REFINEMENTS OF CARLEMAN’S INEQUALITY

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Abstract. In this paper, we obtain a series of refined Carleman’s Inequalities with Arithmetic-Geometric mean inequality by decreasing their weight coefficient.

1. Introduction

Let \( \{a_i\}_{n=1}^{\infty} \) is a nonnegative sequence such that \( 0 \leq \sum_{n=1}^{\infty} a_n < +\infty \), then, we have

\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{n=1}^{\infty} a_n
\]  

(1.1)

The equality in (1.1) holds if and only if \( a_n = 0, n = 1, 2, \cdots \). the coefficient \( e \) is optimal

Inequality (1.1) is called Carleman’s Inequality, for details please refer to [1, 2]. Though the coefficient \( e \) is optimal, we can refine its weight coefficient. In this article we give a series of improved Carleman’s inequalities by decreasing the weight coefficient with the arithmetic-geometric mean inequality.

2. The Two Special Cases

In this section, we give two special cases of refined Carleman’s inequality. First we prove two lemmas.

Lemma 2.1. For \( m = 1, 2, \cdots \), the following inequality

\[
\left( 1 + \frac{1}{m} \right)^m \leq e \left( 1 - \frac{1 - 2/e}{m} \right)
\]

(2.1)

holds, where \( 1 - \frac{2}{e} \approx 0.2642411 \) is best possible.

Proof. Let

\[
\left( 1 + \frac{1}{m} \right)^m \leq e \left( 1 - \frac{\beta}{m} \right)
\]

(2.2)

Then, it is equivalent to

\[
\beta \leq m - \frac{m}{e} \left( 1 + \frac{1}{m} \right)^m,
\]

Let \( f(x) = \frac{1}{x} - \frac{1}{ex} (1 + x)^{\frac{1}{2}} \quad x \in (0, 1] \)

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It’s obvious that the function \( f(x) \) is a monotone decreasing function on interval \((0, 1]\). Consequently, \( \beta = f(1) = 1 - \frac{2}{e} \) is the optimal value of satisfying inequality (2.2), So (2.1) holds. The proof of lemma 2.1 follows.

**Lemma 2.2.** For \( m = 1, 2, \cdots \), the following inequality

\[
\left(1 + \frac{1}{m}\right)^m \leq \frac{e}{(1 + \frac{1}{m})^\frac{2}{m^2} - 1}
\]

holds, where \( \frac{1}{m^2} - 1 \approx 0.442695 \) is the best possible.

*Proof.* Let

\[
\left(1 + \frac{1}{m}\right)^m \leq \frac{e}{(1 + \frac{1}{m})^\alpha}
\]

It is equivalent to

\[
\alpha \leq \frac{1}{\ln(1 + \frac{1}{m})} - m
\]

Let

\[
f(x) = \frac{1}{\ln(1 + x)} - \frac{1}{x} \quad x \in (0, 1]
\]

Because the function \( f(x) \) is a monotone decreasing function on interval \((0, 1]\). Consequently, \( \alpha = f(1) = \frac{1}{m^2} - 1 \) is the optimal value of satisfying inequality (2.4), So (2.3) holds. The proof of lemma 2.2 follows.

**Theorem 2.3.** Let \( \{a_i\}^n_{i=1} \) is a nonnegative sequence such that \( 0 \leq \sum^\infty_{n=1} a_n < +\infty \), we have

\[
\sum^\infty_{n=1} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum^\infty_{m=1} \left(1 - \frac{1 - 2/e}{m}\right) a_m
\]

\[
\sum^\infty_{n=1} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum^\infty_{m=1} \frac{a_m}{(1 + \frac{1}{m})^\frac{2}{m^2} - 1}.
\]

*Proof.* Let \( c_i > 0 \) \((i = 1, 2, \cdots)\), according to arithmetic-geometric mean inequality, we have

\[
(c_1 a_1 c_2 a_2 \cdots c_n a_n)^\frac{1}{n} \leq \frac{1}{n} \sum^n_{m=1} c_m a_m
\]

