A GRÜSS TYPE INEQUALITY FOR ISOTONIC LINEAR FUNCTIONALS AND APPLICATIONS

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Abstract. An inequality for a normalised isotonic linear functional of Grüss type and particular cases for integrals and norms are established. Applications in obtaining a counterpart for the Cauchy-Buniakowski-Schwartz inequality for functionals and Jessen’s inequality for convex functions are also given.

1. Introduction

Let \( L \) be a linear class of real-valued functions \( g : E \to \mathbb{R} \) having the properties

(L1) \( f, g \in L \) imply \( (\alpha f + \beta g) \in L \) for all \( \alpha, \beta \in \mathbb{R} \);

(L2) \( 1 \in L \), i.e., if \( f(t) = 1 \), \( t \in E \), then \( f \in L \).

An isotonic linear functional \( A : L \to \mathbb{R} \) is a functional satisfying

(A1) \( A(\alpha f + \beta g) = \alpha A(f) + \beta A(g) \) for all \( f, g \in L \) and \( \alpha, \beta \in \mathbb{R} \);

(A2) If \( f \in L \) and \( f \geq 0 \), then \( A(f) \geq 0 \).

The mapping \( A \) is said to be normalised if

(A3) \( A(1) = 1 \).

Usual examples of isotonic linear functionals that are normalised are the following ones

\[ A(f) = \frac{1}{\mu(X)} \int_X f(x) \, d\mu(x) \quad \text{if} \quad \mu(X) < \infty \]

or

\[ A_w(f) := \frac{1}{\int_X w(x) \, d\mu(x)} \int_X w(x) f(x) \, d\mu(x), \]

where \( w(x) \geq 0 \), \( \int_X w(x) \, d\mu(x) > 0 \), \( X \) is a measurable space and \( \mu \) a positive measure on \( X \).

In particular, for \( \bar{x} = (x_1, \ldots, x_n) \), \( \bar{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n \) with \( w_i \geq 0 \), \( W_n := \sum_{i=1}^n w_i > 0 \), we have

\[ A(\bar{x}) := \frac{1}{n} \sum_{i=1}^n x_i \]

and

\[ A_w(\bar{x}) := \frac{1}{W_n} \sum_{i=1}^n w_i x_i, \]

are normalised isotonic functionals on \( \mathbb{R}^n \).

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In 1988, D. Andrica and C. Badea [1], proved the following generalisation of the Grüss inequality for isotonic linear functionals.

**Theorem 1.** If \( f, g \in L \) so that \( fg \in L \) and \( m \leq f \leq M, n \leq g \leq N \) where \( m, M, n, N \) are given real numbers, then for any normalised isotonic linear functional \( A : L \to \mathbb{R} \) one has the inequality

\[
|A(fg) - A(f) A(g)| \leq \frac{1}{4} (M - m)(N - n).
\]

The constant \( \frac{1}{4} \) in (1.1) is best possible in the sense that it cannot be replaced by a smaller constant.

In this paper we point out a refinement of the Grüss inequality (1.1) for isotonic linear functionals. Applications for the Cauchy-Buniakowski-Schwartz and Jessen's inequality are also provided.

## 2. A Grüss Type Inequality

The following result holds.

**Theorem 2.** Let \( f, g \in L \) be such that \( fg \in L \) and assume that there exists the real numbers \( n \) and \( N \) so that

\[
n \leq g \leq N.
\]

Then for any normalised isotonic linear functional \( A : L \to \mathbb{R} \) for which \( \|f - A(f) \cdot 1\| \in L \) one has the inequality

\[
|A(fg) - A(f) A(g)| \leq \frac{1}{2} (N - n) A(\|f - A(f) \cdot 1\|).
\]

The constant \( \frac{1}{2} \) in (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof.** Using the linearity property of \( A \), we have

\[
A \left[ (f - A(f) \cdot 1) \left( g - \frac{n + N}{2} \cdot 1 \right) \right] = A[(f - A(f) \cdot 1)g] - \frac{n + N}{2} A[f - A(f) \cdot 1] = A(fg) - A(f) A(g) - \frac{n + N}{2} [A(f) - A(f) \cdot A(1)] = A(fg) - A(f) A(g)
\]

since, by the normality property of \( A \), \( A(1) = 1 \).

