SHARP BOUNDS OF ČEBYŠEV FUNCTIONAL FOR STILOTTES INTEGRALS AND APPLICATIONS

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Abstract. Sharp bounds of the Čebyšev functional for the Stieltjes integrals similar to the Grüss one and applications for quadrature rules are given.

1. Introduction

Consider the weighted Čebyšev functional

\begin{equation}
T_w(f, g) := \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) g(t) dt - \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) dt \cdot \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) g(t) dt
\end{equation}

where \( f, g, w : [a, b] \to \mathbb{R} \) and \( w(t) \geq 0 \) for a.e. \( t \in [a, b] \) are measurable functions such that the involved integrals exist and \( \int_a^b w(t) dt > 0 \).

In [1], the authors obtained, among others, the following inequalities:

\begin{equation}
|T_w(f, g)| \leq \frac{1}{2} (M - m) \left[ \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) g(t) ds \right]^{\frac{1}{p}} \left( p > 1 \right)
\end{equation}

provided

\begin{equation}
-\infty < m \leq f(t) \leq M < \infty \text{ for a.e. } t \in [a, b]
\end{equation}

and the corresponding integrals are finite. The constant \( \frac{1}{2} \) is sharp in all the inequalities in (1.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if

\begin{equation}
-\infty < n \leq g(t) \leq N < \infty \text{ for a.e. } t \in [a, b],
\end{equation}

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then the following refinement of the celebrated Grüss inequality is obtained:

\[ |T_w(f, g)| \leq \frac{1}{2} (M - m) \frac{1}{f_a} \int_a^b w(t) \left| g(t) - \frac{1}{f_a} \int_a^b w(s) g(s) ds \right| dt \]

\[ \leq \frac{1}{2} (M - m) \left[ \frac{1}{f_a} \int_a^b w(t) dt \int_a^b w(t) \right] \left| g(t) - \frac{1}{f_a} \int_a^b w(s) g(s) ds \right|^2 dt \]

\[ \leq \frac{1}{4} (M - m) (N - n). \]

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

In this paper, we extend the above results for Riemann-Stieltjes integrals. A quadrature formula is also considered.

For this purpose, we introduce the following Čebyšev functional for the Stieltjes integral

\[ T(f, g; u) := \frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) du(t) \]

\[ - \frac{1}{u(b) - u(a)} \int_a^b f(t) du(t) \cdot \frac{1}{u(b) - u(a)} \int_a^b g(t) du(t), \]

where $f, g \in C[a, b]$ (are continuous on $[a, b]$) and $u \in BV[a, b]$ (is of bounded variation on $[a, b]$) with $u(b) \neq u(a)$.

For some recent inequalities for Stieltjes integral see [2]-[5].

2. The Results

The following result holds.

**Theorem 1.** Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $u : [a, b] \to \mathbb{R}$ with $u(a) \neq u(b)$. Assume also that there exists the real constants $m, M$ such that

\[ m \leq f(t) \leq M \text{ for each } t \in [a, b]. \]

If $u$ is of bounded variation on $[a, b]$, then we have the inequality

\[ |T(f, g; u)| \leq \frac{1}{2} (M - m) \frac{1}{u(b) - u(a)} \left[ g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] \left[ \sqrt{\int_a^b (u) \right), \]

where $\sqrt{\int_a^b (u)}$ denotes the total variation of $u$ in $[a, b]$. The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller constant.
Proof. It is easy to see, by simple computation with the Stieltjes integral, that the following equality
\begin{equation}
T(f, g; u) = \frac{1}{u(b) - u(a)} \int_a^b \left[ f(t) - \frac{m + M}{2} \right] \times \left[ g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right] \, du(t)
\end{equation}
holds.

Using the known inequality
\begin{equation}
\left| \int_a^b p(t) \, dv(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \int_a^b |v(t)| \, dv(t),
\end{equation}
provided \( p \in C[a,b] \) and \( v \in BV[a,b] \), we have, by (2.3), that
\begin{align*}
|T(f, g; u)| &\leq \sup_{t \in [a,b]} \left| \int_a^b \left[ f(t) - \frac{m + M}{2} \right] \times \left[ g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right] \, du(t) \right| \\
&\leq \frac{1}{|u(b) - u(a)|} \sup_{t \in [a,b]} \left| f(t) - \frac{m + M}{2} \right| \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| \\
&\leq \frac{1}{|u(b) - u(a)|} \sup_{t \in [a,b]} \left| f(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| \leq \frac{1}{|u(b) - u(a)|} \sup_{t \in [a,b]} \left| f(t) - \frac{m + M}{2} \right| \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right|
\end{align*}
and the inequality (2.2) is proved.

