Midpoint Type Rules from an Inequalities Point of View

P.C. Cerone and S.S. Dragomir

Abstract. The article investigates interior point rules which contain the midpoint as a special case, and obtains explicit bounds through the use of a Peano kernel approach and the modern theory of inequalities. Thus the simplest open Newton-Cotes rules are examined. Both Riemann-Stieltjes and Riemann integrals are evaluated with a variety of assumptions about the integrand enabling the characterisation of the bound in terms of a variety of norms. Perturbed quadrature rules are obtained through the use of Grüss, Chebychev and Lupaş inequalities, producing a variety of tighter bounds. The implementation is demonstrated through the investigation of a variety of composite rules based on inequalities developed. The analysis allows the determination of the partition required that would assure that the accuracy the result would be within a prescribed error tolerance. It is demonstrated that the bounds of the approximations are equivalent to those obtained from a Peano kernel that produces Trapezoidal type rules.

1. Introduction

The following inequality is well known in the literature as the midpoint inequality:

$$\left|\int_a^b f(x)\,dx-(b-a)\,f\left(\frac{a+b}{2}\right)\right|\le\frac{(b-a)^3}{24}\left\|f''\right\|_\infty, \tag{1.1}$$

where the mapping $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is assumed to be twice differentiable on the interval (a,b) and having the second derivative bounded on (a,b). That is, $\|f''\|_\infty:=\sup_{x\in(a,b)}|f''(x)|<\infty$.

Now, if we assume that $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(x)\,dx$ by the *midpoint quadrature* formula $A_M(f,I_n)$ having an error given by $R_M(f,I_n)$, where

$$A_M(f,I_n)=\sum_{i=0}^{n-1} f\left(\frac{x_{i+1}+x_i}{2}\right)h_i \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 41A55; Secondary 26D15.
Key words and phrases. Interior point type rules, Analytic Inequalities, *A priori* bounds.

and the remainder $R_M(f, I_n)$ satisfies the estimation

$$|R_M(f, I_n)| \le \frac{\|f''\|_\infty}{24} \sum_{i=0}^{n-1} h_i^3, \tag{1.3}$$

where $h_i = x_{i+1} - x_i$ for $i = 0, 1, 2, ..., n-1$.

Equation (1.2) is known as the midpoint rule for $n = 1$ and as the composite midpoint rule for $n > 1$. The midpoint rule is the most basic open Newton-Cotes quadrature in which function evaluations occur at the midpoints of equispaced intervals.

The current work investigates an interior point (which contains the midpoint as a special case) and obtains explicit bounds through the use of a Peano kernel approach and the modern theory of inequalities. This approach allows for the investigation of quadrature rules that place fewer restrictions on the behaviour of the integrand and thus allow us to cope with larger classes of functions. Expression (1.1) relies on the behaviour of the second derivative whereas bounds for the interior point are obtained in terms of Riemann-Stieltjes integrals in Sections 2, 3 and 4 for functions that are of bounded variation, Lipschitzian and monotonic respectively. In Section 5, interior point rules are obtained for $f^{(n)} \in L_p[a, b]$, implying that

$$\left\|f^{(n)}\right\|_p := \left(\int_a^b \left|f^{(n)}(x)\right|^p dx\right)^{\frac{1}{p}} < \infty \text{ for } p \ge 1$$

and $\left\|f^{(n)}\right\|_\infty := \sup_{x\in[a,b]} \left|f^{(n)}(x)\right|$.
Further, a generalised Taylor series representation is presented that enables an expansion about any point on an interval.

In Section 6, perturbed interior point rules are obtained using what are termed as **premature** variants of Grüss, Chebychev and Lupaş inequalities. Atkinson [**30**] uses an asymptotic error estimate technique to obtain what he defines as a corrected rule. His approach, however, does not readily produce a bound on the error.

Further, in 6.2, alternate Grüss type results are obtained to produce perturbed interior point rules with bounds given in terms of norms associated with $f'(x) - S$, where $S = \frac{f(b)-f(a)}{b-a}$ is the secant slope.

Finally, in Section 7, a perturbed interior point rule is obtained whose perturbation involves S and not $\frac{f'(b)-f'(a)}{b-a}$. The bound relies on the behaviour of $f''(\cdot)$.

The current work brings together results for interior point type rules giving explicit error bounds, using Peano type kernels and results from the modern theory of inequalities. Although bounds through the use of Peano kernels have been obtained in some classical review books on numerical integration such as Stroud [**35**], Engels [**34**], and Davis and Rabinowitz [**33**], these do not seem to be utilised to perhaps the extent that they should be. So much so that even in the more recent comprehensive monograph by Krommer and Ueberhuber [**34**], a constructive approach is taken via Taylor or interpolating polynomials to obtain quadrature results. This approach does not readily provide explicit error bounds but rather gives the order of the approximation.

2. The Ostrowski Inequality for Mappings of Bounded Variation

In this section we develop interior point type quadrature rules for functions that are of bounded variation. It includes the midpoint rule as a special case. Functions of bounded variation include a very large class in contrast to traditional interior or specifically midpoint rules which rely on the second derivative of the function for its error approximation.

2.1. Some Integral Inequalities. The following inequality for mappings of bounded variation holds [**2**]:

THEOREM 1. *Let $u : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then for all $x \in [a,b]$, we have the inequality*

$$\left|\int_a^b u(t)\,dt - (b-a)\,u(x)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u) \ , \tag{2.1}$$

where $\bigvee_a^b (u)$ denotes the total variation of u.
The constant $\frac{1}{2}$ is the best possible one.

PROOF. Using the integration by parts formula for Riemann-Stieltjes integrals we have

$$\int_a^x (t-a)\,du(t) = u(x)(x-a) - \int_a^x u(t)\,dt$$

and

$$\int_x^b (t-b)du(t) = u(x)(b-x) - \int_x^b u(t)\,dt.$$

If we add the above two equalities, we obtain

$$(b-a)\,u(x) - \int_a^b u(t)\,dt = \int_a^b p(x,t)\,du(t) \ , \tag{2.2}$$

where

$$p(x,t) := \begin{cases} t-a \text{ if } t \in [a,x) \\ t-b \text{ if } x \in [x,b] \end{cases},$$

for all $x, t \in [a,b]$.
It is well known [**30**] that if $p : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then

$$\left|\int_a^b p(x)\,dv(x)\right| \le \sup_{x\in[a,b]} |p(x)| \bigvee_a^b (v). \tag{2.3}$$

Applying the inequality (2.3) for $p(x,t)$ as above and $v(x) = u(x)$, $x \in [a,b]$, we get

$$\left|\int_a^b p(x,t)\,du(t)\right| \le \sup_{t\in[a,b]} |p(x,t)| \bigvee_a^b (u) = \max\{x-a, b-x\} \bigvee_a^b (u) = \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u) \tag{2.4}$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1). Now to prove that $\frac{1}{2}$ is the best possible constant assume that the inequality (2.1) holds with a constant $C > 0$. That is,

$$\left|\int_a^b u(t)\,dt - u(x)(b-a)\right| \le \left[C(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u) \tag{2.5}$$

for all $x \in [a,b]$.
Consider the mapping $u : [a,b] \to \mathbb{R}$, given by

$$u(x) = \begin{cases} 0 \text{ if } x \in [a,b] \setminus \{\frac{a+b}{2}\} \\ 1 \text{ if } x = \frac{a+b}{2} \end{cases}$$

in (2.5). Then u is of bounded variation on $[a,b]$, and

$$\bigvee_a^b (u) = 2, \int_a^b u(t)\,dt = 0\,.$$

For $x = \frac{a+b}{2}$, we get in (2.5)

$$1 \le 2C\,,$$

which implies that $C \ge \frac{1}{2}$ and the theorem is completely proved. ∎

The following corollary holds for monotonic mappings.

Corollary 1. *Let $u : [a,b] \to \mathbb{R}$ be a monotonic mapping on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b u(t)\,dt - (b-a)\,u(x)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] |u(b) - u(a)|\,.$$

The case of Lipschitzian mappings is embodied in the following corollary.

Corollary 2. *Let $u : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$. That is, we recall*

$$|u(x) - u(y)| \le L\,|x-y| \text{ for all } x,y \in [a,b]\,.$$

Then, for all $x \in [a,b]$ we have the inequality

$$\left|\int_a^b u(t)\,dt - (b-a)\,u(x)\right| \le L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](b-a)\,,$$

giving a midpoint type rule.

COROLLARY 3. *Let $u : [a,b] \to \mathbb{R}$ be as above. Then we have the inequality:*

$$\left| \int_a^b u(t)\,dx - (b-a)\,u\left(\frac{a+b}{2}\right) \right| \le \frac{1}{2}(b-a) \bigvee_a^b (u). \tag{2.6}$$

Similar inequalities can be found if we assume that u is monotonic or Lipschitzian on $[a,b]$ by taking $x = \frac{a+b}{2}$ in Corollaries 1 and 2 respectively.

REMARK 1. *If we assume that u is continuous differentiable on (a,b) and u' is integrable on (a,b), then, by (2.1), we get*

$$\left| \int_a^b u(t)\,dx - (b-a)\,u(x) \right| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|u'\|_1 ,$$

which is the inequality obtained by Dragomir and Wang in the recent paper [**7**].

REMARK 2. *It is well known that if $f : [a,b] \to \mathbb{R}$ is a convex mapping on $[a,b]$, then the **Hermite-Hadamard** inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{2.7}$$

holds [**1**].
Now, if we assume that $f : I \subset \mathbb{R} \to \mathbb{R}$ is convex on I and $a,b \in Int(I)$, $a < b$; then f'_+ is monotonic nondecreasing on $[a,b]$ and, by Corollary 3, we obtain

$$0 \le \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \le \frac{1}{2} \left\| f'_+ \right\|_1 , \tag{2.8}$$

which gives a counterpart for the first membership of Hadamard's inequality.

Similar results can be obtained if we assume that f is convex and monotonic or convex and Lipschitzian on $[a,b]$.

2.2. A Quadrature Formula of Riemann Type. Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,...,n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f, I_n, \boldsymbol{\xi}) = \sum_{i=0}^{n-1} f(\xi_i)\,h_i ,$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula [**2**].

THEOREM 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and I_n, ξ_i $(i = 0,...,n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)\,dx = R_n(f, I_n, \boldsymbol{\xi}) + W_n(f, I_n, \boldsymbol{\xi}) , \tag{2.9}$$

where the remainder satisfies the estimation

$$\begin{aligned} |W_n(f, I_n, \boldsymbol{\xi})| &\leq \sup_{i=0,\ldots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\bigvee_a^b(f) \\ &\leq \left[\frac{1}{2}\nu(h) + \sup_{i=0,\ldots,n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\bigvee_a^b(f) \\ &\leq \nu(h)\bigvee_a^b(f) \end{aligned} \tag{2.10}$$

for all ξ_i $(i = 0, \ldots, n-1)$ as above, where $\nu(h) := \max\{h_i | i = 0, 1, \ldots, n\}$. The constant $\frac{1}{2}$ is sharp in (2.10).

Proof. Apply Theorem 1 on the interval $[x_i, x_{i+1}]$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i)\,h_i\right| \leq \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f). \tag{2.11}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$\begin{aligned} |W_n(f, I_n, \boldsymbol{\xi})| &\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i)\,h_i\right| \\ &\leq \sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f) \\ &\leq \sup_{i=0,\ldots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f) \\ &= \sup_{i=0,\ldots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\bigvee_a^b(f). \end{aligned}$$

The second inequality follows by the properties of $\sup(\cdot)$.
Now, as

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}h_i$$

for all $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, \ldots, n-1)$ the last part of (2.10) is also proved. ∎

Corollary 4. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic mapping on $[a, b]$ and I_n, ξ_i $(i = 0, \ldots, n-1)$ be as above. Then we have the Riemann quadrature formula (2.9) where the remainder satisfies the estimation*

$$\begin{aligned} |W_n(f, I_n, \boldsymbol{\xi})| &\leq \sup_{i=0,\ldots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]|f(b) - f(a)| \\ &\leq \left[\frac{1}{2}\nu(h) + \sup_{i=0,\ldots,n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]|f(b) - f(a)| \\ &\leq \nu(h)\,|f(b) - f(a)| \end{aligned}$$

for all ξ_i $(i = 0, \ldots, n-1)$ as above.

The case of Lipschitzian mappings is embodied within the following corollary.

Corollary 5. *Let $f:[a,b]\to\mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$ and I_n,ξ_i $(i=0,...,n-1)$ be as above. Then we have the Riemann quadrature formula (2.9) where the remainder satisfies the estimation*

$$\begin{aligned}\left|W_n\left(f,I_n,\boldsymbol{\xi}\right)\right| &\leq L\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]h_i\\ &\leq L\sum_{i=0}^{n-1}h_i^2.\end{aligned}$$

The proof is obvious by Corollary 2 applied on the intervals $[x_i,x_{i+1}]$ and summing the resulting inequalities.

We shall omit the details.

Note that the best estimation we can get from (2.10) is that one for which $\xi_i=\frac{x_i+x_{i+1}}{2}$, obtaining the following midpoint formula for functions of bounded variation.

