A DOUBLE INEQUALITY FOR REMAINDER OF POWER SERIES OF TANGENT FUNCTION

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Abstract. By mathematical induction, an identity and a double inequality for remainder of power series of tangent function are established.

1. Introduction

It is well known that Bernoulli numbers $B_i$ are defined [11] by

$$
\frac{x}{e^x - 1} = 1 - \frac{1}{2} x + \sum_{i=1}^{\infty} \frac{(-1)^i B_i}{(2i)!} x^{2i-1}, \quad |x| < 2\pi.
$$

(1)

About Bernoulli numbers, some new results can be found in [1, 3, 5].

The tangent and cotangent can be expanded into power series with coefficients involving Bernoulli numbers as follows [11, p. 5]:

$$
\tan x = \sum_{i=1}^{\infty} \frac{2^{2i}(2^{2i} - 1) B_i}{(2i)!} x^{2i-1}, \quad |x| < \frac{\pi}{2};
$$

(2)

$$
\cot x = \frac{1}{x} - \sum_{i=1}^{\infty} \frac{2^{2i} B_i}{(2i)!} x^{2i-1}, \quad |x| < \pi.
$$

(3)

Introduce two notations $S_n(x)$ and $r_n(x)$ by

$$
S_n(x) = \sum_{i=1}^{\infty} \frac{2^{2i}(2^{2i} - 1) B_i}{(2i)!} x^{2i-1},
$$

(4)

$$
r_n(x) = \tan x - S_n(x)
$$

(5)
for $0 < x < \frac{\pi}{2}$. Then $\tan x = \lim_{n \to \infty} S_n(x)$. We call $r_n(x)$ the remainder of power series for tangent function.

For elementary functions $\sin x$, $\cos x$, and $e^x$, there are much literature on estimates of their remainder. For examples, see [6, 7, 9]. The methods used in [6, 7, 9] have been applied to construct inequalities of elliptic integrals. See [8, 10]. Some inequalities involving $\tan x$ were researched by the second author and others in [2].

In this article, we will establish a double inequality for remainder $r_n(x)$ of power series for $\tan x$. That is

**Theorem 1.** For $x \in (0, \frac{\pi}{2})$ and $n \in \mathbb{N}$, we have

$$\frac{2^{2n+1}(2^{2n+1} - 1)B_{n+1}x^{2n}}{(2n+2)!} \tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n} x^{2n} \tan x. \tag{6}$$

**Remark 1.** If taking $n = 1$ in (6), we have for $x \in (0, 1)$

$$\frac{\pi}{2} \cdot \frac{x}{1 - \frac{x^2}{17} x^2} < \tan \frac{\pi x}{2} < \frac{\pi}{2} \cdot \frac{x}{1 - x^2}. \tag{7}$$

For $0 < x < \sqrt{3 - \frac{24}{\pi^2}}$, the left inequality in (7) is better than the left inequality in the following Becker-Stark inequality [4, p. 351]:

$$\frac{4}{\pi} \cdot \frac{x}{1 - x^2} < \tan \frac{\pi x}{2} < \frac{\pi}{2} \cdot \frac{x}{1 - x^2}, \quad x \in (0, 1). \tag{8}$$

If taking $n = 2$ in (6), we obtain

$$x + \frac{1}{3} x^3 + \frac{2}{15} x^4 \tan x < \tan x < x + \frac{1}{3} x^3 + \left(\frac{2}{\pi}\right)^4 x^4 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{9}$$

The constants $\frac{2}{15}$ and $\left(\frac{2}{\pi}\right)^4$ in (9) are best possible.