From (2.1) we may easily deduce that

\[
\left| g - \frac{n + N}{2} \cdot 1 \right| \leq \frac{M - n}{2} \cdot 1.
\]

It is known that if \( h \in L \) so that \( |h| \in L \), then, by the monotonicity and linearity of \( A \), one has

\[
|A(h)| \leq A(|h|).
\]
Using this property, the monotonicity property of $A$ and condition (2.4), we deduce

\begin{align}
(2.6) & \quad \left| A \left[ (f - A(f) \cdot 1) \left( g - \frac{n + N}{2} \cdot 1 \right) \right] \right| \\
& \leq A \left( \left| (f - A(f) \cdot 1) \left( g - \frac{n + N}{2} \cdot 1 \right) \right| \right) \\
& \leq \frac{N - n}{2} A \left( |f - A(f) \cdot 1| \right).
\end{align}

Utilising (2.3) and (2.6) we deduce the desired result (2.2).

To prove the sharpness of the constant $\frac{1}{2}$, we assume that (2.2) holds with a constant $c > 0$ for $A = \frac{1}{b-a} \int_a^b$, $L = L[a,b]$ (the Lebesgue space of integrable functions on $[a,b]$) and $g$ satisfying the condition (2.1) on the interval $[a,b]$, i.e., one has the inequality

\begin{align}
(2.7) & \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \\
& \leq c(N - n) \cdot \frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y) dy \right| dx.
\end{align}

If we choose $g = f$ and $f : [a,b] \rightarrow \mathbb{R}$,

\[ f(x) = \begin{cases} 
-1 & \text{if } x \in [a, \frac{a+b}{2}] \\
1 & \text{if } x \in (\frac{a+b}{2}, b]
\end{cases} \]

then

\begin{align*}
\frac{1}{b-a} \int_a^b f^2(x) dx - \left( \frac{1}{b-a} \int_a^b f(x) dx \right)^2 & = 1, \\
\frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y) dy \right| dx & = 1, \\
m = -1, M & = 1.
\end{align*}

and by (2.7) we deduce $c \geq \frac{1}{2}$. ■

The following corollaries are natural consequences of the above result.

**Corollary 1.** Let $f \in L$ be such that $f^2 \in L$ and there exists the real numbers $m, M$ so that

\begin{align}
(2.8) & \quad m \leq f \leq M.
\end{align}

Then for any $A : L \rightarrow \mathbb{R}$ a normalised isotonic linear functional so that $|f - A(f) \cdot 1| \in L$ one has the inequality

\begin{align}
(2.9) & \quad 0 \leq A(f^2) - [A(f)]^2 \leq \frac{1}{2} (M - m) A(|f - A(f) \cdot 1|).
\end{align}

The constant $\frac{1}{2}$ is sharp.
Corollary 2. Let \( f, g \in L \) so that \( fg \in L \) and \( f \) satisfy (2.8) while \( g \) satisfies (2.1). Then for any normalised isotonic linear functional \( A : L \to \mathbb{R} \) so that \( |f - A(f) \cdot 1|, |g - A(g) \cdot 1| \in L \) one has the inequality:

\[
|A(fg) - A(f)A(g)| \leq \frac{1}{2} (M - m) (N - n)^{\frac{1}{2}} [A(|f - A(f) \cdot 1|)A(|g - A(g) \cdot 1|)]^{\frac{1}{2}}.
\]

The constant \( \frac{1}{2} \) is sharp.

Remark 1. Using Hölder’s inequality for isotonic linear functionals, we may state the following inequalities as well

\[
|A(fg) - A(f)A(g)| \\
\leq \frac{1}{2} (N - n) A(|f - A(f) \cdot 1|) \quad \text{if} \quad |f - A(f) \cdot 1| \in L,
\]

\[
\leq \frac{1}{2} (N - n) [A(|f - A(f) \cdot 1|^p)]^{\frac{1}{p}} \quad \text{if} \quad |f - A(f) \cdot 1|^p \in L, \quad p > 1
\]

\[
\leq \sup_{t \in \mathcal{E}} \{|f(t) - A(f)|\};
\]

provided \( f, g \in L \) and \( fg \in L \) while \( g \) satisfies the condition (2.1).

If \( f \) and \( g \) fulfill the conditions (2.8) and (2.1), then we have the following refinement of the Grüss inequality (1.1)

\[
|A(fg) - A(f)A(g)| \leq \frac{1}{2} (N - n) A(|f - A(f) \cdot 1|)
\]

\[
\leq \frac{1}{2} (N - n) [A(f^2) - [A(f)]^2]^{\frac{1}{2}}
\]

\[
\leq \frac{1}{4} (M - m) (N - n).
\]

The constants \( \frac{1}{2}, \frac{1}{2} \) and \( \frac{1}{4} \) are sharp in (2.11).