Corollary 6. *Let f,I_n be as Theorem 2. Then we have the midpoint rule*

$$\int_a^b f(x)\,dx=M_n\left(f,I_n\right)+S_n\left(f,I_n\right)\,,$$

where

$$M_n\left(f,I_n\right)=\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)h_i$$

and the remainder $S_n\left(f,I_n\right)$ satisfies the estimation

$$\left|S_n\left(f,I_n\right)\right|\leq\frac{1}{2}\nu\left(h\right)\bigvee_a^b\left(f\right).$$

Similar results can be obtained from Corollaries 4 and 5.

Remark 3. *If we assume that $f:[a,b]\to\mathbb{R}$ is differentiable on (a,b) and whose derivative f' is integrable on (a,b) we can put instead of $\bigvee_a^b(f)$ the L_1-norm $\|f'\|_1$ obtaining the estimation due to Dragomir and Wang from the paper* [**7**].

3. An Inequality for Monotonic Mappings

Bounds were obtained for monotonic mappings in Corollary 2 as a particular case in the development for functions of bounded variation. This section treats specifically monotonic functions and obtains tighter bounds.

3.1. Integral Inequalities. The following results of Ostrowski type holds [**37**].

Theorem 3. *Let $u : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then for all $x \in [a,b]$, we have the inequality*

$$\begin{aligned}
&\left|(b-a)\,u(x) - \int_a^b u(t)\,dt\right| \\
&\leq [2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)\,dt \\
&\leq (x-a)(u(x)-u(a)) + (b-x)(u(b)-u(x)) \\
&\leq \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right](u(b)-u(a)).
\end{aligned} \tag{3.1}$$

All the inequalities in (3.1) are sharp and the constant $\frac{1}{2}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integrals, we have the identity as given by (2.2).

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i\in\{0,\dots,n-1\}}(x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$. If $p : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then

$$\begin{aligned}
\left|\int_a^b p(x)\,dv(x)\right| &= \left|\lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right]\right| \\
&\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right| \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right| \\
&\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right| \left(v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right) \\
&= \int_a^b |p(x)|\,dv(x).
\end{aligned}$$

As u is monotonic nondecreasing on $[a,b]$, and $p(x,\cdot)$ is continuous on the intervals, then using the above inequality we can state that

$$\left|\int_a^b p(x,t)\,du(t)\right| \leq \int_a^b |p(x,t)|\,du(t). \tag{3.2}$$

Now, let us observe that

$$\begin{aligned}
&\int_a^b |p(x,t)|\,du(t) \\
&= \int_a^x |t-a|\,du(t) + \int_x^b |t-b|\,du(t) \\
&= \int_a^x (t-a)du(t) + \int_x^b (b-t)du(t)
\end{aligned}$$

$$\begin{aligned}
&= (t-a)u(t)\,\Big]_a^x - \int_a^x u(t)\,dt - (b-t)u(t)\,\Big]_x^b + \int_x^b u(t)\,dt \\
&= [2x-(a+b)]u(x) - \int_a^x u(t)\,dt + \int_x^b u(t)\,dt \\
&= [2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)\,dt.
\end{aligned}$$

Using the inequality (3.2) and the identity (2.2) we get the first part of (3.1).

Now let us observe that

$$\int_a^b sgn(t-x)u(t)\,dt = -\int_a^x u(t)\,dt + \int_x^b u(t)\,dt.$$

As u is monotonic nondecreasing on $[a,b]$, we can state that

$$\int_a^x u(t)\,dt \geq (x-a)u(a) \ \text{ and } \ \int_x^b u(t)\,dt \leq (b-x)u(b)$$

so that

$$\int_a^b sgn(t-x)u(t)\,dt \leq (b-x)u(b) - (x-a)u(a).$$

Consequently

$$\begin{aligned}
& [2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)\,dt \\
\leq\ & [2x-(a+b)]u(x) + (b-x)u(b) - (x-a)u(a) \\
=\ & (b-x)(u(b)-u(x)) + (x-a)(u(x)-u(a))
\end{aligned}$$

and the second part of (3.1) is proved.

Finally, let us observe that

$$\begin{aligned}
& (b-x)(u(b)-u(x)) + (x-a)(u(x)-u(a)) \\
\leq\ & \max\{b-x, x-a\}[u(b)-u(x)+u(x)-u(a)] \\
=\ & \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right](u(b)-u(a))
\end{aligned}$$

and the inequality (3.1) is thus proved.

Now for the sharpness of the inequalities, assume that (3.1) holds with a constant $C>0$ instead of $\frac{1}{2}$. That is,

$$\begin{aligned}
(3.3) \qquad & \left|(b-a)\,u(x) - \int_a^b u(t)\,dt\right| \\
\leq\ & [2x-(a+b)]\,u(x) + \int_a^b sgn(t-x)u(t)\,dt \\
\leq\ & (x-a)(u(x)-u(a)) + (b-x)(u(b)-u(x)) \\
\leq\ & \left[C\,(b-a) + \left|x - \frac{a+b}{2}\right|\right](u(b)-u(a)).
\end{aligned}$$

Consider the mapping $u_0 : [a,b] \to \mathbb{R}$ given by

$$u_0(x) := \begin{cases} -1 & \text{if} \quad x = a \\ 0 & \text{if} \quad x \in (a,b] \end{cases}.$$

Putting in (3.3) $u = u_0$ and $x = a$, we have

$$\begin{aligned} & \left| u(x) - \int_a^b u(t)\,dt \right| \\ = & \; [2x - (a+b)]u(x) + \int_a^b sgn(t-x)u(t)\,dt \\ = & \; (x-a)(u(x) - u(a)) + (b-x)(u(b) - u(x)) = 1 \\ \leq & \; \left[C(b-a) + \left| x - \frac{a+b}{2} \right| \right] (u(b) - u(a)) \\ = & \; \left(C + \frac{1}{2} \right)(b-a) \;, \end{aligned}$$

which proves the sharpness of the first two inequalities and the fact that C should not be less than $\frac{1}{2}$. ∎

The following corollaries are interesting.

COROLLARY 7. *Let u be as above. Then we have the midpoint inequality:*

$$\begin{aligned} & \left| (b-a)\,u\left(\frac{a+b}{2}\right) - \int_a^b u(t)\,dt \right| \\ \leq & \int_a^b sgn\left(t - \frac{a+b}{2}\right) u(t)\,dt \\ \leq & \frac{b-a}{2}\left[u(b) - u(a)\right]. \end{aligned} \tag{3.4}$$

Also, we have the following *"trapezoid inequality"* for monotonic nondecreasing mappings.

COROLLARY 8. *Under the above assumptions, we have*

$$\left| \frac{b-a}{2}\left[u(a) + u(b)\right] - \int_a^b u(t)\,dt \right| \leq \frac{b-a}{2}\left[u(b) - u(a)\right]. \tag{3.5}$$

PROOF. Taking $x = a$ and $x = b$ in Theorem 3, summing, using the triangle inequality and dividing by 2, we get the desired inequality (3.5). ∎

3.2. A Quadrature Formula. Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f, I_n, \boldsymbol{\xi}) = \sum_{i=0}^{n-1} f(\xi_i)\,h_i\,,$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula.

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$ and I_n, ξ_i $(i = 0, ..., n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)\,dx = R_n(f, I_n, \boldsymbol{\xi}) + W_n(f, I_n, \boldsymbol{\xi})\;, \tag{3.6}$$

where the remainder satisfies the estimation

$$
\begin{aligned}
(3.7)\qquad & |W_n(f, I_n, \boldsymbol{\xi})| \\
\leq\ & 2\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f(\xi_i) + \int_a^b S(t, I_n, \boldsymbol{\xi}) f(t)\, dt \\
\leq\ & \sum_{i=0}^{n-1}[(\xi_i - x_i)(f(\xi_i) - f(x_i)) + (x_{i+1} - \xi_i)(f(x_{i+1}) - f(\xi_i))] \\
\leq\ & \sup_{i=0,\dots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right](f(b) - f(a)) \\
\leq\ & \left[\frac{1}{2}\nu(h) + \sup_{i=0,\dots,n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right](f(b) - f(a)) \\
\leq\ & \nu(h)(f(b) - f(a))
\end{aligned}
$$

for all ξ_i $(i = 0, ..., n-1)$ *as above, where* $\nu(h) := \max_{i=0,\dots,n}\{h_i\}$ *and*

$$S(t, I_n, \boldsymbol{\xi}) = sgn(t - \zeta_i) \text{ if } \ t \in [x_i, x_{i+1})(i = 0, ..., n-1).$$

Proof. Apply Theorem 3 on the interval $[x_i, x_{i+1}]$ to get

$$
\begin{aligned}
& \left|\int_{x_i}^{x_{i+1}} f(x)\, dx - f(\xi_i) h_i\right| \\
\leq\ & 2(\xi_i - \frac{x_i + x_{i+1}}{2}) f(\xi_i) + \int_{x_i}^{x_{i+1}} S(t, I_n, \boldsymbol{\xi}) f(t)\, dt \\
\leq\ & (\xi_i - x_i)(f(\xi_i) - f(x_i)) + (x_{i+1} - \xi_i)(f(x_{i+1}) - f(\xi_i)) \\
\leq\ & \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right](f(x_{i+1}) - f(x_i)).
\end{aligned}
$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$
\begin{aligned}
& |W_n(f, I_n, \boldsymbol{\xi})| \\
\leq\ & \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\, dx - f(\xi_i) h_i\right| \\
\leq\ & 2\sum_{i=0}^{n-1}\left[\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f(\xi_i) + \int_{x_i}^{x_{i+1}} S(t, I_n, \boldsymbol{\xi}) f(t)\, dt\right] \\
\leq\ & \sum_{i=0}^{n-1}[(\xi_i - x_i)(f(\xi_i) - f(x_i)) + (x_{i+1} - \xi_i)(f(x_{i+1}) - f(\xi_i))] \\
\leq\ & \sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right](f(x_{i+1}) - f(x_i)) \\
\leq\ & \sup_{i=0,\dots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i)) \\
=\ & \sup_{i=0,\dots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right](f(b) - f(a)).
\end{aligned}
$$

The fourth inequality follows by the properties of $\sup(\cdot)$.
Now, as

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \le \frac{1}{2}h_i$$

for all $\xi_i \in [x_i, x_{i+1}](i = 0, ..., n-1)$ the last part of (3.7) is also proved. ∎

Corollary 9. *Let f, I_n be as in Theorem 4. Then we have the midpoint rule*

$$\int_a^b f(x)\,dx = M_n(f, I_n) + S_n(f, I_n)\ ,$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$|S_n(f, I_n)| \le \int_a^b \mu(I_n) f(t)\,dt \le \frac{1}{2}\nu(h)(f(b) - f(a))\,,$$

where

$$\mu(I_n) = sgn\left(t - \frac{x_i + x_{i+1}}{2}\right) \ if\ t \in [x_i, x_{i+1})\ \ (i = 0, ..., n-1).$$

4. Ostrowski Inequality for Lipschitzian Mappings

In Corollary 2, bounds were obtained for an interior point rule for Lipschitzian mappings as a special instance of functions of bounded variation. Treating specifically Lipschitzian mappings, tighter bounds are now obtained.

4.1. Integral Inequalities. The following inequality for Lipschitzian mappings holds [**38**].

Theorem 5. *Let $u : [a, b] \to \mathbb{R}$ be an $L-$Lipschitzian mapping on $[a, b]$. That is,*

$$|u(x) - u(y)|\ | \le L\,|x - y| \ \ for\ all\ x, y \in [a, b].$$

Then we have the inequality

$$\left|\int_a^b u(t)dt - (b-a)u(x)\right| \le L\left[\frac{(b-a)^2}{4} + (x - \frac{a+b}{2})^2\right], \tag{4.1}$$

for all $x \in [a, b]$.
The constant $\frac{1}{4}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integrals we have the identity as given in (2.2).
Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i\in\{0,...,n-1\}}\left(x_{i+1}^{(n)} - x_i^{(n)}\right)$

and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$. If $p : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is L-Lipschitzian on $[a,b]$, then

$$\begin{aligned}
&\left|\int_a^b p(x)dv(x)\right| \\
= &\left|\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right]\right| \\
\leq &\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)}-x_i^{(n)}\right)\left|\frac{v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)}{x_{i+1}^{(n)}-x_i^{(n)}}\right| \\
\leq &L\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)}-x_i^{(n)}\right)
\end{aligned}$$

and so

$$\left|\int_a^b p(x)dv(x)\right| \leq L\int_a^b |p(x)|\,dx. \tag{4.2}$$

Applying the inequality (4.2) for $p(x,t)$ as given in (2.2) and $v(x) = u(x),\ x \in [a,b]$, we get

$$\begin{aligned}
\left|\int_a^b p(x,t)du(t)\right| &\leq L\left[\int_a^x |t-a|\,dt + \int_x^b |t-b|dt\right] \\
&= \frac{L}{2}\left[(x-a)^2+(b-x)^2\right] \\
&= L\left[\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2\right] \qquad (4.3)
\end{aligned}$$

and so by (4.3), via the identity (2.2), we deduce the desired inequality (4.1).
Now to determine the best constant, assume that the inequality (4.1) holds with a constant $C > 0$. That is

$$\left|\int_a^b u(t)dt-(b-a)u(x)\right| \leq L\left[C(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right] \tag{4.4}$$

for all $x \in [a,b]$.
Consider the mapping $f : [a,b] \to \mathbb{R},\ f(x) = x$ in (4.4). Then

$$\left|x-\frac{a+b}{2}\right| \leq C(b-a)^2+\left(x-\frac{a+b}{2}\right)^2$$

for all $x \in [a,b]$; and then for $x = a$, we get

$$\frac{b-a}{2} \leq \left(C+\frac{1}{4}\right)(b-a)$$

which implies that $C \geq \frac{1}{4}$ and the theorem is completely proved. ∎

The following corollary holds, giving a midpoint rule for Lipschitzian functions.

