MEANS, $g$–CONVEX DOMINATED FUNCTIONS & HADAMARD–TYPE INEQUALITIES

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Abstract. Hadamard–type inequalities are derived for $g$–convex dominated maps. Applications are given involving two functionals and some common means.

1. Introduction

The Hermite–Hadamard inequality
\[ g \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b g(x) \, dx \leq \frac{g(a) + g(b)}{2} \]
for a convex real–valued function $g$ on a finite interval $[a, b]$ is central to mathematical analysis and is the subject of a huge literature dealing with various generalisations and refinements. In this note we connect together some disparate threads through a Hermite–Hadamard motif. The first of these threads is the unifying concept of a $g$–convex dominated function (see [6]).

**Definition 1.** Let $g : I \to \mathbb{R}$ be a given convex function. The real function $f : I \to \mathbb{R}$ is called $g$–convex dominated on $I$ if

\[ |\lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y)| \leq \lambda g(x) + (1 - \lambda) g(y) - g(\lambda x + (1 - \lambda) y) \]

for all $x, y \in I$ and $\lambda \in [0, 1]$.

The class of $g$–convex dominated functions on an interval $I$ is manifestly nonempty. If $g$ is convex on $I$ and $e : I \to \mathbb{R}$ is defined by

\[ e(x) := x, \]

then $e$ and $g$ are both $g$–convex dominated on $I$. Indeed there are concave functions which are $g$–convex dominated (for example $-g$) as well as functions which are neither convex nor concave. The concept of $g$–convex dominated functions draws together functions with some convex–like properties. We aim to elucidate some of these properties.

A second thread involves several means in common use for a pair $x, y$ of positive numbers, namely the following. For $x \neq y$ and $p \in \mathbb{R} \setminus \{-1, 0\}$, we define the $p$–logarithmic mean (generalised logarithmic mean) $L_p(x, y)$ by

\[ L_p(x, y) := \left[ \frac{y^{p+1} - x^{p+1}}{(p + 1)(y - x)} \right]^{1/p}. \]
In fact the singularities at $p = -1, 0$ are removable and $L_p$ can be defined for $p = -1, 0$ so as to make $L_p(x, y)$ a continuous function of $p$. In the limit as $p \to 0$ we obtain the identric mean $I(x, y)$, given by

$$I(x, y) := \frac{1}{e} \left( \frac{y}{x} \right)^{1/(y-x)},$$

and in the case $p \to -1$ the logarithmic mean $L(x, y)$, given by

$$L(x, y) := \frac{y - x}{\ln y - \ln x}.$$

In each case we define the mean as $x$ when $y = x$, which occurs as the limiting value of $L_p(x, y)$ for $y \to x$. See [1] Chapter 6, Section 3 for more detail on these means.

In addition we have the arithmetic, geometric and harmonic means, defined respectively by

$$A(x, y) := \frac{x + y}{2}, \quad G(x, y) := \sqrt{xy} \quad \text{and} \quad H(x, y) := \frac{2xy}{x + y}.$$  

The first two arise from $L_p(x, y)$ in the respective cases $p = 1, -2$. Remarkably there is no value of $p$ for which $L_p = H$ (see [1] p. 347). However $H$ is connected with the generalised logarithmic–mean canon by

$$H(x, y) = \left[ A \left( \frac{1}{x}, \frac{1}{y} \right) \right]^{-1}.$$  

The final thread involves two functionals which interpolate between $f((a + b)/2)$ and $\int_a^b f(x)dx/(b - a)$. If $f : [a, b] \to \mathbb{R}$ with $f \in L^1[a, b]$, we define the induced mapping $H_f : [0, 1] \to \mathbb{R}$ by

$$H_f(t) := \frac{1}{b - a} \int_a^b f \left( tx + (1 - t) \frac{a + b}{2} \right) dx.$$  

Similarly for $f : [a, b] \to \mathbb{R}$ integrable on $[a, b]$ we may define $F_f : [0, 1] \to \mathbb{R}$ by

$$F_f(t) := \frac{1}{(b - a)^2} \int_a^b \int_a^b f \left( tx + (1 - t) y \right) dx dy.$$  

For treatments of these functionals see [2]–[6].