The following weighted version of Theorem 2 also holds.

Theorem 3. Let \( f, g, h \in L \) be such that \( h \geq 0, fh, gh, fgh \in L \) and there exists the real constants \( n, N \) so that (2.1) holds. Then for any \( B : L \to \mathbb{R} \) an isotonic linear functional so that \( B(h) > 0, h \left| f - \frac{1}{B(h)} \cdot 1 \right| \in L \) one has the inequality:

\[
|B(fgh) - B(fh)B(gh)| \leq \frac{1}{2} (N - n) \frac{1}{B(h)} B \left[ h \left| f - \frac{1}{B(h)} B(hf) \cdot 1 \right| \right].
\]

The constant \( \frac{1}{2} \) is best possible.

Proof. Apply Theorem 1 for the functional \( A_h : L \to \mathbb{R}, \)

\[
A_h(f) := \frac{1}{B(h)} B(hf),
\]

that is a normalised isotonic linear functional on \( L \). ■

Similar corollaries may be stated from the weighted inequality (2.12), but we omit the details.
3. Applications for Integral and Discrete Inequalities

Let \((\Omega, A, \mu)\) be a measurable space consisting of a set \(\Omega\), a \(\sigma\)-algebra of parts of \(\Omega\) and a countably additive and positive measure \(\mu\) on \(A\) with values in \(\mathbb{R} \cup \{\infty\}\).

For a \(\mu\)-measurable function \(w : \Omega \rightarrow \mathbb{R}\) with \(w(x) \geq 0\) for \(\mu\)-a.e. \(x \in \Omega\), assume \(\int_{\Omega} w(x) \, d\mu(x) > 0\). Consider the Lebesgue space \(L_w(\Omega, \mu) := \{ f : \Omega \rightarrow \mathbb{R}, \ f \text{ is measurable on } \int_{\Omega} w(x) \, |f(x)| \, d\mu(x) < \infty \}\).

If \(f, g : \Omega \rightarrow \mathbb{R}\) are \(\mu\)-measurable functions and \(f, g, fg \in L_w(\Omega, \mu)\), then we may consider the Čebyšev functional

\[
T_w(f, g) := \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) f(x) g(x) \, d\mu(x)
\]

\[
- \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) f(x) \, d\mu(x) \times \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) g(x) \, d\mu(x).
\]

We may also consider the functional

\[
D_w(f) := \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) \, d\mu(x) \times \int_{\Omega} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y) \, d\mu(y)} \int_{\Omega} w(y) f(y) \, d\mu(y) \right| \, d\mu(x).
\]

Applying Theorem 2 for the normalised isotonic linear functional

\[
A(f) := \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) f(x) \, d\mu(x),
\]

\(A : L_w(\Omega, \mu) \rightarrow \mathbb{R}\), we may recapture the following result due to Cerone and Dragomir [2]. Note that the proof of this result in [2] is different to the one in Theorem 2.

**Theorem 4.** Let \(w, f, g : \Omega \rightarrow \mathbb{R}\) be \(\mu\)-measurable functions with \(w \geq 0\) \(\mu\)-a.e. on \(\Omega\) and \(\int_{\Omega} w(x) \, d\mu(x) > 0\). If \(f, g, fg \in L_w(\Omega, \mu)\) and there exists the constants \(n, N\) so that

\[
-\infty < n \leq g(x) \leq N < \infty \text{ for } \mu\text{-a.e. } x \in \Omega,
\]

then we have the inequality

\[
|T_w(f, g)| \leq \frac{1}{2} (N - n) D_w(f).
\]

The constant \(\frac{1}{2}\) is sharp in the sense that it cannot be replaced by a smaller constant.

**Remark 2.** If \(\Omega = [a, b]\) and \(w(x) = 1\) in Theorem 4, then we recapture the result obtained in [3]

\[
\left| \frac{1}{b - a} \int_{a}^{b} f(x) g(x) \, dx - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \cdot \frac{1}{b - a} \int_{a}^{b} g(x) \, dx \right|
\]

\[
\leq \frac{1}{2} (N - n) \cdot \frac{1}{b - a} \int_{a}^{b} \left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(y) \, dy \right| \, dx
\]

provided \(n \leq g(x) \leq N\) for a.e. \(x \in [a, b]\).
Note that the proof in Theorem 2 is different to the one in [3], using only the linearity and monotonicity properties of the functional \( A \). We should also remark that in [3] the authors did not show the sharpness of the constant \( \frac{1}{2} \).

