A note on two inequalities of Telyakovskii type

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1. Introduction


A null-sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class $S$, or briefly $\{a_n\} \in S$ if there exists a monotonically decreasing sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all $n$. Telyakovskii [2], firstly proved that the Sidon’s class is equivalent to the class $S$ and second that $S$ is a $L^1$-integrability class for cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \quad (1.1)$$

Thus the class $S$ is known as Sidon-Telyakovskii class $S$.

**Theorem A** [2]. Let the coefficients of the series (1.1) belong to the class $S$. Then the series (1.1) is a Fourier series of some $f \in L^1(0, \pi)$ and the following inequality holds:

$$\pi \int_0^\pi |f(x)| \, dx \leq C \sum_{n=0}^{\infty} A_n, \quad \text{where} \quad C > 0.$$ 

Similar theorem for sine series

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx \quad (1.2)$$

is also proved for the class $S$ by Telyakovskii [2].

**Theorem B** [2]. Let the coefficients of the series (1.2) belong to the class $S$. Then the following relation holds for $p = 1, 2, 3, \ldots$

$$\frac{\pi}{p+1} \int_0^\pi |g(x)| \, dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O \left( \sum_{n=1}^{\infty} A_n \right).$$

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Key words and phrases: inequalities, Sidon-Telyakovskii class, Fourier series
In particular $g(x)$ is a Fourier series iff \[ \sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty. \]

Very recently, the author of this note [3] defined the following class $S_r$.
A null sequence $\{a_n\}$ belongs to the class $S_r$, $r = 0, 1, 2, \ldots$ if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} n^r A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all $n$.

When $r = 0$ it is clear that $S_r = S$.

In [3] and [4] we obtained $L^1$-estimates of the $r$-th derivatives for the series (1.1) and (1.2), i.e. for the series
\[
\sum_{n=1}^{\infty} n^r a_n \cos \left( nx + \frac{r\pi}{2} \right), \tag{1.3}
\]
\[
\sum_{n=1}^{\infty} n^r a_n \sin \left( nx + \frac{r\pi}{2} \right), \tag{1.4}
\]
Namely, the following theorems were proved by the author in [3], [4].

**Theorem 1** [3]. Let the coefficients of the series (1.1) belongs to the class $S_r$, $r = 0, 1, 2, \ldots$. Then the series (1.3) is a Fourier series of some $f^{(r)} \in L^1(0, \pi)$ and the following inequality holds:

\[
\frac{\pi}{0} |f^{(r)}(x)| dx \leq C \sum_{n=1}^{\infty} n^r A_n, \quad C > 0.
\]

**Theorem 2** [4]. Let the coefficients of the series $g(x)$ satisfy the condition $S_r$, $r = 0, 1, 2, \ldots$. Then the following relation holds for $m = 1, 2, 3, \ldots$

\[
\frac{\pi}{m+1} |g^{(r)}(x)| dx = \sum_{n=1}^{m} |a_n| n^{r-1} + O \left( \sum_{n=1}^{\infty} n^r A_n \right).
\]

In particular (1.4) is a Fourier series iff $\sum_{n=1}^{\infty} n^r A_n < \infty$.

**Corollary** [4]. Let the coefficients of the series $g(x)$ satisfy the condition $S_r$, $r = 1, 2, 3, \ldots$. Then the following estimate holds:

\[
\frac{\pi}{0} |g^{(r)}(x)| dx = O \left( \sum_{n=1}^{\infty} n^r A_n \right).
\]
2. Lemma

For the proofs of the Theorem 1 and Theorem 2, we need the following Lemma.

Lemma 1. \[5\] If \(\{a_n\} \in S_r, \ r = 0, 1, 2, 3, \ldots\) then \(\{n^r a_n\} \in S\).

**Proof.** Let \(\{a_n\} \in S_r\). Then

\[
n^r a_n = n^r \sum_{k=n}^{\infty} \Delta a_k \leq \sum_{k=n}^{\infty} k^r |\Delta a_k| \leq \sum_{k=n}^{\infty} k^r A_k = o(1), \quad n \to \infty,
\]

i.e. \(n^r a_n \to 0, \ n \to \infty\).

