SOME NEW INEQUALITIES OF OSTROWSKI TYPE

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ABSTRACT. By the use of the Cauchy mean value theorem, some new inequalities of Ostrowski type are given.

1. Introduction

The following result is known in the literature as Ostrowski’s inequality [1].

**Theorem 1.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with the property that \( |f'(t)| \leq M \) for all \( t \in (a, b) \). Then

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \left[ \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2} \right] (b-a) M
\]

for all \( x \in [a, b] \).

The constant \( \frac{1}{4} \) is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

\[
f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \frac{1}{b-a} \int_a^b p(x, t) f'(t) \, dt, \quad x \in [a, b],
\]

where

\[
p(x, t) := \begin{cases} 
  t - a & \text{if } a \leq t \leq x \\
  t - b & \text{if } x < t \leq b
\end{cases}
\]

which also holds for absolutely continuous functions \( f : [a, b] \to \mathbb{R} \).

The following Ostrowski type result for absolutely continuous functions holds (see [2], [3] and [4]).
Theorem 2. Let \( f : [a, b] \rightarrow \mathbb{R} \) be absolutely continuous on \([a, b]\). Then, for all \( x \in [a, b] \), we have:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \begin{cases} 
\left[ \frac{1}{4} + \left( \frac{x-a+b}{b-a} \right)^2 \right] (b-a) \| f' \|_{\infty} & \text{if } f' \in L_{\infty} [a, b]; \\
\frac{1}{(p+1)^\frac{p}{p+1}} \left[ \left( \frac{x-a}{b-a} \right)^p + \left( \frac{b-x}{b-a} \right)^p \right] \frac{1}{p} (b-a)^\frac{1}{p} \| f' \|_q & \text{if } f' \in L_q [a, b],
\end{cases}
\]

where \( \| \cdot \|_r \) \((r \in [1, \infty])\) are the usual Lebesgue norms on \( L_r [a, b] \), i.e.,

\[
\| g \|_\infty := \text{ess sup}_{t \in [a, b]} |g(t)|
\]

and

\[
\| g \|_r := \left( \int_a^b |g(t)|^r \, dt \right)^{\frac{1}{r}}, \quad r \in [1, \infty).
\]

The constants \( \frac{1}{4}, \frac{1}{(p+1)^\frac{p}{p+1}} \) and \( \frac{1}{2} \) respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from the Fink result in [5] on choosing \( n = 1 \) and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that \( f \) is Hölder continuous, then one may state the result (see [6]):

Theorem 3. Let \( f : [a, b] \rightarrow \mathbb{R} \) be of \( r - H \) Hölder type, i.e.,

\[
|f(x) - f(y)| \leq H |x - y|^r, \quad \text{for all } x, y \in [a, b],
\]

where \( r \in (0, 1] \) and \( H > 0 \) are fixed. Then for all \( x \in [a, b] \) we have the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{H}{r+1} \left[ \frac{(b-x)^{r+1}}{(b-a)^{r+1}} + \left( \frac{x-a}{b-a} \right)^{r+1} \right] (b-a)^r.
\]

The constant \( \frac{1}{r+1} \) is also sharp in the above sense.

Note that if \( r = 1 \), i.e., \( f \) is Lipschitz continuous, then we get the following version of Ostrowski’s inequality for Lipschitzian functions (with \( L \) instead of \( H \)) (see [7])

\[
f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \leq \frac{1}{4} + \left( \frac{x-a+b}{b-a} \right)^2 (b-a) L.
\]

Here the constant \( \frac{1}{4} \) is also best.
Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8]).

**Theorem 4.** Assume that \( f : [a, b] \to \mathbb{R} \) is of bounded variation and denote by \( \int_{a}^{b} \) its total variation. Then

\[
(1.7) \quad \left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right| \leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] \int_{a}^{b} (f)
\]

for all \( x \in [a, b] \).

The constant \( \frac{1}{2} \) is the best possible.

If we assume more about \( f \), i.e., \( f \) is monotonically increasing, then the inequality (1.7) may be improved in the following manner [9] (see also [10]).

**Theorem 5.** Let \( f : [a, b] \to \mathbb{R} \) be monotonic nondecreasing. Then for all \( x \in [a, b] \), we have the inequality:

\[
(1.8) \quad \left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right| \leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_{a}^{b} \text{sgn}(t-x) f(t) \, dt \right\} \leq \frac{1}{b-a} \{ (x-a) [f(x) - f(a)] + (b-x) [f(b) - f(x)] \} \leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] [f(b) - f(a)].
\]

All the inequalities in (1.8) are sharp and the constant \( \frac{1}{2} \) is the best possible.