Consequently

\[
\sum^\infty_{n=1} (a_1 a_2 \cdots a_n)^{1/n} = \sum^\infty_{n=1} \left(\frac{c_1 a_1 c_2 a_2 \cdots c_n a_n}{c_1 c_2 \cdots c_n}\right)^{1/n}
\]

\[
= \sum^\infty_{n=1} (c_1 c_2 \cdots c_n)^{-1/n} (c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n}
\]

\[
\leq \sum^\infty_{n=1} (c_1 c_2 \cdots c_n)^{-1/n} \frac{1}{n} \sum^n_{m=1} c_m a_m
\]

\[
= \frac{1}{n} \sum^n_{m=1} c_m a_m \sum^\infty_{n=m} (c_1 c_2 \cdots c_n)^{-1/n}
\]
Let $c_m = \frac{(m+1)^m}{m^{m+1}}$ \,(m = 1, 2, \cdots, n), c_1 c_2 \cdots c_n = (n+1)^n$, and
\[
\sum_{n=m}^{+\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n} = \sum_{n=m}^{+\infty} \frac{1}{n(n+1)} = \frac{1}{m}
\]
Therefore
\[
(2.7) \quad \sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq \sum_{m=1}^{+\infty} \frac{c_m}{m} a_m = \sum_{m=1}^{+\infty} \left(1 + \frac{1}{m}\right)^m a_m
\]
According to lemma 2.1 and lemma 2.2, and substituting for $(1 + \frac{1}{m})^m$ of inequality (2.7), we have
\[
\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \left(1 - \frac{2/e}{m}\right) a_m
\]
\[
\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \frac{a_m}{(1 + \frac{1}{m})^{\frac{m}{1+\alpha}}}
\]
The proof is complete.

3. A Series of Refined Carleman’s Inequalities

In this section we give a series of refined Carleman’s inequalities with lemma 3.1.

First we have

**Lemma 3.1.** For $m = 1, 2, \cdots$, the following inequality
\[
(3.1) \quad \left(1 + \frac{1}{m}\right)^m \leq e \left(1 - \frac{\beta}{m}\right)
\]
holds, where $0 \leq \alpha \leq \frac{1}{\ln 2} - 1$, $0 \leq \beta \leq 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$.

**Proof.** Inequality (3.1) is equivalent to
\[
(3.2) \quad \beta \leq m - \frac{m}{e} \left(1 + \frac{1}{m}\right)^{m+\alpha}
\]
Let
\[
f(x) = \frac{1}{x} - \frac{1}{e x} (1 + x)^{\frac{1}{1+\alpha}}, \quad x \in (0, 1], \quad 0 \leq \alpha \leq \frac{1}{\ln 2} - 1
\]
then $f(x)$ is a monotone decreasing function of $x$. Consequently, $\beta = f(1) = 1 - \frac{1}{2} 2^{1+\alpha}$ is the optimal value of satisfying inequality (3.2), i.e. $0 \leq \beta \leq 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$. So (2.3) holds, The proof is complete.

**Remark 3.1.** If $\alpha = 0$, then $\beta = 1 - \frac{2}{e}$, and we obtain lemma 1; if $\beta = 0$, then $\alpha = \frac{1}{\ln 2} - 1$, and we obtain lemma 2.

Similar to theorem 2.3, according to lemma 3.1, we have

**Theorem 3.2.** Let $a_n \geq 0$ \,(n = 1, 2, \cdots), $0 \leq \sum_{m=1}^{+\infty} a_n < +\infty$, we have
\[
\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \frac{(1 - \beta/m)}{(1 + \frac{1}{m})^{\alpha m}} a_m
\]
where $\alpha$, $\beta$ satisfy $0 \leq \alpha \leq \frac{1}{\ln 2} - 1$, $0 \leq \beta \leq 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$. 
Remark 3.2. Theorem 2.3 are two special cases of theorem 3.2, if $\alpha = 0$, $\beta = 1 - \frac{2}{e}$, and $\beta = 0$, $\alpha = \frac{1}{\ln 2} - 1$, we can obtain (2.5) and (2.6) in theorem 2.3 respectively.

References