To prove the sharpness of the constant \( \frac{1}{2} \) in the inequality (2.2), we assume that it holds with a constant \( C > 0 \), i.e.,
\begin{equation}
|T(f, g; u)| \leq C(M - m) \frac{1}{|u(b) - u(a)|} \sup_{t \in [a,b]} \left| f(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| \leq \frac{1}{|u(b) - u(a)|} \sup_{t \in [a,b]} \left| f(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right|.
\end{equation}

Let us consider the functions \( f = g, f : [a, b] \to \mathbb{R}, f(t) = t, t \in [a, b] \) and \( u : [a, b] \to \mathbb{R} \) given by
\begin{equation}
u(t) = \begin{cases}
-1 & \text{if } t = a, \\
0 & \text{if } t \in (a, b), \\
1 & \text{if } t = b.
\end{cases}
\end{equation}
Then \( f, g \) are continuous on \([a, b]\), \( u \) is of bounded variation on \([a, b]\) and

\[
\frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) \, du(t) = \frac{1}{2} \int_a^b t^2 \, du(t) = \frac{1}{2} \left[ t^2 u(t) \bigg|_a^b - 2 \int_a^b t u(t) \, dt \right] = \frac{b^2 + a^2}{2},
\]

\[
\frac{1}{u(b) - u(a)} \int_a^b f(t) \, du(t) = \frac{1}{u(b) - u(a)} \int_a^b g(t) \, du(t) = \frac{1}{2} \int_a^b t \, du(t) = \frac{1}{2} \left[ tu(t) \bigg|_a^b - \int_a^b u(t) \, dt \right] = \frac{b + a}{2},
\]

\[
\left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right\|_\infty = \sup_{t \in [a, b]} \left| t - \frac{a + b}{2} \right| = \frac{b - a}{2}
\]

and

\[
\sqrt{b(a)} = 2, \quad M = b, \quad m = a.
\]

Inserting these values in (2.5), we get

\[
\left| \frac{a^2 + b^2}{2} - \frac{(a + b)^2}{4} \right| \leq C \left( b - a \right) \cdot \frac{1}{2} \cdot \frac{(b - a)}{2} \cdot 2,
\]

giving \( C \geq \frac{1}{2}, \) and the theorem is thus proved. \( \blacksquare \)

The corresponding result for monotonic function \( u \) is incorporated in the following theorem.

**Theorem 2.** Assume that \( f \) and \( g \) are as in Theorem 1. If \( u : [a, b] \to \mathbb{R} \) is monotonic nondecreasing on \([a, b]\), then one has the inequality:

\[
|T(f, g; u)| \leq \frac{1}{2} (M - m) \frac{1}{u(b) - u(a)} \int_a^b g(t) \, du(t),
\]

The constant \( \frac{1}{2} \) is sharp in the sense that it cannot be replaced by a smaller constant.

**Proof.** Using the known inequality

\[
\int_a^b p(t) \, dv(t) \leq \int_a^b |p(t)| \, dv(t),
\]

provided \( p \in C \left[ a, b \right] \) and \( v \) is a monotonic nondecreasing function on \( [a, b] \), we have (by the use of equality (2.3)) that
\[
|T(f, g; u)| \leq \frac{1}{u(b) - u(a)} \int_a^b \left| f(t) - \frac{m + M}{2} \right| \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, ds \right| \, dt
\leq \frac{1}{2} (M - m) \frac{1}{u(b) - u(a)} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, ds \right| \, dt.
\]

Now, assume that the inequality (2.7) holds with a constant \( D > 0 \), instead of \( \frac{1}{2} \), i.e.,
\[
(2.9) \quad |T(f, g; u)| \leq D (M - m) \frac{1}{u(b) - u(a)} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, ds \right| \, dt.
\]

If we choose the same function as in the proof of Theorem 1, we observe that \( f, g \) are continuous and \( u \) is monotonic nondecreasing on \([a, b]\). Then, for these functions, we have
\[
T(f, g; u) = \frac{a^2 + b^2}{2} - \frac{(a + b)^2}{4} = \frac{(b - a)^2}{4},
\]
\[
\int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, ds \right| \, dt
= \int_a^b \left| t - \frac{a + b}{2} \right| \, dt
= \int_a^{a + b/2} \left( \frac{a + b}{2} - t \right) \, dt + \int_{a + b/2}^b \left( t - \frac{a + b}{2} \right) \, dt
= \left[ u(t) \left( \frac{a + b}{2} - t \right) \right]_{a + b/2}^{a + b} + \int_a^{a + b/2} u(t) \, dt
+ \left[ u(t) \left( t - \frac{a + b}{2} \right) \right]_{a + b/2}^b - \int_{a + b/2}^b u(t) \, dt
= b - a,
\]
and then, by (2.9) we get
\[
\frac{(b - a)^2}{4} \leq D (b - a) \frac{1}{2} (b - a)
\]
giving \( D \geq \frac{1}{2} \), and the theorem is completely proved.