Now, if we consider the normalised isotonic linear functional

\[
A_{\bar{w}}(\bar{x}) := \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i,
\]

\( A_{\bar{w}} : \mathbb{R}^n \to \mathbb{R} \), where \( w_i \geq 0 \) \( (i = \overline{1,n}) \) and \( W_n := \sum_{i=1}^{n} w_i > 0 \), the by Theorem 2 we may obtain the following discrete inequality obtained by Cerone and Dragomir in [2].

**Theorem 5.** Let \( \bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n) \in \mathbb{R} \) be such that there exists the constants \( b, B \in \mathbb{R} \) so that

\[
b \leq b_i \leq B \text{ for each } i \in \{1, \ldots, n\}.
\]

Then one has the inequality

\[
\left| \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i b_i - \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i \cdot \frac{1}{W_n} \sum_{i=1}^{n} w_i b_i \right|
\leq \frac{1}{2} \left( B - b \right) \frac{1}{W_n} \sum_{i=1}^{n} w_i \left| a_i - \frac{1}{W_n} \sum_{j=1}^{n} w_j a_j \right|.
\]

The constant \( \frac{1}{2} \) is sharp in (3.6).

4. **A Counterpart of the (CBS)-Inequality**

The following inequality is known in the literature as the Cauchy-Buniakowski-Schwartz’s inequality for isotonic linear functionals or the (CBS)-inequality, for short,

\[
[A(fg)]^2 \leq A(f^2) A(g^2),
\]

provided \( f, g : E \to \mathbb{R} \) are with the property that \( fg, f^2, g^2 \in L \) and \( A : L \to \mathbb{R} \) is any isotonic linear functional.

Making use of the Grüss inequality (2.12), we may prove the following counterpart of the (CBS)-inequality for isotonic linear functionals.

**Theorem 6.** Let \( k, l : E \to \mathbb{R} \) be such that \( k^2, l^2, kl \in L \) and there exists the real constants \( \gamma, \Gamma \in \mathbb{R} \) so that

\[
0 \leq A(k^2) A(l^2) - [A(kl)]^2
\leq \frac{1}{2} \left( \Gamma - \gamma \right) A \left[ |l| A(l^2) k - A(kl) l \right].
\]

The constant \( \frac{1}{2} \) is sharp.
Proof. We choose in (2.12) \( f = g = \frac{k}{l} \), \( h = l^2 \) and \( B = A \) to get
\[
0 \leq \frac{A(k^2)}{A(l^2)} - \left[ \frac{A(kl)}{A(l^2)} \right]^2 \\
\leq \frac{1}{2} (\Gamma - \gamma) \frac{1}{A(l^2)} A \left[ l^2 \frac{k}{l} - \frac{1}{A(l^2)} A(kl) \right],
\]
provided \( A(l^2) \neq 0 \), which is equivalent to
\[
0 \leq \frac{A(k^2)}{A(l^2)} - [A(kl)]^2 \\
\leq \frac{1}{2} (\Gamma - \gamma) \frac{1}{A(l^2)} A \left[ kl - \frac{l^2}{A(l^2)} A(kl) \right],
\]
which is clearly equivalent to (4.3).

The following integral inequality holds.

\textbf{Corollary 3.} Let \( w, f, g : \Omega \to \mathbb{R} \) be a \( \mu \)-measurable function with \( w \geq 0 \) \( \mu \)-a.e. on \( \Omega \). If \( f, g \in L_w^2(\Omega, \mu) := \{ f : \Omega \to \mathbb{R}, \int_{\Omega} w(y) f^2(y) d\mu(y) < \infty \} \) and there exists \( \gamma, \Gamma \) so that
\[
-\infty < \gamma \leq \frac{f}{g} \leq \Gamma < \infty \quad \text{for } \mu \text{-a.e. } x \in \Omega,
\]
then one has the inequality:
\[
0 \leq \int_{\Omega} w(x) f^2(x) d\mu(x) \int_{\Omega} w(x) g^2(x) d\mu(x) \\
- \left[ \int_{\Omega} w(x) f(x) g(x) d\mu(x) \right]^2 \\
\leq \frac{1}{2} (\Gamma - \gamma) \int_{\Omega} w(x) |g(x)| \left| \left( \int_{\Omega} w(y) g^2(y) d\mu(y) \right) f(x) \right.
\left. - g(x) \int_{\Omega} w(y) f(y) g(y) d\mu(y) \right| d\mu(x) \\
= \frac{1}{2} (\Gamma - \gamma) \int_{\Omega} w(x) |g(x)| \left| \int_{\Omega} w(y) g(y) \left| f(x) g(x) \right| d\mu(y) \right| d\mu(x).
\]
The constant \( \frac{1}{2} \) is sharp.

