since

\[ \int_a^b f(t) \, dt = 1 \quad \text{and} \quad \int_a^b (t - E(X)) f(t) \, dt = 0. \]

(i) Using the fact that

\[ \int_a^b (t - a) (b-t) f(t) \, dt \geq 0, \]

it follows that

\[ \sigma^2(X) \leq [b - E(X)] [E(X) - a] \]

and so the first inequality in (2.1) is established.

The second inequality in (2.1) follows from the elementary result that

\[ \alpha \beta \leq \frac{1}{4} (\alpha + \beta)^2, \quad \alpha, \beta \in \mathbb{R} \]
where \( \alpha = b - E(X), \beta = E(X) - a \).

The first inequality in (2.2) follows, since
\[
\int_a^b (t - a) (b - t) f(t) \, dt \leq \|f\|_\infty \int_a^b (t - a) (b - t) \, dt
\]
\[
= \frac{(b - a)^3}{6} \|f\|_\infty.
\]

The second inequality is obvious by Hölder’s integral inequality,
\[
\int_a^b (t - a) (b - t) f(t) \, dt \leq \left( \int_a^b f^p(t) \, dt \right)^{\frac{1}{p}} \left( \int_a^b (t - a)^q (b - t)^q \, dt \right)^{\frac{1}{q}}
\]
\[
= \|f\|_p (b - a)^{2+\frac{1}{q}} [B(q + 1, q + 1)]^{\frac{1}{q}}.
\]

(ii) The inequality (2.3) is obvious, taking into account that if \( m \leq f \leq M \) a.e. on \([a, b]\), then \( m (t - a) (b - t) \leq (t - a) (b - t) f(t) \leq M (t - a) (b - t) \) a.e. on \([a, b]\), and by integrating over \([a, b]\),

To prove (2.4), we use the following “pre-Grüss” inequality established in [1]

\[
\left| \frac{1}{b - a} \int_a^b h(t) g(t) \, dt - \frac{1}{b - a} \int_a^b h(t) \, dt \cdot \frac{1}{b - a} \int_a^b g(t) \, dt \right|
\]
\[
\leq \frac{1}{2} (\phi - \gamma) \left[ \frac{1}{b - a} \int_a^b g^2(t) \, dt - \left( \frac{1}{b - a} \int_a^b g(t) \, dt \right)^2 \right]^{\frac{1}{2}},
\]

provided that the mappings \( h, g : [a, b] \rightarrow \mathbb{R} \) are measurable, all the integrals involved in (2.6) exist and are finite and \( \gamma \leq h \leq \phi \) a.e. on \([a, b]\).

Choose in (2.6), \( h(t) = f(t) \) and \( g(t) = (t - a) (b - t) \), which then gives

\[
\left| \frac{1}{b - a} \int_a^b (t - a) (b - t) f(t) \, dt
\]
\[
- \frac{1}{b - a} \int_a^b (t - a) (b - t) \, dt \cdot \frac{1}{b - a} \int_a^b f(t) \, dt \right|
\]
\[
\leq \frac{1}{2} (M - m) \left[ \frac{1}{b - a} \int_a^b (t - a)^2 (b - t)^2 \, dt
\]
\[
- \left( \frac{1}{b - a} \int_a^b (t - a) (b - t) \, dt \right) \right]^{\frac{1}{2}}.
\]

However,
\[
\int_a^b (t - a) (b - t) \, dt = \frac{(b - a)^3}{6}, \quad \int_a^b f(t) \, dt = 1,
\]
\[
\int_a^b (t - a)^2 (b - t)^2 \, dt = (b - a)^5 \int_0^1 t^2 (1 - t)^2 \, dt = \frac{(b - a)^5}{30}
\]
\[ \int_{a}^{b} (t-a)^2 (b-t)^2 dt - \left( \int_{a}^{b} (t-a) (b-t) dt \right)^2 \]

\[ = \frac{(b-a)^4}{30} - \frac{(b-a)^4}{36} = \frac{(b-a)^4}{180} \]

Consequently, by (2.7), we deduce that

\[ \left| \int_{a}^{b} (t-a) (b-t) f(t) dt - \frac{(b-a)^2}{6} \right| \leq \frac{1}{2} (b-a) (M-m) \left( \frac{(b-a)^4}{180} \right)^{\frac{1}{2}} \]

\[ = \frac{(b-a)^3 (M-m)}{12\sqrt{5}}. \]

Using (2.5), we deduce (2.4).

**Remark 1.** For a different proof of the inequality (2.1) see [2].

With additional information about the derivative of \( f \), we can state the following result which complements (2.4).