In Section 2 we present some general results for $g$–convex dominated functions. In Section 3 a number of simple particular examples are given in which the means $L_p$ and $H$ appear naturally. Finally in Section 4 we show that the two functionals defined above inherit $g$–convex dominated properties.

## 2. $g$–Convex Dominated Maps

We shall make use of the following characterisation of convex–dominated functions established in [6].

**Lemma 1.** Let $g$ be a convex function on $I$ and $f : I \to \mathbb{R}$. Then the following statements are equivalent:

1. $f$ is $g$–convex dominated on $I$;
2. the mappings $g - f$ and $g + f$ are convex on $I$;
(iii) there exist two convex mappings \( h, k \) defined on \( I \) such that

\[
f = \frac{1}{2} (h - k) \quad \text{and} \quad g = \frac{1}{2} (h + k).
\]

Proof. “(i) \( \iff \) (ii).” Condition (1.1) is equivalent to

\[
g(\lambda x + (1 - \lambda) y) - \lambda g(x) - (1 - \lambda) g(y)
\]

\[
\leq \lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y)
\]

\[
\leq \lambda g(x) + (1 - \lambda) g(y) - g(\lambda x + (1 - \lambda) y)
\]

for all \( x, y \in I \) and \( \lambda \in [0, 1] \). The two inequalities may be rearranged as

\[
\lambda [f(x) + g(x)] + (1 - \lambda) [f(y) + g(y)]
\]

\[
\geq f(\lambda x + (1 - \lambda) y) + g(\lambda x + (1 - \lambda) y)
\]

and

\[
\lambda [g(x) - f(x)] + (1 - \lambda) [g(y) - f(y)]
\]

\[
\geq g(\lambda x + (1 - \lambda) y) - f(\lambda x + (1 - \lambda) y)
\]

for all \( x, y \in I \) and \( \lambda \in [0, 1] \), which are equivalent to the convexity of \( g + f \) and \( g - f \) respectively.

The equivalence “(ii) \( \iff \) (iii)” is immediate. \( \square \)

**Proposition 1.** Suppose \( f'' \), \( g'' \) exist and satisfy

\[
|f''(x)| \leq g''(x)
\]

on an interval \( I \). Then \( f \) is \( g \)-convex dominated on \( I \).

Proof. By the given condition

\[
g''(x), \quad f''(x) - f''(x), \quad f''(x) + f''(x)
\]

are all nonnegative on \( I \), so \( g, g - f, g + f \) are all convex on \( I \), whence the stated result follows by Lemma 1. \( \square \)

**Theorem 1.** Let \( g: I \to \mathbb{R} \) be a convex function and \( f: I \to \mathbb{R} \) a \( g \)-convex dominated mapping. Then for all \( a, b \in I \) with \( a < b \),

\[
\left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{1}{b - a} \int_a^b g(x) \, dx - g\left(\frac{a + b}{2}\right)
\]

and

\[
\left| f(a) + f(b) \right| - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{g(a) + g(b)}{2} - \frac{1}{b - a} \int_a^b g(x) \, dx.
\]

Proof. Since \( f \) is \( g \)-convex dominated, we have by Lemma 1 that \( g + f \) and \( g - f \) are convex on \([a, b]\), and so by the classical Hermite–Hadamard inequality

\[
(f + g)\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b (f + g)(x) \, dx
\]

\[
\leq \frac{(f + g)(a) + (f + g)(b)}{2}
\]
and
\[
(g - f) \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b (g - f) \, dx \leq \frac{1}{2} \left( g - f \right)(a) + \left( g - f \right)(b).
\]

These inequalities are equivalent to those in the enunciation. \[\square\]

3. Means

We now give several corollaries that provide examples of convex dominated functions and involve generalised logarithmic means.