\textbf{Remark 3.} In particular, if \( f, g \in L^2(\Omega, \mu) \) and the condition (4.3) holds, then
\[
0 \leq \int_{\Omega} f^2(x) d\mu(x) \int_{\Omega} g^2(x) d\mu(x) - \left[ \int_{\Omega} f(x) g(x) d\mu(x) \right]^2 \\
\leq \frac{1}{2} (\Gamma - \gamma) \int_{\Omega} |g(x)| \left| \int_{\Omega} g(y) \left| f(x) g(y) \right| d\mu(y) \right| d\mu(x).
\]
The constant \( \frac{1}{2} \) is sharp.

The following discrete inequality also holds.

\textbf{Corollary 4.} Let \( \bar{a} = (a_1, \ldots, a_n) \), \( \bar{b} = (b_1, \ldots, b_n) \) and \( \bar{w} = (w_1, \ldots, w_n) \) be the sequences of real numbers so that \( w_i \geq 0 \) \( (i = 1, \ldots, n) \), \( W_n := \sum_{i=1}^{n} w_i > 0 \) and \( a_i \),
\[
\gamma \leq \frac{a_i}{b_i} \leq \Gamma \quad \text{for each } i \in \{1, \ldots, n\}.
\]
Then one has the inequality

\[ 0 \leq \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 - \left( \sum_{i=1}^{n} w_i a_i b_i \right)^2 \]

\[ \leq \frac{1}{2} (\Gamma - \gamma) \sum_{i=1}^{n} w_i b_i \left| \sum_{j=1}^{n} w_j b_j \right| a_i b_j \left| \sum_{j=1}^{n} a_j b_j \right|. \]

The constant \( \frac{1}{2} \) is sharp.

**Remark 4.** If \( \bar{a}, \bar{b} \) satisfy (4.6), then one has the inequality

\[ 0 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 \]

\[ \leq \frac{1}{2} (\Gamma - \gamma) \sum_{i=1}^{n} b_i \left| \sum_{j=1}^{n} b_j \right| a_i b_i \left| \sum_{j=1}^{n} a_j b_j \right|. \]

The constant \( \frac{1}{2} \) is sharp.

### 5. A Converse for Jessen’s Inequality

In [4], the author has proved the following converse of Jessen’s inequality for normalized isotonic linear functionals.

**Theorem 7.** Let \( \Phi : (\alpha, \beta) \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable convex function on \( (\alpha, \beta) \), \( f : E \to (\alpha, \beta) \) so that \( \Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L \). If \( A : L \to \mathbb{R} \) is an isotonic linear and normalised functional, then

\[ 0 \leq A(\Phi \circ f) - \Phi(A(f)) \]

\[ \leq A[(\Phi' \circ f) \cdot f] - A(f) A(\Phi' \circ f) \]

\[ \leq \frac{1}{4} [\Phi'(\beta) - \Phi'(\alpha)] (\beta - \alpha) \quad \text{(if } \alpha, \beta \text{ are finite}). \]

We can state the following result improving the inequality (5.1).

**Theorem 8.** Let \( \Phi : [\alpha, \beta] \to \mathbb{R} \) with \( -\infty < \alpha < \beta < \infty \), and \( f, A \) are as in Theorem 7, then one has the inequality

\[ 0 \leq A(\Phi \circ f) - \Phi(A(f)) \]

\[ \leq A[(\Phi' \circ f) \cdot f] - A(f) A(\Phi' \circ f) \]

\[ \leq \frac{1}{4} [\Phi'(\beta) - \Phi'(\alpha)] A(|f - A(f) \cdot 1|), \]

provided \( |f - A(f) \cdot 1| \in L \).