**Theorem 2.** Assume that the p.d.f. of \( X \) is absolutely continuous on \([a, b]\).

(i) If \( f' \in L_{\infty}[a, b] \), then we have:

\[ \left| b - E(X) \right| |E(X) - a| - \sigma^2(X) - \frac{(b-a)^2}{6} \leq \frac{\sqrt{30}}{720} \| f' \|_{\infty} (b-a)^3. \]

(ii) If \( f' \in L_{2}[a, b] \), then we have:

\[ \left| b - E(X) \right| |E(X) - a| - \sigma^2(X) - \frac{(b-a)^2}{6} \leq \frac{\sqrt{5}}{60\pi} \| f' \|_{2} (b-a)^3. \]

**Proof.** (i) Use is made of the following “pre-Chebychev” inequality proved in [1],

\[ \left| \frac{1}{b-a} \int_{a}^{b} h(t) g(t) dt - \frac{1}{b-a} \int_{a}^{b} h(t) dt \cdot \frac{1}{b-a} \int_{a}^{b} g(t) dt \right| \leq \frac{1}{2\sqrt{3}} \| h' \|_{\infty} \left[ \frac{1}{b-a} \int_{a}^{b} g^2(t) dt - \left( \frac{1}{b-a} \int_{a}^{b} g(t) dt \right)^2 \right]^{\frac{1}{2}}. \]

Provided that \( h, g : [a, b] \to \mathbb{R} \) are measurable on \([a, b]\), the integrals involved in (2.10) exist and are finite, \( h \) is absolutely continuous and \( h' \in L_{\infty}[a, b] \).

Now, if we choose \( h(t) = f(t), g(t) = (t-a) (b-t) \) in (2.10), we get

\[ \left| \int_{a}^{b} (t-a) (b-t) f(t) dt - \frac{(b-a)^2}{6} \right| \leq \frac{\| h' \|_{\infty} (b-a)}{2\sqrt{3}} \cdot \frac{(b-a)^2}{12\sqrt{5}} \]

\[ = \frac{(b-a)^3 \| h' \|_{\infty}}{24\sqrt{30}}. \]

Using (2.5), we deduce (2.8).
For the second part of the theorem, we use the following “pre-Lupaş” inequality as stated in [1]

\[
\begin{align*}
\left| \frac{1}{b-a} \int_{a}^{b} h(t) g(t) \, dt - \frac{1}{b-a} \int_{a}^{b} h(t) \, dt \cdot \frac{1}{b-a} \int_{a}^{b} g(t) \, dt \right| \\
\leq \frac{b-a}{\pi} \|h'|_2 \left[\frac{1}{b-a} \int_{a}^{b} g^2(t) \, dt \right] & \leq \left( \frac{1}{b-a} \int_{a}^{b} g(t) \, dt \right)^2 \\
\end{align*}
\]

provided that \(g, h\) are as above and \(h' \in L_2[a, b]\).

Now if we choose in (2.11) \(h(t) = f(t), g(t) = (t-a)(b-t)\), we obtain the desired inequality (2.9). The details are omitted.

**Theorem 3.** Let \(X\) be a random variable and \(f : [a, b] \to \mathbb{R}\) its p.d.f. If \(f\) is such that \(f^{(n)}(n \geq 0)\) is absolutely continuous on \([a, b]\), then we have the inequality

\[
\left| [E(X) - a][b - E(X)] - \sigma^2(X) - \sum_{k=0}^{n} \frac{(k+1)(b-a)^{k+3}f^{(k)}(a)}{(k+3)!} \right|
\]

\[
\leq \begin{cases} \\
\frac{\|f^{(n+1)}\|_{\infty}}{(n+1)! (n+3)! (n+4)!} (b-a)^{n+4} & \text{if } f^{(n+1)} \in L_\infty[a, b] \\
\frac{\|f^{(n+1)}\|_p (b-a)^{n+3+\frac{1}{p}}}{n!(nq+1)^{\frac{1}{q}} (n+2+\frac{1}{q}) (n+3+\frac{1}{q})} & \text{if } f^{(n+1)} \in L_p[a, b], \ p > 1 \\
\frac{\|f^{(n+1)}\|_1 (b-a)^{n+3}}{n!(n+2)(n+3)} & \text{if } f^{(n+1)} \in L_1[a, b] \\
\end{cases}
\]

where \(\|\cdot\|_p (1 \leq p \leq \infty)\) are the usual Lebesgue norms on \([a, b]\), i.e.,

\[
\|g\|_\infty := \text{ess sup}_{t \in [a,b]} |g(t)|, \quad \|g\|_p := \left( \int_{a}^{b} |g(t)|^p \, dt \right)^{\frac{1}{p}}, \quad (p \geq 1).
\]

**Proof.** The following Taylor’s formula with integral remainder is well known in the literature (see for example [3]):

\[
f(t) = \sum_{k=0}^{n} \frac{(t-a)^k}{k!} f^{(k)}(a) + \frac{1}{n!} \int_{a}^{t} (t-s)^n f^{(n+1)}(s) \, ds
\]

for all \(t \in [a, b]\).