In this paper we point out different Ostrowski type inequalities assuming some special properties for the derivative of the function \( f \) around a given point \( x \in (a, b) \).

### 2. The Results

The following theorem holds.

**Theorem 6.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Let \( p \in (0, \infty) \) and assume, for a given \( x \in (a, b) \), we have that

\[
(2.1) \quad M_p(x) := \sup_{u \in (a, b)} \left\{ |x-u|^{1-p} |f'(u)| \right\} < \infty.
\]

Then we have the inequality

\[
(2.2) \quad \left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right| \leq \frac{1}{p(p+1)(b-a)} \left[ (x-a)^{p+1} + (b-x)^{p+1} \right] M_p(x).
\]

**Proof.** Let \( x \in (a, b) \) and define the mapping \( g_{1,x} : (a, x) \to \mathbb{R} \), \( g_{1,x}(t) = (x-t)^p \).

Applying the Cauchy mean value theorem, for any \( t \in (a, x) \) there exists a \( \eta \in (t, x) \) such that

\[
[f(t) - f(x)] g_{1,x}'(\eta) = [g_{1,x}(t) - g_{1,x}(x)] f' (\eta)
\]
Remark 2. If provided by the classical Ostrowski’s inequality, \( p > 1 \), then obviously that for \( p > 1 \), the accuracy order provided by (2.2) is higher than 1, as provided by the classical Ostrowski’s inequality.

\[
(-p)(f(t) - f(x))(x - t)^{p-1} = (x - t)^p f'(t)
\]

from where we obtain

\[
|f(t) - f(x)| = \frac{(x - t)^p |f'(t)|}{p(x - t)^{p-1}} \leq \frac{(x - t)^p}{p} M_p(x), \quad t \in (a, x).
\]

We define the mapping \( g_{2,x} : (x, b) \rightarrow \mathbb{R}, g_{2,x}(t) = (t - x)^p \). Applying the Cauchy mean value theorem, we can find a \( \xi \in (x, t) \) such that

\[
|f(t) - f(x)| p (\xi - x)^{p-1} = (t - x)^p f'(\xi)
\]

from where we get

\[
|f(t) - f(x)| = \frac{(t - x)^p |f'(\xi)|}{p (\xi - x)^{p-1}} \leq \frac{(t - x)^p}{p} M_p(x), \quad t \in (x, b).
\]

In conclusion, by (2.3) and (2.4) we may write

\[
|f(t) - f(x)| \leq \frac{1}{p} M_p(x) |t - x|^p \quad \text{for all} \quad t \in (a, b).
\]

Integrating (2.5) over \( t \) on \( [a, b] \), we get

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{1}{b - a} \int_a^b |f(t) - f(x)| \, dt \leq \frac{1}{p} M_p(x) \frac{1}{b - a} \int_a^b |t - x|^p \, dt
\]

\[
= \frac{1}{p} M_p(x) \frac{1}{b - a} \left[ \int_a^x (x - t)^p \, dt + \int_x^b (t - x)^p \, dt \right]
\]

\[
= \frac{1}{p} M_p(x) \frac{(x - a)^{p+1} + (b - x)^{p+1}}{(p + 1)(b - a)}
\]

and the inequality (2.2) is proved. \( \blacksquare \)

Remark 1. For \( p = 1 \), we obtain

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{(x - a)^2 + (b - x)^2}{2(b - a)} \|f'\|_{\infty}
\]

\[
= \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b - a} \right)^2 \right] \|f'\|_{\infty} (b - a), \quad x \in [a, b],
\]

where \( \|f'\|_{\infty} := \sup_{t \in (a, b)} |f'(t)| < \infty \), which is Ostrowski’s inequality (1.1). It is obvious that for \( p > 1 \), the accuracy order provided by (2.2) is higher than 1, as provided by the classical Ostrowski’s inequality.

Remark 2. If \( p \in (0, 1) \) and \( f' \in L_{\infty}[a, b] \), then obviously

\[
M_p(x) \leq (\max \{x - a, b - x\})^{1-p} \|f'\|_{\infty}
\]

\[
= \left[ \frac{a + b}{2} + \left| \frac{x - a + b}{2} \right| \right]^{1-p} \|f'\|_{\infty}
\]

for all \( x \in [a, b] \).
The following mid-point formula holds.