**Proof.** Taking into account that \( \alpha \leq f \leq \beta \) and \( \Phi' \) is monotonic on \( [\alpha, \beta] \), we have \( \Phi'(\alpha) \leq \Phi' \circ f \leq \Phi'(\beta) \). Applying Theorem 2, we deduce

\[ A[(\Phi' \circ f) \cdot f] - A(f) A(\Phi' \circ f) \]

\[ \leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] A(|f - A(f) \cdot 1|), \]

and the theorem is proved. \( \square \)

The following corollary addressing the integral case also holds.
Corollary 5. Let $\Phi: [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on $(\alpha, \beta)$ and $f: \Omega \to [\alpha, \beta]$ so that $\Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L_w(\Omega, \mu)$, where $w \geq 0$ $\mu$-a.e. on $\Omega$ with $\int_{\Omega} w(x) \, d\mu(x) > 0$. Then we have the inequality:

$$\begin{align*}
(5.3) \quad 0 & \leq \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) \Phi(f(x)) \, d\mu(x) \\
& \quad - \Phi\left(\frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) f(x) \, d\mu(x)\right) \\
& \quad \leq \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) \Phi'(f(x)) f(x) \, d\mu(x) \\
& \quad - \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) \Phi'(f(x)) \, d\mu(x) \\
& \quad \times \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) f(x) \, d\mu(x) \\
& \leq \frac{1}{2} \left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) \, d\mu(x) \\
& \quad \times \int_{\Omega} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y) \, d\mu(y)} \int_{\Omega} w(y) f(y) \, d\mu(y) \right| \, d\mu(x).
\end{align*}$$

Remark 5. If $\mu(\Omega) < \infty$ and $\Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L_w(\Omega, \mu)$, then we have the inequality:

$$\begin{align*}
(5.4) \quad 0 & \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \Phi(f(x)) \, d\mu(x) - \Phi\left(\frac{1}{\mu(\Omega)} \int_{\Omega} f(x) \, d\mu(x)\right) \\
& \quad \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \Phi'(f(x)) f(x) \, d\mu(x) \\
& \quad - \frac{1}{\mu(\Omega)} \int_{\Omega} \Phi'(f(x)) \, d\mu(x) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} f(x) \, d\mu(x) \\
& \quad \leq \frac{1}{2} \left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{\mu(\Omega)} \int_{\Omega} f(x) - \frac{1}{\mu(\Omega)} \int_{\Omega} f(y) \, d\mu(y) \right| \, d\mu(x).
\end{align*}$$

The case of functions of a real variable is embodied in the following inequality that provides a counterpart for the Jensen’s integral inequality:

$$\begin{align*}
(5.5) \quad 0 & \leq \frac{1}{b-a} \int_{a}^{b} \Phi(f(x)) \, dx - \Phi\left(\frac{1}{b-a} \int_{a}^{b} f(x) \, dx\right) \\
& \quad \leq \frac{1}{b-a} \int_{a}^{b} \Phi'(f(x)) f(x) \, dx \\
& \quad - \frac{1}{b-a} \int_{a}^{b} \Phi'(f(x)) \, dx \cdot \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \\
& \quad \leq \frac{1}{2} \left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{b-a} \int_{a}^{b} f(x) - \frac{1}{b-a} \int_{a}^{b} f(y) \, dy \right| \, dx.
\end{align*}$$

The following discrete inequality is valid as well.

Corollary 6. Let $\Phi: [\alpha, \beta] \to \mathbb{R}$ be a differentiable convex function on $(\alpha, \beta)$. If $x_i \in [\alpha, \beta]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n > 0$, then one has the counterpart of
Jensen’s discrete inequality:

\[
0 \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right)
\]

\[\leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi'(x_i) \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i
\]

\[\leq \frac{1}{2} \left[ \Phi'(\beta) - \Phi'(\alpha) \right] \frac{1}{W_n} \sum_{i=1}^{n} w_i \left| x_i - \frac{1}{W_n} \sum_{j=1}^{n} w_j x_j \right|.
\]

Remark 6. In particular, we get the discrete inequality:

\[
0 \leq \frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)
\]

\[\leq \frac{1}{n} \sum_{i=1}^{n} \Phi'(x_i) x_i - \frac{1}{n} \sum_{i=1}^{n} \Phi'(x_i) \frac{1}{n} \sum_{i=1}^{n} x_i
\]

\[\leq \frac{1}{2} \left[ \Phi'(\beta) - \Phi'(\alpha) \right] \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right|.
\]

References


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