Since

\[
[E(X) - a][b - E(X)] - \sigma^2(X) = \int_{a}^{b} (t-a)(b-t) f(t) \, dt,
\]
then we have

\[
E(X) - a \mid [b - E(X)] - \sigma^2(X) = \int_a^b (t-a)(b-t) \left[ \sum_{k=0}^n \frac{(t-a)^k}{k!} f^{(k)}(a) + \frac{1}{n!} \int_a^t (t-s)^n f^{(n+1)}(s) \, ds \right] \, dt
\]

\[
= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} \int_a^b (t-a)^{k+1} (b-t) \, dt
\]

\[
+ \frac{1}{n!} \int_a^b \left[ (t-a)(b-t) \int_a^t (t-s)^n f^{(n+1)}(s) \, ds \right] \, dt.
\]

Using the transform, \( t = (1-u)a + ub \), we have

\[
\int_a^b (t-a)^{k+1} (b-t) \, dt = (b-a)^{k+3} \int_0^1 u^{k+1} (1-u) \, du = \frac{1}{(k+2)(k+3)}
\]

and by (2.15), we deduce that

\[
\left| E(X) - a \mid [b - E(X)] - \sigma^2(X) - \sum_{k=0}^n \frac{(k+1)(b-a)^{k+3} f^{(k)}(a)}{(k+3)!} \right|
\]

\[
\leq \frac{1}{n!} \int_a^b (t-a)(b-t) \left[ \int_a^t (t-s)^n f^{(n+1)}(s) \, ds \right] \, dt =: M(a,b).
\]

However, for all \( t \in [a,b] \) we have

\[
\left| \int_a^t (t-s)^n f^{(n+1)}(s) \, ds \right| \leq \int_a^t |t-s|^n \left| f^{(n+1)}(s) \right| \, ds
\]

\[
\leq \sup_{s \in [a,b]} \left| f^{(n+1)}(s) \right| \int_a^t (t-s)^n \, ds
\]

\[
\leq \left\| f^{(n+1)} \right\|_\infty \frac{(t-a)^{n+1}}{n+1}.
\]

By Hölder’s integral inequality we have,

\[
\left| \int_a^t (t-s)^n f^{(n+1)}(s) \, ds \right|
\]

\[
\leq \left( \int_a^t \left| f^{(n+1)}(s) \right|^p \, ds \right)^{\frac{1}{p}} \left( \int_a^t (t-s)^{nq} \, ds \right)^{\frac{1}{q}}
\]

\[
\leq \left\| f^{(n+1)} \right\|_p \left[ \frac{(t-a)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \ p > 1
\]

for all \( t \in [a,b] \).

Finally, we observe that

\[
\left| \int_a^t (t-s)^n f^{(n+1)}(s) \, ds \right| \leq \int_a^t (t-s)^n \left| f^{(n+1)}(s) \right| \, ds
\]

\[
\leq (t-a)^n \int_a^t \left| f^{(n+1)}(s) \right| \, ds
\]

\[
\leq (t-a)^n \left\| f^{(n+1)} \right\|_1.
\]
for all $t \in [a, b]$.

Consequently,

$$M(a, b) \leq \frac{1}{n!} \times \left\{ \frac{\|f^{(n+1)}\|_n}{n+1} \int_a^b (t-a)^{n+2} (b-t) \, dt \right\}$$

$$= \left\{ \frac{\|f^{(n+1)}\|_n}{(nq+1)^q} \int_a^b (t-a)^{n+1+\frac{1}{q}} (b-t) \, dt \right\}$$

and as

$$\int_0^1 u^{n+2} (1-u) \, du = \frac{1}{(n+3)(n+4)}$$

$$\int_0^1 u^{n+1+\frac{1}{q}} (1-u) \, du = \frac{1}{\left(n + 2 + \frac{1}{q}\right) \left(n + 3 + \frac{1}{q}\right)}$$

and

$$\int_0^1 u^{n+1} (1-u) \, du = \frac{1}{(n+2)(n+3)}$$

the inequality (2.12) is proved. 

**Remark 2.** A similar result can be obtained if use is made of a Taylor expansion around the point $b$.

**References**


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