Now, we consider the sequence \(\{B_k\}\) defined as follows

\[
B_k = k^r A_k + \sum_{i=k+1}^{\infty} [i^r - (i - 1)^r] A_i. \quad (1.5)
\]

We have:

\[
B_k - B_{k+1} = k^r A_k - (k + 1)^r A_{k+1} + (k + 1)^r A_k - k^r A_{k+1} = k^r \Delta A_k \geq 0,
\]

i.e. \(\{B_k\} \downarrow 0\);

\[
\sum_{k=1}^{\infty} B_k = \sum_{k=1}^{\infty} k^r A_k + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} [(i + 1)^r - i^r] A_i = \sum_{k=1}^{\infty} k^r A_k + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} [(i + 1)^r - i^r] A_i = \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} i[(i + 1)^r - i^r] A_i < \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} [(i + 1)^r - i^{r+1}] A_i \leq \sum_{k=1}^{\infty} k^r A_k + O \left( \sum_{i=1}^{\infty} i^r A_i \right) < \infty, \quad \text{i.e.} \sum_{k=1}^{\infty} B_k < \infty.
\]

Then,

\[
\Delta(n^r a_k) = n^r a_k - (k + 1)^r a_{k+1} = k^r \Delta a_k - ((k + 1)^r - k^r) a_{k+1}.
\]

The function \(h(x) = (x + 1)^r - x^r\) is monotone decreasing on \([0, \infty)\), since

\[
h'(x) = r[(x + 1)^{r-1} - x^{r-1}] \geq 0 \quad \text{for} \quad x \geq 0.
\]
This implies that
\[ |\Delta(k^r a_k)| \leq k^r|\Delta a_k| + ((k + 1)^r - k^r)|a_{k+1}| \leq \]
\[ \leq k^r A_k + ((k + 1)^r - k^r) \sum_{i=k+1}^{\infty} |\Delta a_i| \leq \]
\[ \leq k^r A_k + \sum_{i=k+1}^{\infty} (i^r - (i - 1)^r)|\Delta a_i| \leq \]
\[ \leq k^r A_k + \sum_{i=k+1}^{\infty} (i^r - (i - 1)^r)A_i = B_k , \]
\[ |\Delta(k^r a_k)| \leq B_k , \text{ for all } k. \text{ Thus, } \{n^r a_n\} \in S. \]

3. Proofs

3.1. Proof of the Theorem 1

By Lemma 1, we have \( \{n^r a_n\} \in S. \) Applying the Theorem A, we obtain that the series (1.3) is a Fourier series of some \( f(r) \in L^1(0, \pi) \) and
\[ \frac{\pi}{0} |f(r)(x)|dx \leq C \sum_{n=0}^{\infty} B_n , \]
where \( \{B_n\} \) is the sequence defined by (1.5). Hence,
\[ \frac{\pi}{0} |f(r)(x)|dx \leq C \sum_{n=0}^{\infty} n^r A_n + C \sum_{n=0}^{\infty} \sum_{i=n+1}^{\infty} (i^r - (i - 1)^r)A_i = \]
\[ = C \sum_{n=0}^{\infty} n^r A_n + O \left( \sum_{i=1}^{\infty} n^r A_i \right) = O \left( \sum_{i=1}^{\infty} i^r A_i \right), \]
i.e. the our inequality is satisfied.

3.2. Proof of the Theorem 2

Applying the Lemma 1 and Theorem B, we obtain
\[ \frac{\pi}{0} |g(r)(x)|dx = \sum_{n=1}^{m} |n^r a_n| + O \left( \sum_{n=1}^{\infty} B_n \right), \]
where \( B_n \) is the sequence defined by (1.5). Then,
\[ \frac{\pi}{0} |g(r)(x)|dx = \sum_{n=1}^{m} |a_n| n^{r-1} + O \left( \sum_{n=1}^{\infty} n^r A_n \right) + O \left( \sum_{k=1}^{\infty} \sum_{i=k+1}^{\infty} (i^r - (i - 1)^r)A_i \right) = \]
\[ = \sum_{n=1}^{m} n^{r-1} |a_n| + O \left( \sum_{n=1}^{\infty} n^r A_n \right) + O \left( \sum_{n=1}^{\infty} n^r A_n \right) = \]
\[ = \sum_{n=1}^{m} n^{r-1} |a_n| + O \left( \sum_{n=1}^{\infty} n^r A_n \right) . \]
3.3. Proof of the Corollary

By inequalities

\[\sum_{n=1}^{m} |a_n| n^{r-1} \leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} |\Delta a_k| \leq \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} A_k = \sum_{k=1}^{\infty} A_k \sum_{n=1}^{k} n^{r-1} \leq \sum_{k=1}^{\infty} k^r A_k,\]

and by Theorem 2, we obtain

\[\int_{\pi}^{\pi+\pi} |g^{(r)}(x)|dx = O\left(\sum_{n=1}^{\infty} n^r A_n\right).\]

Letting \(m \to \infty\), the inequality is satisfied.

References


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