EXPONENTIAL STABILITY AND BOUNDED CONVOLUTIONS

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Abstract

We consider a mild solution $u_f$ of a well-posed inhomogeneous, Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \quad u(0) = 0$$

on a Banach space $X$, where $A(\cdot)$ is periodic. We prove that if for every almost periodic $X$-valued functions $f$, with $f(0) = 0$, the solution $u_f$ is almost periodic, then the solution of the well-posed Cauchy problem

$$\dot{u}(t) = A(t)u(t), \quad u(0) = x \in X,$$

is uniformly exponentially stable.

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1. Introduction

Let $X$ be a complex Banach space and $\mathcal{L}(X)$ the space of all bounded and linear operators on $X$. We denote by $|| \cdot ||$ the norms of vectors and operators on $X$. Let $BUC(\mathbb{R}_+, X)$ the Banach space of all $X$-valued bounded and uniformly continuous functions on $\mathbb{R}_+$ endowed with sup-norm and $AP(\mathbb{R}_+, X)$ the space of almost periodic function in the sense of Bohr, i.e. the linear closed hull in $BUC(\mathbb{R}_+, X)$ of the set of all functions

$$\{ e^{ij} x : \mu \in \mathbb{R}, \ x \in X \}.$$

Let $AP_0(\mathbb{R}_+, X)$ the set of all functions $f \in AP(\mathbb{R}_+, X)$ such that $f(0) = 0$. It is clear that $AP_0(\mathbb{R}_+, X)$ is a closed subspace of $AP(\mathbb{R}_+, X)$ or of $BUC(\mathbb{R}_+, X)$. We recall that a strongly continuous semigroup on $X$ is a family $T = \{ T(t) \}_{t \geq 0}$ of bounded linear operators acting on the Banach space $X$ which satisfies the following conditions:
• (i) $T(t + s) = T(t)T(s)$ for all $t, s \in \mathbb{R}_+ := [0, \infty)$;
• (ii) $T(0) = Id$, $Id$ is the identity operator on $\mathcal{L}(X)$;
• (iii) the function $t \mapsto T(t)x : \mathbb{R}_+ \to X$ is continuous on $\mathbb{R}_+$ for all $x \in X$ (or, equivalently this function is continuous in $t = 0$).

Let $T$ be a strongly continuous semigroup on $X$ and $A$ its infinitesimal generator. It is well known that in this case the Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x \in X$$

is well-posed and the mild solution of (1.1) is defined by

$$x(t) = T(t)x \quad (t \geq 0).$$

For a locally integrable function $f : \mathbb{R}_+ \to X$, a mild solution of the inhomogeneous Cauchy problem

$$\dot{u}(t) = Au(t) + f(t) \quad (t \geq 0), \quad u(0) = 0$$

is defined by

$$u_f(t) = \int_0^t T(t - \xi)f(\xi)d\xi, \quad t \geq 0.$$ 

For a well-posed Cauchy problem

$$\dot{x}(t) = A(t)x(t) \quad (t \geq 0), \quad x(0) = x \in X$$

with (unbounded) linear operators $A(t)$ the solution lead to an evolution family $U = \{U(t,s) : t \geq s \geq 0\}$ in the space $\mathcal{L}(X)$, that is

• (e$_1$) $U(t,t) = Id, U(t,\tau)U(\tau,s) = U(t,s)$ for $t \geq \tau \geq s \geq 0$;
• (e$_2$) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$.

When the Cauchy problem (1.2) is periodic, i.e. there exists $q > 0$ such that $A(t + q) = A(t)$ for all $t \in \mathbb{R}_+$, the corresponding evolution family $U$ on $X$ has exponential growth, i.e. there exist $\omega \in \mathbb{R}$ and $M > 0$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)} \quad \forall t \geq s \geq 0,$$  

(1.3)
see [BP, Lemma 4.1] or [DK, Theorem 6.6]. We recall that an evolution family $U$, as above, is called uniformly exponentially stable if there are $\omega < 0$ and $M > 0$ such that (1.3) holds. For a locally integrable function $f : \mathbb{R}_+ \to X$ a mild solution of the well-posed inhomogeneous Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \quad (t \geq 0), \quad u(0) = 0$$

is defined by

$$u_f(t) = \int_0^t U(t, \tau)f(\tau)d\tau \quad (t \geq 0).$$

We shall prove the following two theorems.