The case when \( u \) is a Lipschitzian function is embodied in the following theorem.
Theorem 3. Assume that \( f, g : [a, b] \to \mathbb{R} \) are Riemann integrable functions on \([a, b]\) and \( f \) satisfies the condition (2.1). If \( u : (a, b) \to \mathbb{R} \) (\( u(b) \neq u(a) \)) is Lipschitzian with the constant \( L \), then we have the inequality

\[
|T(f, g; u)| \leq \frac{1}{2} L (M - m) \frac{1}{|u(b) - u(a)|} \left[ g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right] dt.
\]  

The constant \( \frac{1}{2} \) cannot be replaced by a smaller constant.

Proof. It is well known that if \( p : [a, b] \to \mathbb{R} \) is Riemann integrable on \([a, b]\) and \( v : [a, b] \to \mathbb{R} \) is Lipschitzian with the constant \( L \), then the Riemann-Stieltjes integral \( \int_a^b p(t) \, dv(t) \) exists and

\[
\left| \int_a^b p(t) \, dv(t) \right| \leq L \int_a^b |p(t)| \, dt.
\]

Using this fact and the identity (2.3), we deduce

\[
|T(f, g; u)| \leq \frac{L}{|u(b) - u(a)|} \int_a^b \left| f(t) - \frac{m + M}{2} \right| \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| dt,
\]

\[
\leq \frac{1}{2} (M - m) \frac{L}{|u(b) - u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| dt,
\]

and the inequality (2.10) is proved.

Now, assume that (2.10) holds with a constant \( E > 0 \) instead of \( \frac{1}{2} \), i.e.,

\[
|T(f, g; u)| \leq EL (M - m) \frac{1}{|u(b) - u(a)|} \left[ g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right] dt.
\]

Consider the function \( f = g, f : [a, b] \to \mathbb{R} \) with

\[
f(t) = \begin{cases} 
-1 & \text{if } t \in [a, \frac{a+b}{2}] \\
1 & \text{if } t \in (\frac{a+b}{2}, b]
\end{cases}
\]

and \( u : [a, b] \to \mathbb{R}, u(t) = t \). Then, obviously, \( f \) and \( g \) are Riemann integrable on \([a, b]\) and \( u \) is Lipschitzian with the constant \( L = 1 \).
Since
\[
\frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) \, du(t) = \frac{1}{b-a} \int_a^b dt = 1,
\]
\[
\frac{1}{u(b) - u(a)} \int_a^b f(t) \, du(t) = \frac{1}{u(b) - u(a)} \int_a^b g(t) \, du(t) = 0,
\]
\[
\int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| \, dt = \int_a^b dt = b - a
\]
and
\[ M = 1, \quad m = 1 \]
then, by (2.12), we deduce \( E \geq \frac{1}{2} \), and the theorem is completely proved.

### 3. A Quadrature Formula

Let us consider the partition of the interval \([a, b]\) given by
\[
(I_n) : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.
\]
Denote \( v(I_n) := \max \{h_i | i = 0, n-1\} \) where \( h_i := x_{i+1} - x_i, i = 0, n-1 \).

If \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and if we define
\[
M_i := \sup_{t \in [x_i, x_{i+1}]} f(t), \quad m_i := \inf_{t \in [x_i, x_{i+1}]} f(t),
\]
\[
v(f, I_n) = \max_{i=0, n-1} (M_i - m_i),
\]
then, obviously, by the continuity of \( f \) on \([a, b]\), for any \( \varepsilon > 0 \), we may find a division \( I_n \) with norm \( v(I_n) < \delta \) such that \( v(f, I_n) < \varepsilon \).

Consider now the quadrature rule
\[
S_n(f, g; u, I_n) := \sum_{i=0}^{n-1} \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t) \, du(t) \cdot \int_{x_i}^{x_{i+1}} g(t) \, du(t)
\]
provided \( f, g \in C[a, b], u \in BV[a, b] \) and \( u(x_{i+1}) \neq u(x_i), i = 0, \ldots, n-1 \).