**Corollary 1.** Let \( f \) and \( p \) be as in Theorem 6. Assume that

\[
M_p \left( \frac{a+b}{2} \right) := \sup_{u \in (a,b)} \left\{ \left| \frac{a+b}{2} - u \right|^{\frac{1-p}{p}} |f'(u)| \right\} < \infty.
\]

Then we have the midpoint inequality

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{(b-a)^p}{p(p+1)2^p} M_p \left( \frac{a+b}{2} \right).
\]

Before we continue our presentation, we recall the following special means:

(a) The arithmetic mean

\[
A = A(a,b) := \frac{a+b}{2}, \quad a,b \geq 0;
\]

(b) The geometric mean

\[
G = G(a,b) := \sqrt{ab}, \quad a,b \geq 0;
\]

(c) The harmonic mean

\[
H = H(a,b) := \frac{2ab}{a+b}, \quad a,b > 0;
\]

(d) The logarithmic mean

\[
L = L(a,b) := \left\{ \begin{array}{ll}
\frac{a}{2} & \text{if } a = b, \\
\frac{b-a}{\ln b - \ln a} & \text{if } a \neq b,
\end{array} \right. \quad a,b > 0;
\]

(e) The identric mean

\[
I = I(a,b) := \left\{ \begin{array}{ll}
\frac{a^b}{b^a} & \text{if } a = b, \\
\frac{1}{\ln a} & \text{if } a \neq b,
\end{array} \right. \quad a,b > 0;
\]

(f) The \( p \)-logarithmic mean

\[
L_p = L_p(a,b) := \left\{ \begin{array}{ll}
\left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b, \\
\frac{1}{b-a} & \text{if } a = b,
\end{array} \right. \quad a,b > 0;
\]

where \( p \in \mathbb{R} \setminus \{-1, 0\} \).

The following result also holds.

**Theorem 7.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) with \( a > 0 \), and differentiable on \((a, b)\). Let \( p \in \mathbb{R} \setminus \{0\} \) and assume that

\[
K_p(f') := \sup_{u \in (a,b)} \left\{ u^{1-p} |f'(u)| \right\} < \infty.
\]

Then we have the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{K_p(f')}{|p|(b-a)}
\]
Integrating (2.9) over $t$ for all $x \in [a, b]$. 

For $x \in [a, b]$, we have 

$$
\begin{align*}
2x^p(x - A) + (b - x)L_p^p(b, x) - (x - a)L_p^p(x, a) & \quad \text{if } p \in (0, \infty) \\
(x - a)L_p^p(x, a) - (b - x)L_p^p(b, x) - 2x^p(x - A) & \quad \text{if } p \in (-\infty, -1) \cup (-1, 0) \\
(x - a)L^{-1}_p(a, x) - (b - x)L^{-1}_p(b, x) - \frac{2}{x}(x - A) & \quad \text{if } p = -1
\end{align*}
$$

Proof. Consider the mapping $g : [a, b] \to \mathbb{R}$, $g(x) = x^p$. Applying the Cauchy mean value theorem, then for any $x$ and $t \in [a, b]$, there exists a $\eta$ between $x$ and $t$ such that

$$
[f(t) - f(x)]g'(\eta) = [g(t) - g(x)]f'(\eta)
$$

i.e.,

$$(f(t) - f(x))p\eta^{p-1} = (t^p - x^p)f'(\eta)$$

from where we obtain:

$$
|f(t) - f(x)| = \frac{|f'(\eta)||t^p - x^p|}{|p|\eta^{p-1}} \leq \frac{K_p(f')}{|p|}|t^p - x^p|.
$$

In conclusion, for any $t, x \in [a, b]$, we have the inequality 

$$
(2.9) \quad |f(t) - f(x)| \leq \frac{K_p(f')}{|p|}|t^p - x^p|.
$$

Integrating (2.9) over $t$ on $[a, b]$, we get

$$
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{b-a} \int_a^b |f(t) - f(x)| \, dt
$$

\leq \frac{K_p(f')}{p} \frac{1}{b-a} \int_a^b |t^p - x^p| \, dt.
$$

For $p > 0$, we have

$$
\int_a^b |t^p - x^p| \, dt = \int_a^x (x^p - t^p) \, dt + \int_x^b (t^p - x^p) \, dt = 2x^p(x - A) + (b - x)L_p^p(b, x) - (x - a)L_p^p(x, a).
$$