**Corollary 1.** Suppose \([a, b] \subset (0, \infty)\) and \(p \in \mathbb{R} \setminus \{-2, -1\}\). Let \(f : [a, b] \to \mathbb{R}\) be a twice differentiable mapping such that \(|f''(x)| \leq Mx^p\) \((M > 0)\) for \(x \in [a, b]\). Then
\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{M}{(p + 1)(p + 2)} \left[ L_{p+2}^2(a, b)^{p+2} - [A(a, b)]^{p+2} \right]
\]
and
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{M}{(p + 1)(p + 2)} \left[ A(a^{p+2}, b^{p+2}) - [L_{p+2}^2(a, b)]^{p+2} \right].
\]

**Proof.** Define the mapping \(g : [a, b] \to \mathbb{R}\) by
\[
g(x) = \frac{Mx^{p+2}}{(p + 1)(p + 2)}.
\]

Then
\[
g''(x) = Mx^p \geq |f''(x)|
\]
on \([a, b]\). The stated results follow from Proposition 1 and Theorem 1. \[\square\]

In particular we derive the following in the case \(p = 0\).

**Remark 1.** Let \(f\) be twice differentiable on \([a, b]\) and suppose that
\[
M := \sup_{x \in [a, b]} |f''(x)| < \infty.
\]

Then
\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{M}{24} (b - a)^2
\]
and
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{M}{12} (b - a)^2.
\]
Remark 2. In the case $p = -3$, the enunciation yields via (1.2) that
\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{M}{2} \left[ \frac{A(a, b) - L(a, b)}{A(a, b) L(a, b)} \right].
\]

and
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{M}{2} \left[ \frac{L(a, b) - H(a, b)}{H(a, b) L(a, b)} \right].
\]

We now examine the two cases $p = -2, -1$ excluded in the preceding corollary. First we take $p = -2$.

Corollary 2. Suppose $[a, b] \subset (0, \infty)$ and let $f : [a, b] \to \mathbb{R}$ be twice differentiable and such that $|f''(x)| \leq M/x^2$ for all $x \in (a, b)$. Then
\[
\text{exp} \left[ f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right] \leq \left[ \frac{A(a, b)}{I(a, b)} \right]^M
\]
and
\[
\text{exp} \left[ \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right] \leq \left[ \frac{I(a, b)}{G(a, b)} \right]^M.
\]

Proof. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = -M \ln x$. Then $g''(x) = M/x^2$. Proposition 1 and Theorem 1 provide
\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right| \leq M \left[ \ln \frac{a + b}{2} - \frac{\int_{1}^{b} \ln x \, dx}{b - a} \right]
\]
and
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right| \leq M \left[ \frac{\int_{1}^{b} \ln x \, dx}{b - a} - \frac{\ln a + \ln b}{2} \right].
\]
The stated inequalities follow from
\[
\int_{1}^{b} \ln x \, dx = b \ln b - a \ln a - (b - a) = (b - a) \ln I(a, b).
\]

\[\square\]

For $p = -1$ we obtain the following.

Corollary 3. Suppose $[a, b] \subset (0, \infty)$ and let $f : [a, b] \to \mathbb{R}$ be twice differentiable with $|f''(x)| \leq M/x$ for all $x \in (a, b)$. Then
\[
\text{exp} \left[ f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right] \leq \left[ \frac{(b^2/a^2)}{e^{-\frac{2}{M}(b^2-a^2)}} \right]^{\frac{M}{a}}
\]

and
\[
\text{exp} \left[ \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right] \leq \left[ \frac{(a^2b^2)}{e^{-\frac{2}{M}(b^2-a^2)}} \right]^{\frac{M}{a}}.
\]
Proof. Consider the mapping \( g(x) = M \ln x - Mx \). Then \( g''(x) = M/x \). Proposition 1 and Theorem 1 provide