COROLLARY 10. *Let $u : [a,b] \to \mathbb{R}$ be as above. Then we have the inequality:*

$$\left| \int_a^b u(t)dx - (b-a)u\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4}L(b-a)^2. \tag{4.5}$$

REMARK 4. *It is well known that if $f : [a,b] \to \mathbb{R}$ is a convex mapping on $[a,b]$, then the **Hermite-Hadamard** inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{4.6}$$

Now, if we assume that $f : I \subset \mathbb{R} \to \mathbb{R}$ is convex on I and $a,b \in Int(I), a < b$; then f'_+ is monotonic nondecreasing on $[a,b]$ and by Theorem 5 we obtain

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{1}{4}f'_+(b)(b-a)\,, \tag{4.7}$$

which gives a counterpart for the first membership of Hadamard's inequality.

4.2. A Quadrature Formula of Riemann Type. Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,...,n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i)h_i\,,$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula [**38**].

THEOREM 6. *Let $f : [a,b] \to \mathbb{R}$ be an $L-$Lipschitzian mapping on $[a,b]$ and I_n, ξ_i $(i = 0,...,n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx = R_n(f, I_n, \xi) + W_n(f, I_n, \xi)\,, \tag{4.8}$$

where the remainder satisfies the estimation

$$\begin{aligned} |W_n(f, I_n, \xi)| &\leq \frac{1}{4}L\sum_{i=0}^{n-1} h_i^2 + L\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \\ &\leq \frac{1}{2}L\sum_{i=0}^{n-1} h_i^2 \end{aligned} \tag{4.9}$$

for all ξ_i $(i = 0,...,n-1)$ as above. The constant $\frac{1}{4}$ is sharp in (4.9).

PROOF. Apply Theorem 5 on the interval $[x_i, x_{i+1}]$ to get

$$\left| \int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i \right| \leq L\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]. \tag{4.10}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$\begin{aligned} |W_n(f, I_n, \xi)| &\leq \sum_{i=0}^{n-1}\left| \int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i \right| \\ &\leq L\sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + (\xi_i - \frac{x_i + x_{i+1}}{2})^2\right]. \end{aligned}$$

Now, as

$$\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \leq \frac{1}{4} h_i^2$$

for all $\xi_i \in [x_i, x_{i+1}] (i = 0, ..., n-1)$ the second part of (4.9) is also proved. ∎

Note that the best estimation we can get from (4.9) is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$ obtaining the following midpoint formula.

COROLLARY 11. *Let f, I_n be as above. Then we have the midpoint rule*

$$\int_a^b f(x)dx = M_n(f, I_n) + S_n(f, I_n)$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$|S_n(f, I_n)| \leq \frac{1}{4} L \sum_{i=0}^{n-1} h_i^2.$$

REMARK 5. *If we assume that $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and whose derivative f' is bounded on (a, b) we can put instead of L the infinity norm $\|f'\|_\infty$, obtaining the estimation due to Dragomir and Wang from the paper* [**4**].

5. A Generalisation for Derivatives that are Absolutely Continuous

In 1938, Ostrowski (see for example [**1**, p.468]) proved the following integral inequality.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°(I° is the interior of I), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$, then we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [**4**] by S.S. Dragomir and S. Wang.

In 1976, G.V. Milovanović and J.E. Pečarić (see for example [**1**, p. 468]), proved the following generalization of Ostrowski's result.

Let $f:[a,b]\to\mathbb{R}$ be an n-times differentiable function, $n\geq 1$, such that $\left\|f^{(n)}\right\|_{\infty}:=\sup_{t\in(a,b)}\left|f^{(n)}(t)\right|<\infty$. Then

$$\begin{aligned}&\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}\frac{n-k}{k!}\cdot\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a}\right)\right.\\&\left.-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{\left\|f^{(n)}\right\|_{\infty}}{n(n+1)!}\cdot\frac{(x-a)^{n+1}+(b-x)^{n+1}}{b-a}\end{aligned}$$

for all $x\in[a,b]$.

In [**8**], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following Ostrowski type inequality for twice differentiable mappings:

Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) and $f'':(a,b)\to\mathbb{R}$ is bounded, i.e., $\|f''\|_{\infty}=\sup_{t\in(a,b)}|f''(t)|<\infty$. Then we have the inequality:

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)f'(x)\right|\\\leq\ &\left[\frac{1}{24}(b-a)^2+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_{\infty}\leq\frac{(b-a)^2}{6}\|f''\|_{\infty}\end{aligned}$$

for all $x\in[a,b]$.

In this section we establish another generalization of the Ostrowski inequality for n-time differentiable mappings which naturally generalizes the result from [**8**]. Further, work on representation in terms of power series expansion of functions on an interval is presented, giving a generalisation of Taylor series which produces an expansion about a point.

5.1. Integral identities. The following theorem holds [**15**] (see also [**39**]).

THEOREM 7. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$ we have the identity:*

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\\&\quad+(-1)^n\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt\end{aligned}\tag{5.1}$$

where the kernel $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$K_n(x,t):=\begin{cases}\dfrac{(t-a)^n}{n!} & \text{if } t\in[a,x]\\[2ex]\dfrac{(t-b)^n}{n!} & \text{if } t\in(x,b]\end{cases},\ x\in[a,b]\tag{5.2}$$

and n is a natural number, $n\geq 1$.

PROOF. The proof is by mathematical induction [**15**]. For a different argument, see [**39**].

For $n=1$, we have to prove the equality

$$\int_a^b f(t)\,dt = (b-a)\,f(x) - \int_a^b K_1(x,t)\,f^{(1)}(t)\,dt \tag{5.3}$$

where

$$K_1(x,t) := \begin{cases} t-a & \text{if } t\in[a,x] \\ t-b & \text{if } t\in(x,b] \end{cases}.$$

Integrating by parts, we have:

$$\begin{aligned} & \int_a^b K_1(x,t)\,f^{(1)}(t)\,dt \\ = & \int_a^x (t-a)\,f'(t)\,dt + \int_x^b (t-b)\,f'(t)\,dt \\ = & (t-a)\,f(t)|_a^x - \int_a^x f(t)\,dt + (t-b)\,f(t)|_x^b - \int_x^b f(t)\,dt \\ = & (x-a)\,f(x) + (b-x)\,f(x) - \int_a^b f(t)\,dt \\ = & (b-a)\,f(x) - \int_a^b f(t)\,dt \end{aligned}$$

and the identity (5.3) is proved.

Assume that (5.1) holds for "n" and let us prove it for "$n+1$". That is, we have to prove the equality

$$\begin{aligned} \int_a^b f(t)\,dt = & \sum_{k=0}^{n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x) \\ & +(-1)^{n+1}\int_a^b K_{n+1}(x,t)\,f^{(n+1)}(t)\,dt. \end{aligned} \tag{5.4}$$

We have, using (5.2),

$$\begin{aligned} & \int_a^b K_{n+1}(x,t)\,f^{(n+1)}(t)\,dt \\ = & \int_a^x \frac{(t-a)^{n+1}}{(n+1)!}f^{(n+1)}(t)\,dt + \int_x^b \frac{(t-b)^{n+1}}{(n+1)!}f^{(n+1)}(t)\,dt \end{aligned}$$

and integrating by parts gives

$$\begin{aligned} & \int_a^b K_{n+1}(x,t)\,f^{(n+1)}(t)\,dt \\ = & \left.\frac{(t-a)^{n+1}}{(n+1)!}f^{(n)}(t)\right|_a^x - \frac{1}{n!}\int_a^x (t-a)^n f^{(n)}(t)\,dt \\ & + \left.\frac{(t-b)^{n+1}}{(n+1)!}f^{(n)}(t)\right|_x^b - \frac{1}{n!}\int_x^b (t-b)^n f^{(n)}(t)\,dt \\ = & \frac{(x-a)^{n+1}+(-1)^{n+2}(b-x)^{n+1}}{(n+1)!}f^{(n)}(x) - \int_a^b K_n(x,t)\,f^{(n)}(t)\,dt. \end{aligned}$$

That is

$$\int_a^b K_n(x,t) f^{(n)}(t)\,dt = \frac{(x-a)^{n+1} + (-1)^{n+2}(b-x)^{n+1}}{(n+1)!} f^{(n)}(x) - \int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt.$$

Now, using the mathematical induction hypothesis, we get

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) \\ &\quad + \frac{(b-x)^{n+1} + (-1)^n (x-a)^{n+1}}{(n+1)!} f^{(n)}(x) \\ &\quad - (-1)^n \int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt \\ &= \sum_{k=0}^{n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) \\ &\quad + (-1)^{n+1} \int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt.\end{aligned}$$

That is, identity (5.4) and the theorem is thus proved. ∎

Corollary 12. *With the above assumptions, we have the representation*

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{(k+1)!}\right] \frac{(b-a)^{k+1}}{2^{k+1}} f^{(k)}\left(\frac{a+b}{2}\right) \\ &\quad + (-1)^n \int_a^b M_n(t) f^{(n)}(t)\,dt\end{aligned} \tag{5.5}$$

where

$$M_n(t) := \begin{cases} \dfrac{(t-a)^n}{n!} & \text{if } t \in \left[a, \frac{a+b}{2}\right] \\ \dfrac{(t-b)^n}{n!} & \text{if } t \in \left(\frac{a+b}{2}, b\right] \end{cases}.$$

The proof follows by Theorem 7 by choosing $x = \frac{a+b}{2}$ so that $M_n(t) = K_n\left(\frac{a+b}{2}, t\right)$.

Corollary 13. *With the above assumptions, we have the representation:*

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2}\right] \\ &\quad + \int_a^b T_n(t) f^{(n)}(t)\,dt\end{aligned} \tag{5.6}$$

where

$$T_n(t) := \frac{1}{n!}\left[\frac{(b-t)^n + (-1)^n (t-a)^n}{2}\right], \qquad t \in [a,b]. \tag{5.7}$$

Proof. Choose $x=a$ and $x=b$ in (5.1), then summing the resulting identities and dividing by 2, gives

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\\ &\quad+\int_a^b T_n(t)\,f^{(n)}(t)\,dt\end{aligned}$$

and the corollary is proved. ∎

The following Taylor-like formula with integral remainder also holds.

Corollary 14. *Let $g:[a,y]\to\mathbb{R}$ be a mapping such that $g^{(n)}$ is absolutely continuous on $[a,y]$. Then for all $x\in[a,y]$, we have the identity*

$$\begin{aligned}g(y) &= g(a)+\sum_{k=0}^{n-1}\frac{\left[(y-x)^{k+1}+(-1)^k(x-a)^{k+1}\right]}{(k+1)!}g^{(k+1)}(x)\\ &\quad+(-1)^n\int_a^y K_n(y,t)\,g^{(n+1)}(t)\,dt. \end{aligned}\tag{5.8}$$

The proof is obvious by Theorem 7 choosing $f=g'$, and $b=y$.

Remark 6. *If we choose $n=1$ in (5.1), we get the identity (5.3) which is the identity employed by S.S. Dragomir and S. Wang to prove an Ostrowski type inequality in paper* [**4**].

If in (5.5) we choose $n=1$, then we get

$$\int_a^b f(t)\,dt=(b-a)\,f\left(\frac{a+b}{2}\right)-\int_a^b M_1(t)\,f'(t)\,dt \tag{5.9}$$

where

$$M_1(t)=\begin{cases} t-a & \text{if } t\in\left[a,\frac{a+b}{2}\right]\\ t-b & \text{if } t\in\left(\frac{a+b}{2},b\right]\end{cases}$$

which gives the midpoint type identity useful in Numerical Analysis, although here only the first derivative is involved.

Also, if we put $n=1$ in (5.6), we get the trapezoid identity

$$\int_a^b f(t)\,dt=\frac{b-a}{2}\left(f(a)+f(b)\right)+\int_a^b T_1(t)\,f'(t)\,dt \tag{5.10}$$

where

$$T_1(t)=\frac{a+b}{2}-t,\qquad t\in[a,b].$$

Finally, if in the Taylor-like formula (5.8) we put $n=1$, we get

$$g(y)=g(a)+(y-a)\,g'(x)-\int_a^y K_1(y,t)\,g^{(2)}(t)\,dt$$

where $x\in[a,y]$.

Remark 7. *If we choose $n = 2$ in (5.1), we get the identity:*

$$\int_a^b f(t)\,dt = (b-a)\,f(x) - \left(x - \frac{a+b}{2}\right) f'(x) + \int_a^b K_2(x,t)\,f''(t)\,dt \tag{5.11}$$

where $K_2(x,t)$ is as given in (5.2), which is the identity employed by P. Cerone, S.S. Dragomir and J. Roumeliotis to prove some Ostrowski type inequalities for twice differentiable mappings in the paper [**8**].