For $p \in (-\infty, -1) \cup (-1, 0)$, we have

$$
\int_a^b |x^p - t^p| \, dt = \int_a^x (t^p - x^p) \, dt + \int_x^b (x^p - t^p) \, dt = (x - a)L_p^p(x, a) - (b - x)L_p^p(b, x) - 2x^p(x - A)
$$

and, finally, for $p = -1$, we have

$$
\int_a^b \left| \frac{1}{x} - \frac{1}{t} \right| \, dt = \int_a^x \left( \frac{1}{t} - \frac{1}{x} \right) \, dt + \int_x^b \left( \frac{1}{x} - \frac{1}{t} \right) \, dt = (x - a)L^{-1}_p(a, x) - (b - x)L^{-1}_p(b, x) - \frac{2}{x}(x - A)
$$

and the theorem is proved. $\blacksquare$

The following corollary is natural.
Corollary 2. With the assumptions in Theorem 7, we have the midpoint inequality

\begin{equation}
| f (A) - \frac{1}{b - a} \int_a^b f (t) \, dt | \leq \frac{K_p (f')}{|p|} \times \begin{cases} 
\frac{1}{2} (L_p^p (b, A) - L_p^p (A, a)) & \text{if } p > 0; \\
\frac{1}{2} (L_p^p (A, a) - L_p^p (A, b)) & \text{if } p \in (-\infty, -1) \cup (-1, 0); \\
\frac{1}{2} (L^{-1} (a, A) - L^{-1} (A, b)) & \text{if } p = -1.
\end{cases}
\end{equation}

The following theorem also holds.

Theorem 8. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) (with \( a > 0 \)) and differentiable on \((a, b)\). If

\begin{equation}
P (f') := \sup_{u \in (a, b)} |uf' (u)| < \infty
\end{equation}

then we have the inequality

\begin{equation}
| f (x) - \frac{1}{b - a} \int_a^b f (t) \, dt | \leq \frac{P (f')}{b - a} \left[ \ln \left( \frac{|I (x, b)|^{b-x}}{|I (a, x)|^{a-x}} \right) + 2 (x - A) \ln x \right]
\end{equation}

for all \( x \in [a, b] \).

**Proof.** Consider the mapping \( g : [a, b] \to \mathbb{R} \), \( g (t) = \ln t \). Applying the Cauchy mean value theorem for any \( x \) and \( t \in [a, b] \) there exists a \( \eta \) between \( x \) and \( t \) such that

\[(f (t) - f (x)) g' (\eta) = (g (t) - g (x)) f' (\eta)\]

i.e.,

\[(f (t) - f (x)) \frac{1}{\eta} = (\ln t - \ln x) f' (\eta)\]

from where we get

\[|f (t) - f (x)| = |\eta f' (\eta)| |\ln t - \ln x| \leq P (f') |\ln t - \ln x| .\]

In conclusion, for any \( t, x \in [a, b] \), we have the inequality

\begin{equation}
|f (t) - f (x)| \leq P (f') |\ln t - \ln x| .
\end{equation}
Integrating (2.13) over \( t \) on \([a, b]\), we get
\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{b-a} \int_a^b |f(t) - f(x)| \, dt \leq P(f') \frac{1}{b-a} \int_a^b |\ln t - \ln x| \, dt
\]
\[
= P(f') \frac{1}{b-a} \left[ \int_a^x (\ln x - \ln t) \, dt + \int_x^b (\ln t - \ln x) \, dt \right]
\]
\[
= P(f') \frac{1}{b-a} [(x-a) \ln x - (x-a) \ln I(a, x) + (b-x) \ln I(b, x) - (b-x) \ln x]
\]
\[
= \frac{1}{b-a} [2(x-A) \ln x + (b-x) \ln I(x, b) - (x-a) \ln I(a, x)] P'(f)
\]
and the theorem is proved.

The following corollary is natural.

**Corollary 3.** With the assumptions of Theorem 8, we have the inequality
\[
(2.14) \quad \left| f(A) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{2} P(f') \ln \left[ \frac{I(A, b)}{I(a, A)} \right],
\]
where \( A = A(a, b) = \frac{a+b}{2} \).

**References**


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