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{1}{b-a} \int_a^b g(x) \, dx - g \left( \frac{a+b}{2} \right)
\]

and

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) \, dx.
\]

We have

\[
\int_a^b (x \ln x - x) \, dx = \ln \left( \left( \frac{b^2}{a^2} \right) e^{-\frac{3}{4}(b^2-a^2)} \right)^M,
\]

\[
g \left( \frac{a+b}{2} \right) = \ln \left[ \left( \frac{a+b}{2} \right)^{\frac{a+b}{2}} e^{-\frac{a+b}{2}} \right]^M,
\]

\[
\frac{1}{b-a} \int_a^b g(x) \, dx = \ln \left[ \left( \frac{b^2}{a^2} \right)^{\frac{a+b}{2}} e^{-\frac{a+b}{2}} \right]^{\frac{M}{b-a}},
\]

and

\[
\frac{g(a) + g(b)}{2} = \ln \left( (a^b)^{\frac{1}{2}} e^{-\frac{a+b}{2}} \right)^M.
\]

These yield

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \ln \left\{ \left[ \left( \frac{b^2}{a^2} \right) e^{-\frac{3}{4}(b^2-a^2)} \right]^M \right\},
\]

and

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \ln \left\{ \left[ (a^b)^{\frac{1}{2}} e^{-\frac{a+b}{2}} \right]^M \right\},
\]

whence the desired results. \( \square \)

The remarks in the introduction suggest that the results of Corollaries 2 and 3 may derive in the limit from those of Corollary 1 by letting \( p \to -2 \) and \( p \to -1 \) respectively.

We shall have the first inequality in Corollary 2, or rather the version of it obtained by taking natural logarithms of both sides, as a limiting form of the first inequality in Corollary 1 if we can show that

\[
\frac{1}{(p+1)(p+2)} \left[ |L_{p+2} (a,b)|^{p+2} - [A(a,b)]^{p+2} \right] \to \ln \frac{A(a,b)}{I(a,b)}
\]

as \( p \to -2 \). In extenso this reads as

\[
\frac{1}{(p+1)(p+2)} \left[ \frac{b^{p+3} - a^{p+3}}{(p+3)(b-a)} - \left( \frac{a+b}{2} \right)^{p+2} \right] \to \ln \frac{a+b}{2} - \left[ \frac{b \ln b - a \ln a}{b-a} - 1 \right].
\]
Set \( p = -2 + h \). It suffices to show that
\[
\frac{1}{h} \left[ \left( \frac{a + b}{2} \right)^h - \frac{b^{1+h} - a^{1+h}}{(1 + h)(b - a)} \right] \to \ln \frac{a + b}{2} - \left[ \frac{b \ln b - a \ln a}{b - a} - 1 \right]
\]
as \( h \to 0 \). By L'Hôpital's rule, the left-hand side has limit equal to
\[
\frac{d}{dx} \left( \frac{a + b}{2} \right) \bigg|_{x=0} - \frac{d}{dx} \frac{b^x - a^x}{x(b - a)} \bigg|_{x=1},
\]
which is readily verified to reduce to the right-hand side of (3.1).

The second inequality of Corollary 2 and the two inequalities of Corollary 3 may be derived similarly.

4. Functionals

We now examine how the two induced maps defined in the introduction inherit \( g \)-convex dominated properties. We make use of the following result (see [2] and [3]).

**Proposition 2.** If \( g \) is convex, then

(a) \( H_g \) is convex;

(b) \( H_g \) is monotone nondecreasing.