If in (5.5) we choose $n = 2$, then we get

$$\int_a^b f(t)\,dt = (b-a)\,f\left(\frac{a+b}{2}\right) + \int_a^b M_2(t)\,f''(t)\,dt \tag{5.12}$$

where

$$M_2(t) = \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in \left[a, \frac{a+b}{2}\right] \\[2ex] \dfrac{(t-b)^2}{2} & \text{if } t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

which is the classical midpoint identity.

Also, if we put $n = 2$ in (5.6), we get the identity

$$\int_a^b f(t)\,dt = \frac{b-a}{2}\left(f(a) + f(b)\right) + \frac{(b-a)^2}{2} \cdot \frac{f'(a) - f'(b)}{2} + \int_a^b T_2(t)\,f''(t)\,dt \tag{5.13}$$

where $T_2(t)$ is as given in (5.7).

Finally, if we put $n = 2$ in (5.8), we get

$$g(y) = g(a) + (y-a)\,g'(x) - (y-a)\left(x - \frac{a+y}{2}\right) g''(x) + \int_a^y K_2(y,t)\,g^{(3)}(t)\,dt, \tag{5.14}$$

where K_2 is given in (5.2) and $a \le x \le y$.

5.2. Some integral inequalities. The following theorem holds (see also [**15**] for the case of the $\|\cdot\|_\infty$ norm).

THEOREM 8. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$, we have the inequalities*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right|$$

$$(5.15)\quad \leq \begin{cases} \frac{\|f^{(n)}\|_\infty}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right] & \text{if} \quad f^{(n)} \in L_\infty[a,b], \\ \frac{\|f^{(n)}\|_p}{n!} \left[\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1} \right]^{\frac{1}{q}} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f^{(n)} \in L_p[a,b], \\ \frac{\|f^{(n)}\|_1}{n!} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^n & \text{if} \quad f^{(n)} \in L_1[a,b], \end{cases}$$

where

$$\left\| f^{(n)} \right\|_\infty := \sup_{t \in [a,b]} \left| f^{(n)}(t) \right| < \infty \ \text{and} \ \left\| f^{(n)} \right\|_p := \left(\int_a^b \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}}.$$

PROOF. Using the identity (5.1), we have:

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ = & \left| \int_a^b K_n(x,t) f^{(n)}(t)\,dt \right| := Q(x). \end{aligned}$$

Now, observe that

$$Q(x) \leq \left\| f^{(n)} \right\|_\infty \|K_n(x,\cdot)\|_1 = \left\| f^{(n)} \right\|_\infty \int_a^b |K_n(x,t)|\,dt$$

and so using (5.2),

$$\begin{aligned} Q(x) &\leq \left\| f^{(n)} \right\|_\infty \left[\int_a^x \frac{(t-a)^n}{n!} dt + \int_x^b \frac{(b-t)^n}{n!} dt \right] \\ &= \frac{\|f^{(n)}\|_\infty}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right], \end{aligned}$$

and the first part of inequality (5.15) is proved.

Further, using Hölder's integral inequality, we have

$$\begin{aligned} Q(x) &\leq \left\| f^{(n)} \right\|_p \left(\int_a^b |K_n(x,t)|^q\,dt \right)^{\frac{1}{q}} \\ &= \frac{\|f^{(n)}\|_p}{n!} \left[\int_a^x (t-a)^{nq}\,dt + \int_x^b (b-t)^{nq}\,dt \right]^{\frac{1}{q}}, \end{aligned}$$

and so, on evaluation of the above integrals, the second inequality in (5.15) is proven.

Finally, let us observe that,

$$\begin{aligned} Q(x) &\leq \|K_n(x,\cdot)\|_\infty \left\|f^{(n)}\right\|_1 \\ &= \left\|f^{(n)}\right\|_1 \sup_{t\in[a,b]} |K_n(x,t)| \\ &= \frac{\left\|f^{(n)}\right\|_1}{n!} \left[\max\{x-a, b-x\}\right]^n \\ &= \frac{\left\|f^{(n)}\right\|_1}{n!} \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n \end{aligned}$$

and the theorem is completely proved. ∎

Remark 8. *It may be noticed that the expressions for the bounds of a generalised interior point rule such as that given by (5.15) are upper bounded by taking $x = a$ or $x = b$ in the bound while keeping a general $x \in [a,b]$ for the rule. The sharpest bound is obtained by taking $x = \frac{a+b}{2}$, giving the result expressed in the following corollary.*

Corollary 15. *Let the conditions of Theorem 8 hold. Then*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left(\frac{b-a}{2}\right)^{k+1} \left[1 + (-1)^k\right] \frac{f^{(k)}\left(\frac{a+b}{2}\right)}{(k+1)!}\right| \tag{5.16}$$
$$\leq \begin{cases} \frac{(b-a)^{n+1}}{2^n (n+1)!} \left\|f^{(n)}\right\|_\infty & \text{if} \quad f^{(n)} \in L_\infty[a,b], \\ \frac{(b-a)^{n+\frac{1}{q}}}{2^n n! (nq+1)^{\frac{1}{q}}} \left\|f^{(n)}\right\|_p & \text{if} \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{and} \quad f^{(n)} \in L_p[a,b], \\ \frac{(b-a)^n}{2^n n!} \left\|f^{(n)}\right\|_1 & \text{if} \quad f^{(n)} \in L_1[a,b]. \end{cases}$$

Remark 9. *Taking $n = 1$ in Theorem 8 and Corollary 14 reproduces some of the results obtained by Dragomir and Wang ([**4**]-[**7**]) while $n = 2$ reproduces the results of Cerone, Dragomir and Roumeliotis ([**8**]-[**11**]). It is important to note that assuming that the behaviour of the first derivatives determines the bound on the rule allows greater flexibility than assumptions about the second derivative. Taking $n = 2$ allows a comparison with traditional midpoint $\left(x = \frac{a+b}{2}\right)$ or interior point rules. It should further be noted that only even derivatives occur in the rule given in (5.16).*

Remark 10. *It is most interesting to observe that the bounds given in (5.15) for a generalised interior point method obtained from investigating various norms of $K_n(x,t)$ as given by (5.2) are the same as the bounds obtained from the generalised trapezoidal type rule resulting from various norms of the Peano kernel given by $\frac{(x-t)^n}{n!}$.*

The following corollary is a generalisation of the trapezoidal inequality [**15**].

Corollary 16. *With the above assumptions, we have the inequality:*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{5.17}$$
$$\leq \frac{(b-a)^{n+1}}{(n+1)!}\left\|f^{(n)}\right\|_\infty\times\begin{cases}1 & \text{if } n=2r\\ \dfrac{2^{2r+1}-1}{2^{2r}} & \text{if } n=2r+1\end{cases}.$$

Proof. Using the identity (5.6), we get

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{5.18}$$
$$=\left|\int_a^b T_n(t)\,f^{(n)}(t)\,dt\right|\leq\left\|f^{(n)}\right\|_\infty\int_a^b|T_n(t)|\,dt.$$

If $n=2r$, then

$$\int_a^b|T_n(t)|\,dt \tag{5.19}$$
$$=\frac{1}{(2r)!}\int_a^b\left[\frac{(b-t)^{2r}+(t-a)^{2r}}{2}\right]dt$$
$$=\frac{1}{(2r)!}\cdot\frac{1}{2}\left[\frac{(b-a)^{2r+1}}{(2r+1)}+\frac{(b-a)^{2r+1}}{(2r+1)}\right]=\frac{(b-a)^{2r+1}}{(2r+1)!}.$$

For $n=2r+1$, put $h_{2r+1}(t):=(b-t)^{2r+1}-(t-a)^{2r+1},\quad t\in[a,b]$. Observe that $h_{2r+1}(t)=0$ iff $t=\frac{a+b}{2}$ and $h_{2r+1}(t)>0$ if $t\in[a,\frac{a+b}{2})$ and $h_{2r+1}(t)<0$ if $t\in(\frac{a+b}{2},b]$.

Then

$$\int_a^b|T_{2r+1}(t)|\,dt$$
$$=\int_a^{\frac{a+b}{2}}\left[(b-t)^{2r+1}-(t-a)^{2r+1}\right]dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^{2r+1}-(b-t)^{2r+1}\right]dt$$
$$=2\frac{(b-a)^{2r+2}}{2r+2}-\frac{4\left(\frac{b-a}{2}\right)^{2r+2}}{2r+2}$$
$$=\frac{1}{2r+2}\left[2(b-a)^{2r+2}-\frac{(b-a)^{2r+2}}{2^{2r}}\right]$$
$$=\frac{(b-a)^{2r+2}}{2r+2}\left(2-\frac{1}{2^{2r}}\right)=\frac{(b-a)^{2r+2}}{2r+2}\cdot\frac{2^{2r+1}-1}{2^{2r}}.$$

Using (5.17) we get the desired inequality (5.18). ∎

The following inequalities for the Taylor like expansion (5.8) also hold.

COROLLARY 17. *Let g be as in Corollary 14. Then we have the inequality:*

$$\left| g(y) - g(a) - \sum_{k=0}^{n-1} \frac{\left[(y-x)^{k+1} + (-1)^k (x-a)^{k+1} \right]}{(k+1)!} g^{(k+1)}(x) \right|$$

$$(5.20) \quad \leq \begin{cases} \frac{\left\| g^{(n+1)} \right\|_\infty}{(n+1)!} \left[(x-a)^{n+1} + (y-x)^{n+1} \right], & g^{(n+1)} \in L_\infty [a,b], \\ \frac{\left\| g^{(n+1)} \right\|_p}{n!} \left[\frac{(x-a)^{nq+1} + (y-x)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, & g^{(n+1)} \in L_p [a,b] \\ & \frac{1}{p} + \frac{1}{q} = 1,\ p > 1, \\ \frac{\left\| g^{(n+1)} \right\|_1}{n!} \left[\frac{1}{2}(y-a) + \left| x - \frac{a+y}{3} \right| \right]^n, & g^{(n+1)} \in L_1 [a,b] \end{cases}$$

for all $a \leq x \leq y$.

PROOF. From equation (5.8) and using norms, or else from (5.15) on choosing $f = g'$ and $b = y$ readily produces the above result. ∎

REMARK 11. *Since the right hand side of (5.20) are convex functions, then upper bounds may be found by taking either $x = a$ or b on the right.*
It is well known that for the classical Taylor expansion around a we have the inequality

$$(5.21) \quad \left| g(y) - \sum_{k=0}^{n} \frac{(y-a)^k}{k!} g^{(k)}(a) \right| \leq \frac{(y-a)^{n+1}}{(n+1)!} \left\| g^{(n+1)} \right\|_\infty$$

for all $y \geq a$. It is clear now that the above approximation (5.20) *around the arbitrary point $x \in [a,y]$ provides a better approximation for the mapping g at the point y than the classical Taylor expansion around the point a.*

If in (5.20) we choose $x = \frac{a+y}{2}$, then we get

$$\left| g(y) - g(a) - \sum_{k=1}^{n} \frac{\left[1 + (-1)^{k-1} \right]}{k!} \frac{(y-a)^k}{2^k} g^{(k)}\left(\frac{a+y}{2} \right) \right|$$

$$(5.22) \quad \leq \begin{cases} \frac{(y-a)^{n+1}}{(n+1)!2^n} \left\| g^{(n+1)} \right\|_\infty, & g^{(n+1)} \in L_\infty [a,b], \\ \frac{(y-a)^{n+\frac{1}{q}}}{2^n n! (nq+1)^{\frac{1}{q}}} \left\| g^{(n+1)} \right\|_p, & g^{(n+1)} \in L_p [a,b] \\ & \text{with } \frac{1}{p} + \frac{1}{q} = 1,\ p > 1, \\ \frac{(y-a)^n}{2^n n!} \left\| g^{(n+1)} \right\|_1, & g^{(n+1)} \in L_1 [a,b]. \end{cases}$$

The above inequality (5.22) shows that for $g \in C^\infty [a,b]$ the series

$$g(a) + \sum_{k=0}^{\infty} \frac{\left[1 + (-1)^k \right]}{(k+1)!} \frac{(y-a)^{k+1}}{2^{k+1}} g^{(k+1)}\left(\frac{a+y}{2} \right)$$

converges more rapidly to $g(y)$ than the usual one

$$\sum_{k=0}^{\infty} \frac{(y-a)^k g^{(k)}(a)}{k!}$$

which comes from Taylor's expansion. Further, it should be noted that the Taylor-like expansion in (5.22) only involves odd derivatives of g.

Remark 12. *If in the inequality* (5.15) *we choose* $n=1$ *we get*

$$\left|\int_a^b f(t)\,dt-(b-a)\,f(x)\right|\le\frac{(x-a)^2+(b-x)^2}{2}\left\|f'\right\|_\infty .$$

As a simple calculation shows that

$$\frac{1}{2}\left[(x-a)^2+(b-x)^2\right]=\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2$$

consequently we obtain the Ostrowski inequality:

$$\left|\int_a^b f(t)\,dt-(b-a)\,f(x)\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)^2\left\|f'\right\|_\infty \tag{5.23}$$

for all $x\in[a,b]$.

