A REVERSE OF BESSEL'S INEQUALITY IN 2-INNER PRODUCT SPACES AND SOME GRÜSS TYPE RELATED RESULTS WITH APPLICATIONS

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Abstract. A reverse of Bessel's inequality in 2-inner product spaces and companions of Grüss inequality with applications for determinantal integral inequalities are given.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2\mathrm{I}_1$) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,

($2\mathrm{I}_2$) $(x, x|z) = (z, z|x)$,

($2I_3$) $(y, x|z) = \overline{(x, y|z)}$,

($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,

($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-*inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product spaces can be immediately obtained as follows [2]:

(1) If $\mathbb{K} = \mathbb{R}$, then ($2\mathrm{I}_3$) reduces to

$$(y, x|z) = (x, y|z).$$

(2) From ($2\mathrm{I}_3$) and ($2\mathrm{I}_4$), we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha}(x, y|z). \tag{1.1}$$

(3) Using ($2I_2$)–($2I_5$), we have

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$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{1.2}$$

In the real case $\mathbb{K} = \mathbb{R}$, (1.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{1.3}$$

and, using this formula, it is easy to see that, for any $\alpha \in \mathbb{R}$,

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{1.4}$$

In the complex case, using (1.1) and (1.2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (1.2), yields

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] + \frac{i}{4}[(z, z|x + iy) - (z, z|x - iy)]. \tag{1.5}$$

Using the above formula and (1.1), we have, for any $\alpha \in \mathbb{C}$,

$$(x, y|\alpha z) = |\alpha|^2(x, y|z). \tag{1.6}$$

However, for $\alpha \in \mathbb{R}$, (1.6) reduces to (1.4).

Also, from (1.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By $(2I_1)$, we know that $(u, u|z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as,

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{1.7}$$

For $x = z$, (1.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) holds too. Thus (1.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (1.7), it follows that

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{1.9}$$

Using (1.8), it is easy to check that (1.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (1.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot | \cdot))$, we can define a function $\| \cdot | \cdot \|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{1.10}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:

(2N$_1$) $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

(2N$_2$) $\|z|x\| = \|x|z\|$,

(2N$_3$) $\|\alpha x|z\| = |\alpha| \|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

(2N$_4$) $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\| \cdot | \cdot \|$ defined on $X \times X$ and satisfying the conditions $(2N_1)$–$(2N_4)$ is called a 2-*norm* on X and $(X, \| \cdot | \cdot \|)$ is called a *linear* 2-*normed space* [4]. Whenever a 2-inner product space $(X, (\cdot, \cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \| \cdot | \cdot \|)$ with the 2-norm defined by (1.10).

2. A Reverse of Bessel's Inequality

Let $(X; (\cdot, \cdot|\cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1 \leq i \leq n}$ are linearly independent vectors in the 2-inner product space X, and, for a given $z \in X, (f_i, f_j|z) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(f_i)_{1 \leq i \leq n}$ is $z-$*orthonormal*), then the following inequality is the corresponding *Bessel's inequality* (see for example [2]) for the $z-$*orthonormal family* $(f_i)_{1 \leq i \leq n}$ in the 2-inner product space $(X; (\cdot, \cdot|\cdot))$:

$$\sum_{i=1}^{n} |(x, f_i|z)|^2 \leq \|x|z\|^2 \tag{2.1}$$

for any $x \in X$. For more details on this inequality, see the recent paper [2] and the references therein.

The following lemma holds.

Lemma 1. *Let $\{e_i\}_{i \in I}$ be a family of z-orthornormal vectors in X, F a finite part of I and ϕ_i, Φ_i $(i \in F)$, real or complex numbers. The following statements are equivalent for $x \in X$:*

(i) $\operatorname{Re} \left(\sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i | z \right) \geq 0$,

(ii) $\left\| x - \sum_{i \in F} \frac{\phi_i + \Phi_i}{2} e_i | z \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}$.

Proof. It is easy to see that for $y, a, A \in X$, the following are equivalent

(a) $\operatorname{Re} (A - y, y - a|z) \geq 0$ and

(aa) $\left\| y - \frac{a+A}{2} |z \right\| \leq \frac{1}{2} \|A - a|z\|$.

Now, for $a = \sum_{i\in F} \phi_i e_i$, $A = \sum_{i\in F} \Phi_i e_i$, we have

$$\begin{aligned}\|A - a|z\| &= \left\|\sum_{i\in F} (\Phi_i - \phi_i)\, e_i|z\right\| = \left(\left\|\sum_{i\in F} (\Phi_i - \phi_i)\, e_i|z\right\|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \|e_i|z\|^2\right)^{\frac{1}{2}} = \left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}},\end{aligned}$$

which gives, for $y = x$, the desired equivalence. ∎

The following reverse of Bessel's inequality holds.

Theorem 1. *Let $\{e_i\}_{i\in I}$, F, ϕ_i, Φ_i, $i \in F$ and $x, z \in X$ so that either (i) or (ii) of Lemma 1 holds. Then we have the inequality:*

$$\begin{aligned}0 &\le \|x|z\|^2 - \sum_{i\in F} |(x, e_i|z)|^2 \\ &\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2 - \mathrm{Re}\left(\sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i|z\right) \\ &\left(\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2\right).\end{aligned} \tag{2.2}$$

The constant $\frac{1}{4}$ is best in both inequalities.