For $x \in (0, \frac{\pi}{6})$, the Djokvič inequality states [4, p. 350] that

$$x + \frac{1}{3} x^3 < \tan x < x + \frac{4}{9} x^3. \tag{10}$$

Since

$$\frac{1}{3} + \left(\frac{2}{\pi}\right)^4 x \tan x < \frac{1}{3} + \left(\frac{2}{\pi}\right)^4 \cdot \frac{\pi}{6} \cdot \frac{1}{\sqrt{3}} < \frac{4}{9},$$

thus, the inequality in (9) is better than those in (10).
2. Proof of Theorem

Let

\[ h_n(x) = \frac{\tan x - S_n(x)}{x^{2n} \tan x} \]  

(11)

for \( n \in \mathbb{N} \). Then we have the following lemma.

**Lemma 1.** For \( x \in (0, \frac{\pi}{2}) \) and \( n \in \mathbb{N} \), we have

\[ h_n(x) = \sum_{j=1}^{n} \frac{2^{2(n-j+1)}[2^{2(n-j+1)} - 1]B_{n-j+1}}{(2(n-j+1))!} \sum_{k=j}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2k-1} \cdot \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2(k-1)} \]  

(12)

**Proof.** We shall prove this lemma by mathematical induction on \( n \).

For \( n = 1 \), we have

\[ h_1(x) = \frac{\tan x - S_1(x)}{x^2 \tan x} = \frac{1}{x^2} - \frac{\cot x}{x} = \frac{1}{x^2} - \frac{1}{x} \left( \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2k-1} \right) \]

\[ = \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2(k-1)} \],

the formula (12) holds for \( n = 1 \).

For \( n = 2 \), we have

\[ h_2(x) = \frac{\tan x - S_2(x)}{x^4 \tan x} = \frac{1}{x^4} - \frac{\cot x}{x^3} - \frac{\cot x}{3x} \]

\[ = \frac{1}{x^4} - \frac{1}{x^3} \left( \frac{1}{x} - \frac{1}{3x} - \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2k-1} \right) \]

\[ = \sum_{k=2}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2(k-2)} + \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{3 \cdot (2k)!} x^{2(k-1)} \],

the formula (12) holds for \( n = 2 \).

Assume formula (12) holds for \( n = m \). Then for \( n = m + 1 \), we have

\[ h_{m+1} = \frac{\tan x - S_{m+1}(x)}{x^{2(m+1)} \tan x} = \frac{\tan x - S_m(x) - \frac{2^{2(m+1)}(2^{2(m+1)} - 1)B_{m+1}}{[2^{2(m+1)}]!} x^{2m+1}}{x^{2(m+1)} \tan x} \]
By induction, the proof of Lemma 1 is complete. □
Now we give a proof of Theorem 1.

Proof of Theorem 1. From (12), it is deduced that $h_n'(x) > 0$, and $h_n(x)$ is strictly increasing in $(0, \frac{\pi}{2})$. Easy computing yields

$$h_n(0 + 0) = \frac{2^{2n+2}(2^{2n+2} - 1)B_{n+1}}{(2n + 2)!},$$

$$h \left( \frac{\pi}{2} - 0 \right) = \left( \frac{2}{\pi} \right)^{2n}.$$

Therefore, we have

$$\frac{2^{2n+2}(2^{2n+2} - 1)B_{n+1}}{(2n + 2)!} < h_n(x) < \left( \frac{2}{\pi} \right)^{2n}.$$  \hspace{1cm} (17)

Inequalities in (17) are equivalent to the double inequality (6).

In [4, p. 421], the following inequalities are given

$$\frac{2}{\pi^{2n+4n}} < \frac{B_{2n}}{(2n)!} < \frac{2}{\pi^{2n(4n - 2)}}.$$ \hspace{1cm} (18)

Then we have

$$\frac{4^{n+1}(4^{n+1} - 1)B_{2n+2}}{(2n + 2)!} > \left( 2 - \frac{1}{2^{2n+1}} \right) \left( \frac{2}{\pi} \right)^{2n+2}.$$ \hspace{1cm} (19)

The first inequality in (6) follows from (19).

The proof of Theorem 1 is complete. \hfill \Box

References


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