If in (5.18) we put $n=1$, we get the midpoint inequality

$$\left|\int_a^b f(t)\,dt-(b-a)\,f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{4}(b-a)^2\left\|f'\right\|_\infty . \tag{5.24}$$

From the inequality (5.17), for $n=1$, we get the trapezoid inequality

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{2}\left[f(a)+f(b)\right]\right|\le\frac{1}{2}(b-a)^2\left\|f'\right\|_\infty . \tag{5.25}$$

Also, from (5.20) we deduce

$$\left|g(y)-g(a)-(y-a)\,g'(x)\right|\le\left[\frac{(y-a)^2}{4}+\left(x-\frac{a+y}{2}\right)^2\right]\left\|g''\right\|_\infty \tag{5.26}$$

for all $a\le x\le y$.

Remark 13. *If in the inequality* (5.15) *we choose* $n=2$, *then we get*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-(b-a)\,f(x)+(b-a)\left(x-\frac{a+b}{2}\right)f'(x)\right|\\ \le\;&\frac{1}{6}\left[(x-a)^3+(b-x)^3\right]\left\|f''\right\|_\infty\end{aligned}$$

for all $x\in[a,b]$.

From which, on noting that

$$(x-a)^3+(b-x)^3=(b-a)\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right],$$

we recapture the result obtained in [**8**], namely

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-(b-a)\,f(x)+(b-a)\left(x-\frac{a+b}{2}\right)f'(x)\right|\\ \le\;&\left[\frac{1}{24}+\frac{1}{2}\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)^3\left\|f''\right\|_\infty .\end{aligned} \tag{5.27}$$

For $f'' \in L_\infty[a,b]$ with $n=2$ in (5.16), we get the classical midpoint inequality

$$\left|\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{24}(b-a)^3\|f''\|_\infty, \tag{5.28}$$

while from (5.17), we get a perturbed trapezoid inequality,

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{2}[f(a)+f(b)]-\frac{(b-a)^2}{4}[f'(b)-f'(a)]\right| \le \frac{(b-a)^3}{6}\|f''\|_\infty. \tag{5.29}$$

Finally, if we put $n=2$ in (5.20) for $g''' \in L_\infty[a,b]$, then we get the inequality:

$$\left|g(y)-g(a)-(y-a)g'(x)+(y-a)\left(x-\frac{a+y}{2}\right)g''(x)\right| \le \left[\frac{1}{24}+\frac{1}{2}\cdot\frac{\left(x-\frac{a+y}{2}\right)^2}{(y-a)^2}\right](y-a)^3\|g'''\|_\infty,$$

valid for any $x\in[a,y]$.

Equivalent results may also be produced which involve the $L_p[a,b]$ spaces for $p\ge 1$.

The following particular case for euclidean norms is of particular interest.

Corollary 18. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b) and $f''\in L_2(a,b)$. Then we have the following inequality*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-(b-a)\left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right]\right| \\ &\le \frac{(b-a)^{\frac12}}{2\sqrt5}\left[\left(\frac{b-a}{2}\right)^4+10\left(\frac{b-a}{2}\right)^2\left(x-\frac{a+b}{2}\right)^2+5\left(x-\frac{a+b}{2}\right)^4\right]^{\frac12} \\ &\le \frac{(b-a)^{\frac52}}{2\sqrt5}.\end{aligned} \tag{5.30}$$

Proof. Let $p=q=2$ and $n=2$ in (5.15) to give the left hand side of (5.30) with the right hand side given by

$$\frac{\left\|f^{(2)}\right\|_2}{2!\sqrt5}\left[(x-a)^5+(b-x)^5\right]^{\frac12}.$$

Now, expansion of $(x-a)^5+(b-x)^5$ in a Taylor series about $x=\frac{a+b}{2}$ readily produces the first bound in (5.30). The coarser bound is obtained from evaluating at either of the end points. Hence, the corollary is proved. ∎

Remark 14. *The optimal rule from (5.30) in terms of obtaining the tightest bound occurs on taking $x=\frac{a+b}{2}$ since the bound is symmetric and convex.*

Now, using the celebrated Hermite-Hadamard integral inequality for convex functions, $g:[a,b]\to\mathbb{R}$, which may be written as

$$g\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b g(x)\,dx\le\frac{g(a)+g(b)}{2}, \tag{5.31}$$

we obtain the following theorem [**40**].

Theorem 9. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable and $\gamma \leq f(x) \leq \Gamma$ for all $x \in (a,b)$. Then we have the following double inequality:*

$$\frac{\gamma (b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma (b-a)^2}{24} \tag{5.32}$$

and the estimation

$$\left| \int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) - \frac{(\gamma+\Gamma)(b-a)^3}{48} \right| \leq \frac{(\Gamma-\gamma)(b-a)^3}{48}. \tag{5.33}$$

Proof. Let us choose in (5.31) $g(x) = f(x) - \frac{\gamma x^2}{2}$, then $g(x)$ is a convex function in x, since $g''(x) \geq 0$, and hence

$$f\left(\frac{a+b}{2}\right) - \frac{\gamma (a+b)^2}{8} \leq \frac{1}{b-a}\left(\int_a^b f(x)\,dx - \frac{\gamma\left(b^3-a^3\right)}{6}\right),$$

which is equivalent to

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) &\geq \frac{\gamma}{2}\left(\frac{b^3-a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2\right) \\ &= \frac{\gamma (b-a)^2}{24}, \end{aligned}$$

and the first part of (5.32) is therefore obtained. For the second part, let $g(x) = \frac{x^2\Gamma}{2} - f(x)$, and similar manipulations, as previously lead to the second part of (5.32). The inequality (5.33) is now obvious by (5.32). The details have been omitted. ∎

5.3. Applications for numerical integration. Consider the partition $I_m : a = x_0 < x_1 < ... < x_{m-1} < x_m = b$ of the interval $[a,b]$ and the intermediate points $\boldsymbol{\xi} = \left(\xi_0, ..., \xi_{m-1}\right)$ where $\xi_j \in [x_j, x_{j+1}]$, $j = 0, ..., m-1$. Define the formula

$$F_{m,n}(f, I_m, \boldsymbol{\xi}) := \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{\left[\left(x_{j+1}-\xi_j\right)^{k+1} + (-1)^k \left(\xi_j - x_j\right)^{k+1}\right]}{(k+1)!} f^{(k)}\left(\xi_j\right)$$

which can be regarded as a perturbation of Riemann's sum

$$\Gamma(f, I_m, \boldsymbol{\xi}) = \sum_{j=0}^{m-1} f\left(\xi_j\right) h_j$$

where $h_j := x_{j+1} - x_j, \quad j = 0, ..., m-1$.

The following theorem holds.

Theorem 10. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$is absolutely continuous on $[a,b]$ and I_m a partitioning of $[a,b]$ as above. Then we have the quadrature formula*

$$\int_a^b f(x)\,dx = F_{m,n}(f, I_m, \boldsymbol{\xi}) + R_{m,n}(f, I_m, \boldsymbol{\xi}) \tag{5.34}$$

where $F_{m,n}$ is defined above and the remainder $R_{m,n}$ satisfies the estimation:

$$
|R_{m,n}(f, I_m, \boldsymbol{\xi})| \tag{5.35}
$$
$$
\leq \begin{cases} \frac{\|f^{(n)}\|_\infty}{(n+1)!} \sum_{j=0}^{m-1} \left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1} - \xi_j\right)^{n+1} \right] \\ \text{for } \ f^{(n)} \in L_\infty [a,b], \\ \\ \frac{\|f^{(n)}\|_p}{n!(nq+1)^{\frac{1}{q}}} \left[\sum_{j=0}^{m-1} \left(\xi_j - x_j\right)^{nq+1} + \sum_{j=0}^{m-1} \left(x_{j+1} - \xi_j\right)^{nq+1} \right]^{\frac{1}{q}} \\ \text{for } \ f^{(n)} \in L_p [a,b], \ \text{where } \ \frac{1}{p} + \frac{1}{q} = 1, \ p > 1, \\ \\ \frac{\|f^{(n)}\|_1}{n!} \left[\frac{1}{2}\nu(h) + \max_{j=0,...,m-1} \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n \\ \text{for } \ f^{(n)} \in L_1 [a,b], \end{cases}
$$

for all $\boldsymbol{\xi}$ as above and $\nu(h) = \max\{h_j | j = 0, ..., m-1\}$.

Proof. Apply Theorem 8 on the interval $[x_j, x_{j+1}]$ to get

$$
\left| \int_{x_j}^{x_{j+1}} f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{\left[\left(x_{j+1} - \xi_j\right)^{k+1} + (-1)^k \left(\xi_j - x_j\right)^{k+1} \right]}{(k+1)!} \right] f^{(k)}\left(\xi_j\right) \right|
$$
$$
\leq \begin{cases} \frac{1}{(n+1)!} \sup_{t \in [x_j, x_{j+1}]} \left| f^{(n)}(t) \right| \left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1} - \xi_j\right)^{n+1} \right], \\ \\ \frac{1}{n!} \left(\int_{x_j}^{x_{j+1}} \left| f^{(n)}(s) \right|^p ds \right)^{\frac{1}{p}} \left[\frac{\left(\xi_j - x_j\right)^{nq+1} + \left(x_{j+1} - \xi_j\right)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \\ \\ \frac{1}{n!} \left(\int_{x_j}^{x_{j+1}} \left| f^{(n)}(s) \right| ds \right) \left[\frac{1}{2} h_j + \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n. \end{cases}
$$

Summing over j from 0 to $m-1$ and using the generalized triangle inequality, we have

$$
|R_{m,n}(f, I_n, \boldsymbol{\xi})|
$$
$$
\leq \begin{cases} \frac{1}{(n+1)!} \sum_{j=0}^{m-1} \sup_{t \in [x_j, x_{j+1}]} \left| f^{(n)}(t) \right| \left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1} - \xi_j\right)^{n+1} \right], \\ \\ \frac{1}{n!} \sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} \left| f^{(n)}(s) \right|^p ds \right)^{\frac{1}{p}} \left[\frac{\left(\xi_j - x_j\right)^{nq+1} + \left(x_{j+1} - \xi_j\right)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \\ \\ \frac{1}{n!} \sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} \left| f^{(n)}(s) \right| ds \right) \left[\frac{1}{2} h_j + \left| \xi_j - \frac{x_j - x_{j+1}}{2} \right| \right]^n. \end{cases}
$$

As $\sup_{t \in [x_j, x_{j+1}]} \left| f^{(n)}(t) \right| \leq \left\| f^{(n)} \right\|_\infty$, the first inequality in (5.35) readily follows.

Now, using the discrete Hölder inequality, we have

$$
\begin{aligned}
& \frac{1}{(nq+1)^{1/q}} \sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} \left| f^{(n)}(s) \right|^p ds \right)^{\frac{1}{p}} \left[\left(\xi_j - x_j\right)^{nq+1} + \left(x_{j+1} - \xi_j\right)^{nq+1} \right]^{\frac{1}{q}} \\
\leq \; & \frac{1}{(nq+1)^{1/q}} \left[\sum_{j=0}^{m-1} \left[\left(\int_{x_j}^{x_{j+1}} \left| f^{(n)}(s) \right|^p ds \right)^{\frac{1}{p}} \right]^p \right]^{\frac{1}{p}} \\
& \times \left[\sum_{j=0}^{m-1} \left[\left[\left(\xi_j - x_j\right)^{nq+1} + \left(x_{j+1} - \xi_j\right)^{nq+1} \right]^{\frac{1}{q}} \right]^q \right]^{\frac{1}{q}} \\
= \; & \frac{1}{(nq+1)^{1/q}} \left\| f^{(n)} \right\|_p \left[\sum_{j=0}^{m-1} \left(\xi_j - x_j\right)^{nq+1} + \sum_{j=0}^{m-1} \left(x_{j+1} - \xi_j\right)^{nq+1} \right]^{\frac{1}{q}}
\end{aligned}
$$

and thus the second inequality in (5.35) is proved.

Finally, let us observe that

$$
\begin{aligned}
& \frac{1}{n!} \sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} \left| f^{(n)}(s) \right| ds \right) \left[\frac{1}{2} h_j + \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n \\
\leq \; & \max_{j=0,\dots,m-1} \left[\frac{1}{2} h_j + \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n \sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} \left| f^{(n)}(s) \right| ds \right) \\
\leq \; & \left[\frac{1}{2} h_j + \max_{j=0,\dots,m-1} \left| \xi_j - \frac{x_j + x_{j+1}}{2} \right| \right]^n \left\| f^{(n)} \right\|_1
\end{aligned}
$$

and the last part of (5.35) is proved. ∎

Remark 15. *As $(x-a)^{\alpha} + (b-x)^{\alpha} \leq (b-a)^{\alpha}$ for $\alpha \geq 1$, $x \in [a,b]$, then we remark that the first branch of (5.35) can be bounded by*

$$
\frac{1}{(n+1)!} \left\| f^{(n)} \right\|_{\infty} \sum_{j=0}^{m-1} h_j^{n+1}. \tag{5.36}
$$

The second branch can be bounded by

$$
\frac{1}{n!(nq+1)^{1/q}} \left\| f^{(n)} \right\|_p \left[\sum_{j=0}^{m-1} h_j^{nq+1} \right]^{\frac{1}{q}} \tag{5.37}
$$

and finally, the last branch in (5.35) can be bounded by

$$
\frac{1}{n!} \left[\nu(h)\right]^n \left\| f^{(n)} \right\|_1 . \tag{5.38}
$$

Note that all the bounds provided by (5.36)-(5.38) are uniform bounds for $R_{m,n}(f, I_m, \boldsymbol{\xi})$ in terms of the intermediate points $\boldsymbol{\xi}$.