**Theorem 2.** Let \( g : [a, b] \to \mathbb{R} \) be convex and \( f : [a, b] \to \mathbb{R} \) a \( g \)-convex dominated mapping on \([a, b] \). Then

(i) \( H_f \) is \( H_g \)-convex dominated on \([0, 1] \);

(ii) for all \( 0 \leq t_1 < t_2 \leq 1 \) we have

\[
0 \leq |H_f(t_2) - H_f(t_1)| \leq H_g(t_2) - H_g(t_1);
\]

(iii) for all \( t \in [0, 1] \)

\[
0 \leq \left| f \left( \frac{a + b}{2} \right) - H_f(t) \right| \leq H_g(t) - g \left( \frac{a + b}{2} \right)
\]

and

\[
0 \leq \left| \frac{1}{b - a} \int_a^b f(x) \, dx - H_f(t) \right| \leq \frac{1}{b - a} \int_a^b g(x) \, dx - H_g(t).
\]

**Proof:** (i) Since \( f \) is \( g \)-convex dominated on \([a, b] \), it follows from Lemma 1 that \( g - f \) and \( g + f \) are convex on \([a, b] \) and so by Proposition 2 \( H_{g-f} \) and \( H_{g+f} \) are convex on \([0, 1] \). By the linearity of the mapping \( f \mapsto H_f \), we have \( H_{g-f} = H_g - H_f \) and \( H_{g+f} = H_g + H_f \). Since \( H_g \) is convex, Lemma 1 yields that \( H_f \) is \( H_g \)-dominated on \([0, 1] \).
(ii) By Proposition 2 \( H_{g-f} \) and \( H_{g+f} \) are monotone nondecreasing on \([0,1]\) and thus
\[
H_g(t_1) - H_f(t_1) = H_{g-f}(t_1) \leq H_{g-f}(t_2) = H_g(t_2) - H_f(t_2)
\]
and
\[
H_g(t_1) + H_f(t_1) = H_{g+f}(t_1) \leq H_{g+f}(t_2) = H_g(t_2) + H_f(t_2).
\]
Therefore
\[
H_f(t_2) - H_f(t_1) \leq H_g(t_2) - H_g(t_1)
\]
and
\[
H_g(t_2) - H_g(t_1) \geq -[H_f(t_2) - H_f(t_1)],
\]
which are equivalent to (4.1).

(iii) Since
\[
H_g(0) = g\left(\frac{a+b}{2}\right) \quad \text{and} \quad H_f(0) = f\left(\frac{a+b}{2}\right),
\]
the first inequality in (iii) occurs as a special case of (4.1). Likewise the second arises with \( t = 1 \).

\(\Box\)

Similarly we have the following result for the second functional, \( F_f \) (see [3]).

**Proposition 3.** If \( g \) is convex, then

(a) \( F_g \) is convex;

(b) \( F_g \) is monotone nonincreasing on \([0,1/2]\) and monotone nondecreasing on \([1/2,1]\).

This leads to the following result.

**Theorem 3.** Let \( g : [a, b] \rightarrow \mathbb{R} \) be convex and \( f : [a, b] \rightarrow \mathbb{R} \) a \( g \)-convex dominated function on \([a, b]\). Then

(i) \( F_f \) is \( F_g \)-convex dominated on \([0,1]\);

(ii) we have
\[
0 \leq |F_f(t_2) - F_f(t_1)| \leq F_g(t_2) - F_g(t_1) \quad \text{for} \quad \frac{1}{2} \leq t_1 < t_2 \leq 1
\]
and
\[
0 \leq |F_f(t_2) - F_f(t_1)| \leq F_g(t_1) - F_g(t_2) \quad \text{for} \quad 0 \leq t_1 < t_2 \leq \frac{1}{2};
\]

(iii) for all \( t \in [0,1] \) we have
\[
0 \leq \left| \frac{1}{b-a} \int_a^b f(x) \, dx - F_f(t) \right| \leq \frac{1}{b-a} \int_a^b g(x) \, dx - F_g(t) ,
\]
\[
0 \leq \left| \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) \, dxdy - F_f(t) \right|
\]
\[
\leq F_g(t) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) \, dxdy
\]
and
\[ 0 \leq |F_f(t) - H_f(t)| \leq F_g(t) - H_g(t). \]

References