**THEOREM 1.** Let $T = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on $X$. The following statements are equivalent:

- **(1)** $T$ is uniformly exponentially stable, i.e. its growth bound

  $$\omega_0(T) := \lim_{t \to \infty} \frac{\ln ||T(t)||}{t}$$

  is negative;

- **(2)** the function $t \mapsto \int_0^t T(\xi)f(t - \xi)d\xi : \mathbb{R}_+ \to X$ belongs to $A\mathcal{P}_0(\mathbb{R}_+, X)$ for all $f \in A\mathcal{P}_0(\mathbb{R}_+, X)$;

- **(3)** $\sup_{t \geq 0} ||\int_0^t T(\xi)f(t - \xi)d\xi|| = M_f < \infty$, $\forall f \in A\mathcal{P}_0(\mathbb{R}_+, X)$.

**THEOREM 2.** Let $U = \{U(t,s) : t \geq s \geq 0\}$ be a q-periodic evolution family on $X$. The following statements are equivalent:

- **(i)** $U$ is uniformly exponentially stable;

- **(ii)** the function $t \mapsto u_f(t) = \int_0^t U(t,\xi)f(\xi)d\xi : \mathbb{R}_+ \to X$ belongs to $A\mathcal{P}_0(\mathbb{R}_+, X)$ for all $f \in A\mathcal{P}_0(\mathbb{R}_+, X)$;

- **(iii)** $\sup_{t \geq 0} ||\int_0^t U(t,\xi)f(\xi)d\xi|| = K_f < \infty$, for all $f \in A\mathcal{P}_0(\mathbb{R}_+, X)$. 


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2. Proofs of the theorems

Proof of Theorem 1. The proof of implications (1) ⇒ (3) and (2) ⇒ (3) are obvious and we omit the details. The proof of (3) ⇒ (1) is based on the following result which has been proved in [BDL, Proposition 4], see also [VS, Corollary 4.5 and its Reformulation] for a related result:

If \( \sup_{t>0} ||f^t_0 e^{-i\mu \xi} T(t-\xi) g(\xi) d\xi|| < \infty \) for every \( g \in P^0_q(\mathbb{R}_+, X) \) and some \( \mu \in \mathbb{R} \) then \( T(g) \) is power bounded and \( e^{i\mu} \in \rho(T(q)) \). Here \( P^0_q(\mathbb{R}_+, X) \) is the set of all \( X \)-valued continuous functions such that \( f(t+q) = f(t) \) for any \( t \geq 0 \) and \( f(0) = 0 \).

Now we prove that (1) implies (2). Let \( T = \{T^t\}_{t \geq 0} \) the evolution semigroup associated of \( T \) on the space \( AP_0(\mathbb{R}_+, X) \), i.e.,

\[
(T^t f)(s) = \begin{cases} 
T(t) f(s-t), & s \geq t \\
0, & 0 \leq s \leq t
\end{cases}
\]

for every \( f \in AP_0(\mathbb{R}_+, X) \). It is easy to see that \( T^t \) acts on \( AP_0(\mathbb{R}_+, X) \) for all \( t \geq 0 \) and, in addition, \( T \) is strongly continuous, see [NM, Lemma 2]. Let \( (G, D(G)) \) the infinitesimal generator of \( T \) and \( u, f \in AP_0(\mathbb{R}_+, X) \).

As in [MRS, Lemma 1.1] it is can be proves that \( u \in D(G) \) and \( Gu = -f \) if and only if \( u = u_f \). Moreover, if \( T \) is uniformly exponentially stable then the growth bound of \( T \) is negative, hence \( G \) is invertible. It follows that \( u_f \in D(G) \subset AP_0(\mathbb{R}_+, X) \) and the proof of Theorem 1 is finished.

