ON NEW PROOFS OF INEQUALITIES INVOLVING TRIGONOMETRIC FUNCTIONS

BAI-NI GUO, WEI LI, BAO-MIN QIAO, AND FENG QI

ABSTRACT. In the note, some new proofs for inequalities involving trigonometric functions are given.

1. Introduction

In [9], J. B. Wilker proposed that
(a) If $0 < x < \frac{\pi}{2}$, then
\[(\sin x)^2 + \frac{\tan x}{x} > 2.\]
(b) There exists a largest constant $c$ such that
\[(\sin x)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x\]
for $0 < x < \frac{\pi}{2}$.

In [8], the inequality (1) was proved, and the following inequalities were also obtained
\[2 + \frac{8}{45} x^3 \tan x > (\sin x)^2 + \frac{\tan x}{x} > 2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x.\]
The constants $\frac{8}{45}$ and $(\frac{2}{\pi})^4$ are best possible, that is, they cannot be replaced by smaller or larger numbers respectively.

The inequalities in (1) and (3) are called Wilker’s inequalities in [3].
In this note, we will give new proofs for the inequalities in (1) and (3).

2. A New Proof of Inequality (1)

The inequality (1) can be rewritten as
\[\sin^2 x \cos x + x \sin x > 2x^2 \cos x.\]

Let
\[g(x) = \sin^2 x \cos x + x \sin x - 2x^2 \cos x, \quad x \in \left(0, \frac{\pi}{2}\right),\]
\[h(x) = 2 \sin x \cos^2 x - 3x \cos x + (1 + x^2) \sin x, \quad x \in \left(0, \frac{\pi}{2}\right).\]

Direct calculation yields
\[g'(x) = 2 \sin x \cos^2 x - \sin^3 x + \sin x + x \cos x - 4x \cos x + 2x^2 \sin x\]
for 0 < x < \frac{\pi}{2}. Easy computation yields

\begin{align*}
\psi(x) &= \sin 2x = \frac{2}{2x^5} + \frac{1}{x^3} - \frac{2 \cot x}{x^3} \\
\psi'(x) &= -\frac{5 \sin 2x}{2x^6} + \frac{\cos 2x}{x^5} - \frac{4}{x^3} + \frac{6 \cos x}{x^4} \sin x + \frac{2}{x^3 \sin^2 x}.
\end{align*}

It is well-known [1, p. 226–227] that

\begin{align*}
\sin 2x &= 2x - \frac{4}{3} x^3 + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+5} x^{2n+5}}{(2n+5)!}, \\
\cot x &= \frac{1}{x} - \frac{1}{3} x - \sum_{n=0}^{\infty} \frac{2^{2n+4} B_{2n+2}}{(2n+4)!} x^{2n+3},
\end{align*}

where \( B_n \) denotes the \( n \)-th Bernoulli number, which is defined in [1, p. 228] by

\begin{equation}
\frac{t}{e^t-1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_k}{(2k)!} x^{2k}, \quad |t| < 2\pi.
\end{equation}

Therefore, by direct computation, we have

\begin{equation}
\psi(x) = \sum_{n=0}^{\infty} \frac{2^{2n+4}}{(2n+5)!} \{2(2n+5)B_{n+2} + (-1)^n\} x^{2n}.
\end{equation}

From the identity in [1, p. 231]

\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{\pi^{2n} \cdot 2^{2n-1}}{(2n)!} B_n,
\end{equation}

by mathematical induction, for \( n > 2 \), we have

\begin{equation}
2(2n+5)B_{n+2} = 4 \cdot (2n+5)! \sum_{k=1}^{\infty} \frac{1}{k^{2n+4}} > 4 \cdot (2n+5)! > 1,
\end{equation}

then \( \phi''(x) \geq 0 \), where \( \phi(x) = \psi(\sqrt{x}) \), and \( \phi'(x) \) is increasing on \( (0, \frac{\pi^2}{4}) \). Since \( \phi'(\left(\frac{\pi}{2}\right)^2) = \psi'(\frac{\pi}{2}) = 2 \cdot (\frac{\pi}{2})^3 \cdot (1 - \frac{10}{\pi^2}) < 0 \), hence \( \phi'(x) < 0 \), and then \( \phi(x) \) is decreasing, that is \( \psi(x) \) is decreasing on \( (0, \frac{\pi}{2}) \), then we have

\begin{equation}
\frac{8}{45} = \psi(0) > \psi(x) > \psi\left(\frac{\pi}{2}\right) = \frac{16}{\pi^4}, \quad x \in \left(0, \frac{\pi}{2}\right).
\end{equation}

Inequalities in (15) are equivalent to those in (3). The proof of inequalities in (3) is complete.

Remark 1. For details about Bernoulli numbers, also please refer to [2, 5, 7].
Remark 2. Using Tchebysheff’s integral inequality, many inequalities involving the function $\frac{\sin x}{x}$ are constructed in [6].

REFERENCES

[2] Sen-Lin Guo and Feng Qi, Recursion formulae for $\sum_{m=1}^{n} m^k$, Zeitschrift für Analysis und ihre Anwendungen 18 (1999), no. 4, 1123–1130.

DEPARTMENT OF MATHEMATICS, JIAOZUO INSTITUTE OF TECHNOLOGY, JIAOZUO CITY, HENAN 454000, THE PEOPLE’S REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, THE FIRST TEACHER’S COLLEGE OF LOYANG, LOYANG CITY, HENAN PROVINCE, CHINA

DEPARTMENT OF MATHEMATICS, SHANGQIU EDUCATION COLLEGE, SHANGQIU CITY, HENAN PROVINCE, CHINA

DEPARTMENT OF MATHEMATICS, JIAOZUO INSTITUTE OF TECHNOLOGY, JIAOZUO CITY, HENAN 454000, THE PEOPLE’S REPUBLIC OF CHINA

E-mail address: qifeng@jzit.edu.cn
URL: http://rgmia.vu.edu.au/qi.html