We may now state the following result in approximating the Stieltjes integral
\[
\int_a^b f(t) g(t) \, du(t).
\]

**Theorem 4.** Let \( f, g \in C[a, b] \) and \( u \in BV[a, b] \). If \( I_n \) is a division of the interval \([a, b]\) and \( u(x_{i+1}) \neq u(x_i), i = 0, \ldots, n-1 \), then we have:
\[
\int_a^b f(t) g(t) \, du(t) = S_n(f, g; u, I_n) + R_n(f, g; u, I_n),
\]
where \( S_n(f, g; u, I_n) \) is as defined in (3.2) and the remainder \( R_n(f, g; u, I_n) \) satisfies the estimate
\[
|R_n(f, g; u, I_n)| \leq \frac{1}{2} v(f, I_n)
\]
\[
\times \max_{i=0, n-1} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) \, du(s) \right\| \cdot \sqrt{a}^\infty (u).
\]
The constant \( \frac{1}{2} \) is sharp in (3.4) in the sense that it cannot be replaced by a smaller constant.
Proof. Applying the inequality (2.2) on the intervals \([x_i, x_{i+1}]\), \(i = 0, \ldots, n-1\), we have

\[
(3.5) \quad \left| \int_{x_i}^{x_{i+1}} f(t) g(t) \, du(t) \right| \leq \frac{1}{2} \left( M_i - m_i \right) \sup_{t \in [x_i, x_{i+1}]} \left| g(t) - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) \, ds \right| \sqrt{(u)} .
\]

Summing the inequalities (3.5) over \(i\) from 0 to \(n-1\), and using the generalised triangle inequality, we have

\[
(3.6) \quad |R_n(f, g; u, I_n)| \leq \frac{1}{2} \sum_{i=0}^{n-1} (M_i - m_i) \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) \, ds \right\|_{[x_i, x_{i+1}], \infty} \times \sqrt{(u)} \times \sqrt{(u)}
\]

\[
= \frac{1}{2} v(f, I_n) \max_{i=0, n-1} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) \, ds \right\|_{[x_i, x_{i+1}], \infty} \times \sum_{i=0}^{n-1} \sqrt{(u)}
\]

and the estimate (3.4) is obtained. 

Remark 1. Similar results may be stated for either \(u\) monotonic or Lipschitzian. We omit the details.

4. SOME PARTICULAR CASES

For \(f, g, w : [a, b] \to \mathbb{R}\), integrable and with the property that \(\int_a^b w(t) \, dt \neq 0\), reconsider the weighted Cébyšev functional

\[
(4.1) \quad T_w(f, g) := \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) f(t) g(t) \, dt - \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) f(t) \, dt \cdot \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) g(t) \, dt.
\]

1. If \(f, g, w : [a, b] \to \mathbb{R}\) are continuous and there exists the real constants \(m, M\) such that

\[
(4.2) \quad m \leq f(t) \leq M \text{ for each } t \in [a, b],
\]
then one has the inequality

\[ |T_w(f, g)| \leq \frac{1}{2} (M - m) \frac{1}{\int_a^b w(s) \, ds} \times \left\| g - \frac{1}{\int_a^b w(s) \, ds} \int_a^b g(s) w(s) \, ds \right\| \int_a^b |w(s)| \, ds. \]

The proof follows by Theorem 1 on choosing \( u(t) = \int_a^t w(s) \, ds \).

2. If \( f, g, w \) are as in 1 and \( w(s) \geq 0 \) for \( s \in [a, b] \), then one has the inequality

\[ |T_w(f, g)| \leq \frac{1}{2} (M - m) \frac{1}{\int_a^b w(s) \, ds} \times \left\| g(t) - \frac{1}{\int_a^b w(s) \, ds} \int_a^b g(s) w(s) \, ds \right\| \int_a^b |w(s)| \, ds. \]

The proof follows by Theorem 2 on choosing \( u(t) = \int_a^t w(s) \, ds \).

3. If \( f, g \) are Riemann integrable on \([a, b]\) and \( f \) satisfies (4.2), and \( w \) is continuous on \([a, b] \), then one has the inequality

\[ |T_w(f, g)| \leq \frac{1}{2} \|w\|_{[a, b], \infty} (M - m) \frac{1}{\int_a^b w(s) \, ds} \times \left\| g(t) - \frac{1}{\int_a^b w(s) \, ds} \int_a^b g(s) w(s) \, ds \right\| ds. \]

The proof follows by Theorem 3 on choosing \( u(t) = \int_a^t w(s) \, ds \).

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