As an interesting particular case, we can consider the following perturbed midpoint formula

$$
M_{m,n}(f, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \left[\frac{1 + (-1)^k}{(k+1)!} \right] \frac{h_j^{k+1}}{2^{k+1}} f^{(k)} \left(\frac{x_j + x_{j+1}}{2} \right),
$$

which in effect involves only even k.

We state the following result concerning the estimation of the remainder term. The traditional midpoint rule (1.2), (1.3) is obtained if we take $n=2$ and $f'' \in L_2[a,b]$.

Corollary 19. *Let f and I_m be as in Theorem 10. Then we have*

$$\int_a^b f(t)\,dt = M_{m,n}(f,I_m) + R_{m,n}(f,I_m) \tag{5.39}$$

and the remainder term $R_{m,n}$ satisfies the estimation

$$|R_{m,n}(f,I_m)| \le \begin{cases} \frac{\left\|f^{(n)}\right\|_\infty}{2^n(n+1)!}\sum\limits_{j=0}^{m-1} h_j^{n+1}, & f^{(n)} \in L_\infty[a,b] \\ \frac{\left\|f^{(n)}\right\|_p}{2^n n!(nq+1)^{1/q}}\left[\sum\limits_{j=0}^{m-1} h_j^{n+1}\right]^{\frac{1}{q}}, & f^{(n)} \in L_p[a,b] \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1 \\ \frac{\left\|f^{(n)}\right\|_1}{2^n n!}[\nu(h)]^n, & f^{(n)} \in L_1[a,b]. \end{cases} \tag{5.40}$$

We can consider the following perturbed version of the trapezoid formula:

$$T_{m,n}(f,I_m) := \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{h_j^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})}{2}\right].$$

By the use of Corollary 16, we have the following approximation of the integral $\int_a^b f(t)\,dt$ in terms of $T_{m,n}(f,I_m)$:

Corollary 20. *Let f and I_m be as in Theorem 10. Then we have*

$$\int_a^b f(t)\,dt = T_{m,n}(f,I_m) + \tilde{R}_{m,n}(f,I_m) \tag{5.41}$$

and the remainder $\tilde{R}_{m,n}(f,I_m)$ satisfies the inequality

$$\left|\tilde{R}_{m,n}(f,I_m)\right| \le \frac{C_n}{(n+1)!}\left\|f^{(n)}\right\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}$$

where

$$C_n := \begin{cases} 1 & \text{if } \ n=2r \\ \frac{2^{2r+1}-1}{2^{2r}} & \text{if } \ n=2r+1 \end{cases}.$$

Remark 16. a) *If we choose $n=1$ in the above quadrature formulae* (5.34) *and* (5.39), *we recapture some results from the paper* [**4**].

b) *If we put $n=2$, then by the above Theorem 10 and Corollary 19, we recover some results from the paper* [**8**].

We omit the details.

5.4. Application of Taylor like Expansions for some particular mappings.

a) Consider $g:\mathbb{R}\to\mathbb{R}$, $g(x)=e^x$. Then $g^{(n)}(x)=e^x$, $n\in\mathbb{N}$ and

$$\left\|g^{(n+1)}\right\|_\infty=\sup_{t\in[a,y]}\left|g^{(n+1)}(t)\right|=e^y.$$

Using inequality (5.20), we have

$$\left|e^y-e^a-e^x\sum_{k=0}^{n-1}\frac{\left[(y-x)^{k+1}+(-1)^k(x-a)^{k+1}\right]}{(k+1)!}\right| \tag{5.42}$$
$$\le \frac{e^y}{(n+1)!}\left[(y-x)^{n+1}+(x-a)^{n+1}\right]$$
$$\le \frac{e^y}{(n+1)!}(y-a)^{n+1}$$

for all $a\le x\le y$.

In particular, if we choose $a=0$, then we get

$$\left|e^y-1-e^x\sum_{k=0}^{n-1}\frac{\left[(y-x)^{k+1}+(-1)^k x^{k+1}\right]}{(k+1)!}\right| \tag{5.43}$$
$$\le \frac{e^y}{(n+1)!}\left[(y-x)^{n+1}+x^{n+1}\right]\le\frac{e^y}{(n+1)!}y^{n+1}.$$

Moreover, if we choose $x=\frac{y}{2}$, then we get

$$\left|e^y-1-e^{\frac{y}{2}}\sum_{k=0}^{n-1}\frac{1+(-1)^k}{(k+1)!}\cdot\frac{y^{k+1}}{2^{k+1}}\right|\le\frac{e^y y^{n+1}}{2^n(n+1)!} \tag{5.44}$$

for all $y\ge 0$.

b) Consider $g:(0,\infty)\to\mathbb{R}$, $g(x)=\ln x$. Then

$$g^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n},\qquad n\ge 1, x>0$$

and

$$\left\|g^{(n+1)}\right\|_\infty=\sup_{t\in[a,y]}\left|\frac{(-1)^n n!}{t^{n+1}}\right|=\frac{n!}{a^{n+1}},\qquad a>0.$$

Using the inequality (5.20) we can state:

$$\left|\ln y-\ln a-\sum_{k=0}^{n-1}\frac{(y-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\cdot\frac{(-1)^k k!}{x^{k+1}}\right|$$
$$\le\frac{n!}{(n+1)!a^{n+1}}\left[(y-x)^{n+1}+(x-a)^{n+1}\right]$$
$$\le\frac{n!}{(n+1)!a^{n+1}}(y-a)^{n+1}$$

which is equivalent to

$$\begin{aligned}(5.45)\qquad & \left|\ln\left(\frac{y}{a}\right)-\sum_{k=0}^{n-1}\frac{1}{k+1}\cdot\frac{(x-a)^{k+1}+(-1)^k(y-x)^{k+1}}{x^{k+1}}\right| \\ \leq\ & \frac{(y-x)^{n+1}+(x-a)^{n+1}}{(n+1)\,a^{n+1}} \\ \leq\ & \frac{1}{(n+1)\,a^{n+1}}(y-a)^{n+1}.\end{aligned}$$

Now, if we choose in (5.45) $y=z+1, x=w+1, a=1, z\geq w\geq 0$, then we get

$$\begin{aligned}(5.46)\qquad & \left|\ln(z+1)-\sum_{k=0}^{n-1}\frac{1}{k+1}\cdot\frac{w^{k+1}+(-1)^k(z-w)^{k+1}}{(w+1)^{k+1}}\right| \\ \leq\ & \frac{(z-w)^{n+1}+w^{n+1}}{n+1}\leq\frac{1}{(n+1)}z^{n+1}.\end{aligned}$$

Finally, if we choose in (5.45), $y=ua, x=wa$ with $u\geq w>1$, then we have

$$\begin{aligned}& \left|\ln u-\sum_{k=0}^{n-1}\frac{1}{k+1}\frac{(w-1)^{k+1}+(-1)^k(u-w)^{k+1}}{w^{k+1}}\right| \\ \leq\ & \frac{(u-w)^{n+1}+(w-1)^{m+1}}{n+1}\leq\frac{(u-1)^{n+1}}{(n+1)}.\end{aligned}$$

6. Perturbed Interior Point Rules Through Grüss Type Inequalities

In 1935, G. Grüss (see for example [**14**]), proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of integrals.

Theorem 11. *Let $f,g:[a,b]\to\mathbb{R}$ be two integrable mappings so that $\phi\leq h(x)\leq\Phi$ and $\gamma\leq g(x)\leq\Gamma$ for all $x\in[a,b]$, where ϕ,Φ,γ,Γ are real numbers. Then we have*

$$(6.1)\qquad |T(h,g)|\leq\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma)$$

where

$$(6.2)\qquad T(h,g)=\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this fact as well as for extensions, generalisations, discrete variants and other associated material, see [**14**], and the papers [**23**]-[**28**] where further references are given.

A *premature* Grüss inequality is embodied in following theorem which was proved in the paper [**21**]. It provides a sharper bound than the above Grüss inequality. The term *premature* is used to denote the fact that the result is obtained from not completing the proof of the Grüss inequality if one of the functions is known explicitly.

Theorem 12. *Let h,g be integrable functions defined on $[a,b]$ and let $d \leq g(t) \leq D$. Then*

$$|T(h,g)| \leq \frac{D-d}{2}\left[T(h,h)\right]^{\frac{1}{2}}, \tag{6.3}$$

where $T(h,g)$ is as defined in (6.2).

The above Theorem 12 will now be used to provide a perturbed generalised interior point rule.

6.1. Perturbed Rules From Premature Inequalities. We start with the following result.

Theorem 13. *Let $f:[a,b]\to\mathbb{R}$ be such that the derivative $f^{(n-1)}$, $n\geq 1$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma,\Gamma\in\mathbb{R}$ such that $\gamma\leq f^{(n)}(t)\leq\Gamma$ a.e on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}
|P_M(x)| \quad : \quad &= \left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{\left[(b-x)^{k+1}+(-1)^k(x-a)^{k+1}\right]}{(k+1)!}f^{(k)}(x)\right.\\
&\quad\left.-\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\right|\\
&\leq \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}I(x,n)\\
&\leq \frac{\Gamma-\gamma}{2}\cdot\frac{n}{n+1}\cdot\frac{(b-a)^{n+1}}{\sqrt{2n+1}}, \qquad (6.4)
\end{aligned}$$

where

$$\begin{aligned}
I(x,n) &= \frac{1}{(n+1)\sqrt{2n+1}}\left\{n^2(b-a)\left[(x-a)^{2n+1}+(b-x)^{2n+1}\right]\right.\\
&\quad\left.+(2n+1)(x-a)(b-x)\left[(x-a)^n-(x-b)^n\right]^2\right\}^{\frac{1}{2}}. \qquad (6.5)
\end{aligned}$$

Proof. Applying the premature Grüss result (6.3) by associating $f^{(n)}(t)$ with $g(t)$ and $h(t)$ with $K_n(x,t)$, from (5.2), gives

$$\left|\int_a^b K_n(x,t)f^{(n)}(t)\,dt - \left(\int_a^b K_n(x,t)\,dt\right)\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right|$$

$$\leq (b-a)\frac{\Gamma-\gamma}{2}\left[T(K_n,K_n)\right]^{\frac{1}{2}}, \tag{6.6}$$

where from (6.2)

$$T(K_n,K_n)=\frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt-\left(\frac{1}{b-a}\int_a^b K_n(x,t)\,dt\right)^2.$$

Now, from (5.2),

$$\begin{aligned}
\frac{1}{b-a}\int_a^b K_n(x,t)\,dt &= \frac{1}{b-a}\left[\int_a^x\frac{(t-a)^n}{n!}dt+\int_x^b\frac{(t-b)^n}{n!}dt\right]\\
&= \frac{1}{(b-a)(n+1)!}\left[(x-a)^{n+1}+(-1)^n(b-x)^{n+1}\right]
\end{aligned}$$

and

$$\begin{aligned}\frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt &= \frac{1}{(b-a)(n!)^2}\left[\int_a^x (t-a)^{2n}\,dt + \int_x^b (b-t)^{2n}\,dt\right] \\ &= \frac{1}{(b-a)(n!)^2}\left[\frac{(x-a)^{2n+1}+(b-x)^{2n+1}}{(2n+1)}\right].\end{aligned}$$

Hence, substitution into (6.6) gives

$$\begin{aligned}&\left|\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt \right. \\ &\left. -\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!}\cdot\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\right| \\ &\leq \frac{\Gamma-\gamma}{2}\frac{1}{n!}J(x,n) \qquad (6.7)\end{aligned}$$

where

$$\begin{aligned}J^2(x,n) &= \frac{1}{(2n+1)(n+1)^2}\left\{(n+1)^2(A+B)\left(A^{2n+1}+B^{2n+1}\right)\right. \\ &\left. -(2n+1)\left(A^{n+1}+(-1)^n B^{n+1}\right)^2\right\}\end{aligned}$$

with $A=x-a,\ B=b-x$.

Now, in the proof of Theorem 31 in the previous chapter relating to the trapezoid type inequalities it was shown that $J(x,n)=(n+1)\sqrt{2n+1}I(x,n)$, where $I(x,n)$ is as given by (6.5). Thus, using identity (5.1) into (6.7) readily produces the result (6.4) and the first part of the theorem is proved. The upper bound results on noticing that $I(x,n)$ is convex and symmetric so that the maximum occurs at either of the end points. Thus, the theorem is now completely proven. ∎

Corollary 21. *Let the conditions of Theorem 31 hold. Then the following result holds*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[1+(-1)^k\right]f^{(k)}\left(\frac{a+b}{2}\right)\right. \\ &\left. -\left(\frac{b-a}{2}\right)^{n+1}\frac{\left[1+(-1)^n\right]}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\right| \\ &\leq \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}\left(\frac{b-a}{2}\right)^{n+1}\cdot\frac{1}{\sqrt{2n+1}}\cdot\begin{cases}\frac{2n}{n+1}, & n \text{ even} \\ 2, & n \text{ odd}\end{cases}. \qquad (6.8)\end{aligned}$$

Proof. Taking $x=\frac{a+b}{2}$ in (6.4) gives (6.8), where

$$I\left(\frac{a+b}{2},n\right)=\frac{1}{(n+1)\sqrt{2n+1}}\left(\frac{b-a}{2}\right)^{n+1}\left\{4n^2+(2n+1)\left[1+(-1)^n\right]^2\right\}^{\frac{1}{2}}.$$

Examining the above expression for n even or n odd readily gives the result (6.8). ∎

Remark 17. *For n odd, then the third term in the modulus sign vanished and thus there is no perturbation to the generalised midpoint rule (6.8). Further, it may be noticed that only even derivatives are present.*
For $n = 1$ then there is no perturbation term, giving

$$\left| \int_a^b f(t)\, dt - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{\Gamma - \gamma}{\sqrt{3}} \left(\frac{b-a}{2}\right)^2,$$

where $\gamma \leq f'(t) \leq \Gamma$.
The above result may be compared with (5.24).
If $n = 2$ is taken, then there is a perturbation term giving

$$\begin{aligned} & \left| \int_a^b f(t)\, dt - (b-a) f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} \left(f'(b) - f'(a)\right) \right| \\ \leq & \frac{\Gamma - \gamma}{5\sqrt{5}} \left(\frac{b-a}{2}\right)^3, \end{aligned}$$

where $\gamma \leq f''(t) \leq \Gamma$.