Proof of Theorem 2. The proof of (iii) ⇒ (i) follows from the fact that if \( u_{e^{-\mu t} g(\cdot)} \) is bounded for every \( g \in P^0_q(\mathbb{R}_+, X) \) and some \( \mu \in \mathbb{R} \) then the monodromy operator \( V := U(q,0) \) is power bounded and \( e^{i\mu} \in \rho(V) \), see [B, Proof of Theorem 4]. Here \( \rho(V) \) is the resolvent set of \( V \). The proofs of (i) ⇒ (iii) and (ii) ⇒ (iii) are obvious and the proof of (i) ⇒ (ii) follows along the lines of (1) ⇒ (2) from Theorem 1. Another proof for the implication (ii) ⇒ (i) we will give here. This proof is based on a method indicated in [CLMR, Theorem 2.5]. Let \( h : \mathbb{R} \rightarrow AP_0(\mathbb{R}_+, X) \), \( (G, D(G)) \) the infinitesimal generator of the evolutionary semigroup \( E \) associated to \( U \) on \( AP_0(\mathbb{R}_+, X) \), and

\[
[(\tilde{G} h)(\theta)](t) := [G h(\theta)](t) = \int_0^t U(t, t-\xi) (h(\theta))(t-\xi)d\xi, \quad \theta \in \mathbb{R}, t \geq 0.
\]
It is easy to see that the function
\[
\theta \mapsto \int_0^\infty U(\cdot, \cdot - \xi)h(\theta)(\cdot - \xi)d\xi
\]
belongs to \(AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))\) for all \(h \in AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))\), i.e. \(\tilde{G}\) is a linear operator on \(AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))\). Moreover \(\tilde{G}\) is bounded on \(AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))\), because
\[
||\tilde{G}h||_{AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))} = \sup_{\theta \in \mathbb{R}} ||Gh(\theta)||_{AP_0(\mathbb{R}_+, X)} \leq ||G||_{L(AP_0(\mathbb{R}_+, X))} ||h||_{AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))}.
\]

For the isometry \(J\) defined on the space \(AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))\) by
\[
[(Jh)(\theta)](t) = (h(\theta + t))(t),
\]
we have
\[
[(J^{-1}\tilde{G}Jh)(\theta)](t) = \int_0^t U(t, t - \xi)(h(\theta - \xi))(t - \xi)d\xi
\]
Let \(E = \{E^t\}_{t \geq 0}\) be the evolution semigroup on \(AP_0(\mathbb{R}_+, X)\), defined by
\[
(E^t f)(\xi) = \begin{cases} U(t, t - \xi)f(t - \xi), & t \geq \xi \\ 0, & 0 \leq t \leq \xi \end{cases}
\]
and
\[
(G_\ast h)(\theta) := \int_0^\infty E^\tau h(\theta - \tau)d\tau \quad \theta \in \mathbb{R}, \quad h \in AP(\mathbb{R}, AP_0(\mathbb{R}_+, X)).
\]
A simple calculus show that \(G_\ast = J^{-1}\tilde{G}J\), therefore \(G_\ast\) is a bounded operator on \(AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))\). Each function \(g_+ \in AP_0(\mathbb{R}_+, AP_0(\mathbb{R}_+, X))\) can be extented to a function \(g \in AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))\) by setting
\[
g(\theta) = \begin{cases} g_+(\theta), & \text{if } \theta \geq 0 \\ 0, & \text{if } \theta < 0 \end{cases}
\]
It is clear that $G_\ast g \in AP(\mathbb{R}, AP_0(\mathbb{R}_+, X))$. Consider the function $f_+: \mathbb{R}_+ \to AP_0(\mathbb{R}_+, X)$, defined by

$$
f_+(r) = \int_0^r E^\tau g_+(r-\tau)d\tau \quad (r \geq 0).
$$

It is easy to see that for all $t \geq 0$, we have

$$
[f_+(\theta))(t) = \int_0^{\min(\theta,t)} U(t, t-\tau)(g_+(\theta-\tau))(t-\tau)d\tau, \quad \theta \geq 0,
$$

and

$$
[(G_\ast g)(\theta))(t) = \begin{cases} 
(f_+(\theta))(t), & \text{if } \theta \geq 0 \\
0, & \text{if } \theta < 0
\end{cases}
$$

Then $G_\ast g|_{\mathbb{R}_+} = f_+$ belongs to $AP_0(\mathbb{R}_+, AP_0(\mathbb{R}_+, X))$. From Theorem 1 ((3) $\Rightarrow$ (1)) with $T$ replaced by $E$ and $X$ replaced by $AP_0(\mathbb{R}_+, X)$ it results that $E$ is uniformly exponentially stable. Now is easy to see that $\mathcal{U}$ is uniformly exponentially stable, cf. [CLMR, Theorem 2.2].

**References**


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