Theorem 14. *Let the condition of Theorem 13 be satisfied. Further, suppose that $f^{(n)}$ is differentiable and is such that*

$$\left\| f^{(n+1)} \right\|_\infty := \sup_{t \in [a,b]} \left| f^{n+1}(t) \right| < \infty.$$

Then

$$|P_M(x)| \leq \frac{b-a}{\sqrt{12}} \left\| f^{(n+1)} \right\|_\infty \cdot \frac{1}{n!} I(x,n), \tag{6.9}$$

where $P_M(x)$ is the perturbed interior point rule given by the left hand side of (6.4) and $I(x,n)$ is as given by (6.5).

Proof. Let $h, g : [a,b] \to \mathbb{R}$ be absolutely continuous and h', g' be bounded. Then Chebychev's inequality holds (see [**14**])

$$|T(h,g)| \leq \frac{(b-a)^2}{\sqrt{12}} \sup_{t \in [a,b]} |h'(t)| \cdot \sup_{t \in [a,b]} |g'(t)|.$$

Matić, Pečarić and Ujević [**21**] using a *premature* Grüss type argument proved that

$$|T(h,g)| \leq \frac{(b-a)}{\sqrt{12}} \sup_{t \in [a,b]} |g'(t)| \sqrt{T(h,h)}. \tag{6.10}$$

Thus, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $K(x,\cdot)$, from (5.2), with $h(\cdot)$ in (6.10) produces (6.9) where $I(x,n)$ is as given by (6.5). ∎

Theorem 15. *Let the conditions of Theorem 13 be satisfied. Further, suppose that $f^{(n)}$ is locally absolutely continuous on (a,b) and let $f^{(n+1)} \in L_2(a,b)$. Then*

$$|P_M(x)| \leq \frac{b-a}{\pi} \left\| f^{(n+1)} \right\|_2 \cdot \frac{1}{n!} I(x,n), \tag{6.11}$$

where $P_M(x)$ is the perturbed generalised interior point rule given by the left hand side of (6.4) and $I(x,n)$ is as given in (6.5).

Proof. The following result was obtained by Lupaş (see [**14**]). for $h, g : (a,b) \to \mathbb{R}$ locally absolutely continuous on (a,b) and $h', g' \in L_2(a,b)$, then

$$|T(h,g)| \leq \frac{(b-a)^2}{\pi^2} \|h'\|_2 \|g'\|_2,$$

where

$$\|k\|_2 := \left(\frac{1}{b-a} \int_a^b |k(t)|^2 \right)^{\frac{1}{2}} \text{ for } k \in L_2(a,b).$$

Matić, Pečarić and Ujević [**21**] further show that

$$|T(h,g)| \leq \frac{b-a}{\pi} \|g'\|_2 \sqrt{T(h,h)}. \tag{6.12}$$

Now, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $K(x,\cdot)$, from (5.2) with $h(\cdot)$ in (6.12) gives (6.11), where $I(x,n)$ is as found in (6.5). ∎

Remark 18. *Results (6.9) and (6.11) are not readily comparable to that obtained in Theorem 13 since the bound now involves the behaviour of $f^{(n+1)}(\cdot)$ rather than $f^{(n)}(\cdot)$.*

6.2. Alternate Grüss Type Results for Perturbed Interior Point Rules.

Let

$$\sigma(h(x)) = h(x) - \mathcal{M}(h) \tag{6.13}$$

where

$$\mathcal{M}(h) = \frac{1}{b-a} \int_a^b h(u)\, du. \tag{6.14}$$

Then from (6.2),

$$\mathcal{T}(h,g) = \mathcal{M}(hg) - \mathcal{M}(h)\mathcal{M}(g). \tag{6.15}$$

Dragomir and McAndrew [**36**] showed effectively, that

$$\mathcal{T}(h,g) = \mathcal{T}(\sigma(h), \sigma(g)) \tag{6.16}$$

and proceeded to obtain bounds for the trapezoidal rule. We now apply identity (6.16) to obtain interior point rules.

Theorem 16. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f(\cdot)$ is absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$*

$$\left| \int_a^b f(t)\, dt - (b-a) f(x) + (b-a)\left(x - \frac{a+b}{2}\right) S \right|$$

$$\leq \begin{cases} \left(\frac{b-a}{2}\right)^2 \|\sigma(f')\|_\infty, & \text{if } \; f' \in L_\infty[a,b] \\ \\ \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|\sigma(f')\|_p, & \text{if } \; f' \in L_p[a,b] \\ & \text{with } \; \frac{1}{p}+\frac{1}{q}=1, \;\; p>1, \\ \frac{b-a}{2} \|\sigma(f')\|_1, & \text{if } \; f' \in L_1[a,b], \end{cases} \tag{6.17}$$

where $S = \frac{f(b)-f(a)}{b-a}$, the secant slope.

Proof. Using identity (6.16), associate with $h(t)$, $K_1(x,t)$ from (5.2) and $f'(t)$ for $g(t)$ then

$$\begin{aligned} & \int_a^b K_1(x,t) f'(t)\,dt - (b-a)\mathcal{M}(K_1(x,\cdot))S \\ = & \int_a^b [K_1(x,t) - \mathcal{M}(K_1(x,\cdot))][f'(t)-S]\,dt, \end{aligned} \tag{6.18}$$

where, from (5.2),

$$\begin{aligned} \mathcal{M}(K_1(x,t)) &= \frac{1}{b-a}\left[\int_a^x (t-a)\,dt + \int_x^b (t-b)\,dt\right] \\ &= x - \frac{a+b}{2} \end{aligned} \tag{6.19}$$

and from (6.13),

$$\sigma(K_1(x,\cdot)) = \begin{cases} t-a+\frac{a+b}{2}-x, & t\in[a,x] \\ t-b+\frac{a+b}{2}-x, & t\in(x,b] \end{cases}. \tag{6.20}$$

Thus, (6.18), on utilising (6.19), (6.20) and (5.3) may be written, on taking the modulus, as

$$\begin{aligned} & \left|\int_a^b f(t)\,dt - (b-a)f(x) + (b-a)\left(x-\frac{a+b}{2}\right)S\right| \\ = & \left|\int_a^b \sigma(K_1(x,t))(f'(t)-S)\,dt\right| := B(x). \end{aligned} \tag{6.21}$$

Now, observe that

$$\begin{aligned} B(x) &\le \|f'(\cdot)-S\|_\infty \int_a^b |\sigma(K_1(x,t))|\,dt \\ &= \|f'(\cdot)-S\|_\infty \left[\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}} |u|\,du + \int_{-\left(\frac{b-a}{2}\right)}^{\frac{a+b}{2}-x} |v|\,dv\right] \end{aligned}$$

and so after some algebra, the first bound in (6.17) is obtained.

Now, from (6.21) using Hölder's inequality we have that

$$B(x) \le \|\sigma(f')\|_p \left(\int_a^b |\sigma(K_1(x,t))|^q\,dt\right)^{\frac{1}{q}}. \tag{6.22}$$

Now,

$$\int_a^b |\sigma(K_1(x,t))|^q\,dt = \int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}} |u|^q\,du + \int_{-\left(\frac{b-a}{2}\right)}^{\frac{a+b}{2}-x} |v|^q\,dv.$$

That is,

$$L = L_1 + L_2, \quad \text{say.}$$

For $x < \frac{a+b}{2}$,

$$(q+1) L_1 = \int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}} u^q du = \left(\frac{b-a}{2}\right)^{q+1} - \left(\frac{a+b}{2} - x\right)^{q+1}$$

and

$$\begin{aligned}(q+1) L_2 &= \int_{-\frac{b-a}{2}}^{0} |v|^q dv + \int_0^{\frac{a+b}{2}-x} v^q dv \\ &= \left(\frac{b-a}{2}\right)^{q+1} - \left(\frac{a+b}{2} - x\right)^{q+1}.\end{aligned}$$

Hence,

$$L = \frac{b-a}{q+1} \cdot \left(\frac{b-a}{2}\right)^q.$$

A similar argument holds for $x > \frac{a+b}{2}$ and so from (6.22) and (6.21), the second inequality in (6.17) is obtained.

For the third inequality we note from (6.21) that

$$\begin{aligned}B(x) &\leq \|\sigma(f')\|_1 \|\sigma(K(x,\cdot))\|_\infty \\ &= \sup_{t\in[a,b]} |\sigma(K(x,t))| \cdot \|\sigma(f')\|_1 .\end{aligned} \tag{6.23}$$

Now,

$$\sup_{t\in[a,b]} |\sigma(K(x,t))| = \max\left\{\frac{b-a}{2}, \left|x - \frac{b-a}{2}\right|\right\}$$

and using the result

$$\max\{X, Y\} = \frac{X+Y}{2} + \frac{1}{2}|Y - X|$$

gives, on treating the cases $x > \frac{a+b}{2}$ and $x < \frac{a+b}{2}$ separately,

$$\max\left\{\frac{b-a}{2}, \left|x - \frac{b-a}{2}\right|\right\} = \frac{b-a}{2}.$$

Thus, on substitution into (6.23), we obtain the third result in (6.17), and the theorem is completely proved. ∎

Remark 19. *The results of this section allow the consideration of $f' \in L_p[a,b]$, $p \in [1,\infty)$ whereas the results of Section 6.1 to produce the perturbed rules are valid using $\|\cdot\|_\infty$. The working, however, for explicit results with $K_n(x,t)$ as given by (5.2), is somewhat more difficult with the methodology of the current section.*

This section considers results involving $\sigma(f')$ which may be more useful when information is known about deviations of the slope from its mean.

7. An Ostrowski Type Inequality for Mappings Whose Second Derivatives Are Bounded

In [**4**], S.S. Dragomir and S. Wang obtained the Ostrowski type inequality using for the proof essentially, the identity

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt \tag{7.1}$$

for all $x \in [a,b]$, where f is as above and the kernel, $p(\cdot,\cdot) : [a,b]^2 \to \mathbb{R}$, is given by

$$p(x,t) := \begin{cases} t-a & \text{if } t \in [a,x] \\ \\ t-b & \text{if } t \in (x,b] \end{cases} . \tag{7.2}$$

Identity (7.1) is easily proven from considering $\int_a^b p(x,t) f'(t)\,dt$ and integrating by parts.

The main aim of this section is to obtain a perturbed interior point rule in which the perturbation does not involve derivative evaluations.

7.1. A New Integral Inequality. The following results hold (see for example [**12**]).

THEOREM 17. *Let $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ and twice differentiable function on (a,b), whose second derivative $f'' : (a,b) \to \mathbb{R}$ is bounded on (a,b). Then we have the inequality*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \\ \le\; & \frac{1}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right]^2 + \frac{1}{12}\right\}(b-a)^2 \|f''\|_\infty \\ \le\; & \frac{\|f''\|_\infty}{6}(b-a)^2 \end{aligned} \tag{7.3}$$

for all $x \in [a,b]$.

PROOF. Applying the identity (7.1) for $f'(\cdot)$ we can state

$$f'(t) = \frac{1}{b-a}\int_a^b f'(s)\,ds + \frac{1}{b-a}\int_a^b p(t,s) f''(s)\,ds,$$

which is equivalent to

$$f'(t) = \frac{f(b)-f(a)}{b-a} + \frac{1}{b-a}\int_a^b p(t,s) f''(s)\,ds.$$

Substituting $f'(t)$ in the right hand side of (7.1) we get

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\quad + \frac{1}{b-a}\int_a^b p(x,t)\left[\frac{f(b)-f(a)}{b-a} + \frac{1}{b-a}\int_a^b p(t,s) f''(s)\,ds\right]dt \\ &= \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{f(b)-f(a)}{(b-a)^2}\int_a^b p(x,t)\,dt \\ &\quad + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)\,p(t,s) f''(s)\,ds\,dt \end{aligned}$$

and as

$$\begin{aligned}\int_a^b p(x,t)\,dt &= \int_a^x (t-a)\,dt + \int_x^b (t-b)\,dt \\ &= (b-a)\left(x - \frac{a+b}{2}\right),\end{aligned}$$

the integral identity

$$\begin{aligned}f(x) &= \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right) \\ &\quad + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)\,p(t,s)\,f''(s)\,dsdt \end{aligned} \tag{7.4}$$

results for all $x \in [a,b]$.

Now, using the identity (7.4), we get

$$\begin{aligned}&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b |p(x,t)\,p(t,s)|\,|f''(s)|\,dsdt \\ &\leq \frac{\|f''\|_\infty}{(b-a)^2}\int_a^b\int_a^b |p(x,t)|\,|p(t,s)|\,dsdt \\ &: = \frac{\|f''\|_\infty}{(b-a)^2}A(x).\end{aligned} \tag{7.5}$$

We have

$$\int_a^b |p(t,s)|\,ds = \frac{(t-a)^2 + (b-t)^2}{2}$$

and so, from (7.5),

$$\begin{aligned}A(x) &= \int_a^b |p(x,t)|\left[\frac{(t-a)^2+(b-t)^2}{2}\right]dt \\ &= \frac{1}{2}\left[\int_a^x (t-a)\left[(t-a)^2+(b-t)^2\right]dt + \int_x^b (b-t)\left[(t-a)^2+(b-t)^2\right]dt\right] \\ &= \frac{1}{2}\left[\int_a^x \left[(t-a)^3 + (t-a)(b-t)^2\right]dt + \int_x^b \left[(t-a)^2(b-t)+(b-t)^3\right]dt\right].\end{aligned}$$

Note that

$$\begin{aligned}\int_a^x (t-a)^3\,dt &= \frac{(x-a)^4}{4},\quad \int_x^b (t-b)^3\,dt = \frac{(x-b)^4}{4}, \\ \int_a^x (t-a)(b-t)^2\,dt &= -\frac{1}{3}(b-x)^3(x-a) - \frac{1}{12}(b-x)^4 + \frac{1}{12}(b-x)^4 \\ \text{and } \int_x^b (t-b)(t-a)^2\,dt &= \frac{1}{3}(x-a)^3(b-x) - \frac{1}{12}(b-a)^4 + \frac{1}{12}(x-a)^4.\end{aligned}$$

Consequently, we have

$$\begin{aligned} A(x) &= \frac{1}{12}\Big[(x-a)^4 - 2(b-x)^3(x-a) - 2(x-a)^3(b-x) \\ &\quad + (b-x)^4 + (b-a)^4\Big], \end{aligned}$$

which may be simplified in a variety of ways to give

$$A(x) = \frac{1}{12}\left[6\left(x-\frac{a+b}{2}\right)^4 + 3(b-a)^2\left(x-\frac{a+b}{2}\right)^2 + \frac{7}{8}(b-a)^4\right].$$

Now, using the inequality (7.5) and simple algebraic manipulations, we get the first result in (7.3).

The second part is obvious from the fact that

$$\left|x-\frac{a+b}{2}\right| \le \frac{b-a}{2}$$

for all $x \in [a,b]$. ∎

7.2. Applications in Numerical Integration. Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,1,...,n-1)$ a sequence of intermediate points and $h_i := x_{i+1} - x_i$ $(i = 0,1,...,n-1)$. As in [**12**], consider the perturbed Riemann's sum defined by

$$A_G(f, I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} f(\xi_i) h_i - \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)(f(x_{i+1}) - f(x_i)). \tag{7.6}$$

In that paper Dragomir and Wang [**5**] proved the following result:

THEOREM 18. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) and assume that*

$$\gamma \le f'(x) \le \Gamma \quad \text{for all } x \in (a,b). \tag{7.7}$$

Then we have the quadrature formula:

$$\int_a^b f(x)\,dx = A_G(f, I_n, \boldsymbol{\xi}) + R_G(f, I_n, \boldsymbol{\xi}), \tag{7.8}$$

where the remainder $R_G(f, I_n, \boldsymbol{\xi})$ satisfies the estimation

$$|R_G(f, I_n, \boldsymbol{\xi})| \le \frac{1}{4}(\Gamma - \gamma)\sum_{i=0}^{n-1} h_i^2, \tag{7.9}$$

for all $\boldsymbol{\xi} = (\xi_0, ..., \xi_{n-1})$ as above.

Here, we prove another type of estimation for the remainder $R_G(f, I_n, \boldsymbol{\xi})$ in the case when f is twice differentiable [**12**].

THEOREM 19. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and a twice differentiable function on (a,b), whose second derivative, $f'' : (a,b) \to \mathbb{R}$, is bounded on (a,b). Denote $\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the quadrature formula*

(7.8)*, where the remainder* $R_G(f, I_n, \boldsymbol{\xi})$ *satisfies the estimation:*

$$
\begin{aligned}
(7.10)\qquad & |R_G(f, I_n, \boldsymbol{\xi})| \\
& \leq \frac{\|f''\|_\infty}{2} \sum_{i=0}^{n-1} \left\{ \left[\frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2}{h_i^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} h_i^3 \\
& \leq \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} h_i^3,
\end{aligned}
$$

for all ξ_i *as above.*

Proof. Apply Theorem 17 on the interval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to obtain

$$
\begin{aligned}
& \left| f(\xi_i) h_i - \int_{x_i}^{x_{i+1}} f(t)\, dt - \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)(f(x_{i+1}) - f(x_i)) \right| \\
\leq & \frac{\|f''\|_\infty}{2} \left[\left[\frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2}{h_i^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right] h_i^3 \leq \frac{\|f''\|_\infty}{6} h_i^3
\end{aligned}
$$

for all $\xi_i \in [x_i, x_{i+1}]$ and $i \in \{0, ..., n-1\}$.

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired inequality (7.10).

We omit the details. ∎

8. Concluding Remarks

The current work has demonstrated the development of interior point rules which contains the midpoint rule as a special case. Identities are obtained by using a Peano kernel approach which enables us, through the use of the modern theory of inequalities, to obtain bounds in terms of a variety of norms. This is useful in practice as the behaviour of the function would necessitate the use of one norm over another. Although not all inequalities have been developed into composite quadrature rules, we believe that enough demonstrations have been given that would enable the reader to proceed further.

It has been shown that the bounds for interior point rules are the same as those obtained for the trapezoidal rules of the previous chapter as highlighted in Remark 10.

Rules have also been developed that do not necessarily require the second derivative to be well behaved or indeed, exist, thus allowing the treatment of a much larger class of functions. Rules have been developed by examining the Riemann-Stieltjes integral. Additionally, the results also allow for a non-uniform partition, thus giving the user the option of choosing a partition that minimises the bound or enabling the calculation of the bound given a particular partition.

If we wish to approximate the integral $\int_a^b f(x)\, dx$ using a quadrature rule $Q(f, I_n)$ with bound $B(n)$, where I_n is a uniform partition for example, with an accuracy of $\varepsilon > 0$, then we will need $n_\varepsilon \in \mathbb{N}$ where

$$n_\varepsilon \geq \left[B^{-1}(\varepsilon)\right] + 1$$

with $[x]$ denoting the integer part of x.

This approach enables the user to predetermine the partition required to *assure* that the result is within a certain tolerance rather than utilizing the commonly used method of halving the mesh size and comparing the resulting estimation.

We conclude the work by bringing to the attention of the reader that three-point rules may be obtained by taking a convex combination of trapezoidal type identities, $I_T(x)$ of the previous Chapter and interior point identities of the current chapter, $I_M(x)$. That is,

$$\lambda I_T(x) + (1-\lambda) I_M(x).$$

Simpson type rules would result from taking $\lambda = \frac{1}{3}$ and $x = \frac{a+b}{2}$.

This will not be presented here.

For a three-point quadrature rule involving at most the first derivative, see Cerone and Dragomir [**19**].

References

[1] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities for Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.

[2] S.S. DRAGOMIR, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *submitted.*

[3] K.E. ATKINSON, *An Introduction to Numerical Analysis*, J. Wiley and Sons, Singapore, 1998, (2nd Edition).

[4] S.S. DRAGOMIR and S. WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[5] S.S. DRAGOMIR and S. WANG, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.,* **33** (1997), 15-20.

[6] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_p- norm, *Indian Journal of Mathematics,* (in press).

[7] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_1 norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28**(1997), 239-244.

[8] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS[1], An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications. *RGMIA Research Report Collection*, **1** (1) (1998), 33-39.

[9] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, On Ostrowski type for mappings whose second derivatives belong to $L_1(a,b)$ and applications, *RGMIA Research Report Collection*, **1** (2) (1998), 53-60.

[10] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, An Ostrowski type inequality for mappings whose second derivatives belong to $L_p(a,b)$ and applications, *RGMIA Research Report Collection,* **1** (1) (1998), 41-50.

[11] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, An inequality of Ostrowski-Grüss type for twice differentiable mappings and applications, *RGMIA Research Report Collection,* **1** (2) (1998), 61-63.

[12] S.S. DRAGOMIR and N.S. BARNETT, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications. *RGMIA Research Report Collection*, **1** (2) (1999), 67-76.

[13] S.S. DRAGOMIR and A. SOFO, An integral inequality for twice differentiable mappings and applications, *RGMIA Research Report Collection,* **2** (2) (1999).

[14] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[1] All the papers from RGMIA Res. Rep. Coll. are available at the address: http://rgmia.vu.edu.au

[15] P. CERONE, S. S. DRAGOMIR and J. ROUMELIOTIS, Some Ostrowski type inequalities for n-time differentiable mappings and applications , *Preprint. RGMIA Res. Rep Coll., Vol.1, No.1 (1998), 51-66.* `http://rgmia.vu.edu.au`, *Demonstratio Math.,* in press.

[16] S.S. DRAGOMIR, J.E. PEČARIĆ and S. WANG, The Unified treatment of trapezoid, Simpson and Ostrowski type inequality for monotonic mappings and applications, *RGMIA Res. Rep. Coll.*, 4, **2**(1999), Article 3.

[17] V. ČULJAK, C.E.M. PEARCE and J. E. PEČARIĆ, The unified treatment of some inequalities of Ostrowski and Simpson's type, *RGMIA Res. Rep. Coll.*, 4, **2**(1999), Article 8.

[18] S.S. DRAGOMIR, A Taylor like formula and application in numerical integration, *submitted.*

[19] P. CERONE and S. S. DRAGOMIR, Three point quadrature rules involving, at most, a first derivative, *submitted, RGMIA Res. Rep. Coll.*, 4, **2**(1999), Article 8.

[20] J.E. PEČARIĆ, F. PROSCHAN and Y.L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, 1992.

[21] M. MATIĆ, J.E. PEČARIĆ and N. UJEVIĆ, On new estimation of the remainder in generalised Taylor's formula, *M.I.A.,* Vol. **2** No. 3 (1999), 343-361.

[22] S.S. DRAGOMIR, Y.J. CHO and S.S. KIM, Some remarks on the Milovanović-Pečarić inequality and in applications for special means and numerical integration*., Tamkang Journal of Mathematics,* accepted.

[23] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* accepted.

[24] S.S. DRAGOMIR, Grüss inequality in inner product spaces, *The Australian Math. Gazette, 2,* **26** (1999), 66-70.

[25] S.S. DRAGOMIR, New estimation of the remainder in Taylor's formula using Grüss' type inequalities and applications, *Mathematical Inequalities and Applications, 2,* **2** (1999), 183-194.

[26] S.S. DRAGOMIR and I. FEDOTOV, An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Maths.*, (4) **29** (1998), 287-292.

[27] I. FEDOTOV and S.S. DRAGOMIR, Another approach to quadrature methods, *Indian Journal of Pure and Applied Mathematics*, (in press).

[28] S.S. DRAGOMIR, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang Journal of Mathematics,* accepted.

[29] K.E. ATKINSON, *An Introduction to Numerical Analysis,* Wiley and Sons, Second Edition, 1989.

[30] T. M. APOSTOL, *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975.

[31] P.J. DAVIS and P. RABINOWITZ, *Methods of Numerical Integration*, 2nd Ed., Academic Press, New York, 1994.

[32] H. ENGELS, *Numerical Quadrature and Cubature*, Academic Press, New York, 1980.

[33] A.H. STROUD, *Numerical Quadrature and Solution of Ordinary Differential Equations*, Springer-Verlag, New York, 1974.

[34] A.R. KROMMER and C.W. UEBERHUBER, *Numerical Integration on Advanced Computer Systems*, Lecture Notes in Computer Science, 848, Springer-Verlag, Berlin, 1994.

[35] W.H. PRESS, B.P. FLANNERY, S.A. TELKOLSKY and W.T. VERLING, *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge, 1986.

[36] S.S. DRAGOMIR and A. McANDREW, On trapezoid inequality via Grüss type results and applications, *RGMIA Res. Rep. Coll.,* **2** (2) (1999), Article 6.

[37] S.S. DRAGOMIR, Ostrowski's Inequality for monotonic mappings and applications, *JK-SIAM,* in press.

[38] S.S. DRAGOMIR, On the Ostrowski's integral inequality for Lipschitzian mappings and applications, *Computers and Mathematics with Appl.,* in press.

[39] J. SÁNDOR, On the Jensen-Hadamard inequality, *Studia Univ. Babes-Bolyai, Math.,* **36** (1) (1991), 9-15.

[40] S.S. DRAGOMIR, P. CERONE and A. SOFO, Some remarks on the midpoint rule in numerical integration, *RGMIA Res. Rep. Coll.,* **1** (2) (1998), Article 4.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: pc@matilda.vu.edu.au

E-mail address: sever.dragomir@vu.edu.au
URL: http://matilda.vu.edu.au/~rgmia/dragomirweb.html