Proof. Define

$$I_1 := \sum_{i\in X} \mathrm{Re}\left[(\Phi_i - (x, e_i|z))\left(\overline{(x, e_i|z)} - \overline{\phi_i}\right)\right]$$

and

$$I_2 := \mathrm{Re}\left[\left(\sum_{i\in H} \Phi_i e_i - x, x - \sum_{i\in H} \phi_i e_i|z\right)\right].$$

Observe that

$$\begin{aligned}I_1 &= \sum_{i\in H} \mathrm{Re}\left[\Phi_i \overline{(x, e_i|z)}\right] + \sum_{i\in H} \mathrm{Re}\left[\overline{\phi_i}\,(x, e_i|z)\right] \\ &\quad - \sum_{i\in H} \mathrm{Re}\left[\Phi_i \overline{\phi_i}\right] - \sum_{i\in H} |(x, e_i|z)|^2\end{aligned}$$

and

$$\begin{aligned}I_2 &= \mathrm{Re}\left[\sum_{i\in H} \Phi_i \overline{(x, e_i|z)} + \sum_{i\in H} \overline{\phi_i}\,(x, e_i|z) - \|x|z\|^2 - \sum_{i\in H}\sum_{j\in H} \Phi_i \overline{\phi_i}\,(e_i, e_j|z)\right] \\ &= \sum_{i\in H} \mathrm{Re}\left[\Phi_i \overline{(x, e_i|z)}\right] + \sum_{i\in H} \mathrm{Re}\left[\overline{\phi_i}\,(x, e_i|z)\right] - \|x|z\|^2 - \sum_{i\in H} \mathrm{Re}\left[\Phi_i \overline{\phi_i}\right].\end{aligned}$$

Consequently, subtracting I_2 from I_1, we deduce the following equality that is useful in its turn

$$\|x|z\|^2 - \sum_{i\in F} |(x,e_i|z)|^2 = \sum_{i\in H} \operatorname{Re}\left[(\Phi_i - (x,e_i|z))\left(\overline{(x,e_i|z)} - \overline{\phi_i}\right)\right] - \operatorname{Re}\left[\left(\sum_{i\in H}\Phi_i e_i - x, x - \sum_{i\in H}\phi_i e_i|z\right)\right]. \tag{2.3}$$

Using the following elementary inequality for complex numbers

$$\operatorname{Re}\left[a\bar{b}\right] \le \frac{1}{4}|a+b|^2, \qquad a,b\in\mathbb{K},$$

for the choices $a = \Phi_i - (x,e_i|z)$, $b = (x,e_i|z) - \phi_i$ $(i\in F)$, we deduce

$$\sum_{i\in H} \operatorname{Re}\left[(\Phi_i - (x,e_i|z))\left(\overline{(x,e_i|z)} - \overline{\phi_i}\right)\right] \le \frac{1}{4}\sum_{i\in H}|\Phi_i - \phi_i|^2. \tag{2.4}$$

Making use of (2.3), (2.4) and the assumption (i), we deduce (2.2).

To prove the sharpness of the constant $\frac{1}{4}$, assume that there is a $c>0$ such that

$$\begin{aligned} 0 &\le \|x|z\|^2 - \sum_{i\in F}|(x,e_i|z)|^2 \\ &\le c\sum_{i\in F}|\Phi_i - \phi_i|^2 - \operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i|z\right), \end{aligned} \tag{2.5}$$

provided ϕ_i, Φ_i, x and F satisfy (i) or (ii).

Now, let $F = \{1\}, e_1 = e, \|e|z\| = 1$ and $m \in X$ so that $\|m|z\| = 1$ end $(m,e|z) = 0$. For $\Phi_1 = \Phi, \phi_1 = \phi, \Phi \ne \phi$, define the vector

$$x := \frac{\Phi+\phi}{2}e + \frac{\Phi-\phi}{2}m.$$

A simple calculation shows that

$$(\Phi e - x, x - \phi e|z) = \left|\frac{\Phi-\phi}{2}\right|^2 (e - x, x - e|z) = 0$$

and thus the condition (i) of Lemma 1 holds true for $F = \{1\}$.

Observe also that

$$\begin{aligned} \|x|z\|^2 &= \left\|\frac{\Phi+\phi}{2}e + \frac{\Phi-\phi}{2}m|z\right\|^2 \\ &= \left|\frac{\Phi+\phi}{2}\right|^2 + \left|\frac{\Phi-\phi}{2}\right|^2 \end{aligned}$$

and

$$(x,e|z) = \left(\frac{\Phi+\phi}{2}e + \frac{\Phi-\phi}{2}m, e|z\right) = \frac{\Phi+\phi}{2}.$$

Consequently, by (2.5), we deduce

$$\left|\frac{\Phi-\phi}{2}\right|^2 \le c|\Phi-\phi|^2,$$

which gives $c \ge \frac{1}{4}$, and the proof is completed. ∎

3. A Refinement of the Grüss Inequality

The following result holds.

Theorem 2. *Let $\{e_i\}_{i\in I}$ be a family of z-orthornormal vectors in X, F a finite part of I and ϕ_i, Φ_i, $\gamma_i, \Gamma_i \in \mathbb{K}$, $i \in F$ and $x, y \in X$. If either*

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i|z\right) \geq 0, \tag{3.1}$$
$$\operatorname{Re}\left(\sum_{i\in F}\Gamma_i e_i - y, y - \sum_{i\in F}\gamma_i e_i|z\right) \geq 0,$$

or equivalently,

$$\left\|x - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2}e_i|z\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}, \tag{3.2}$$
$$\left\|y - \sum_{i\in F}\frac{\Gamma_i + \gamma_i}{2}e_i|z\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}},$$

hold, then we have the inequalities

$$\begin{aligned}
&\left|(x,y|z) - \sum_{i\in F}(x,e_i|z)(e_i,y|z)\right| \\
\leq\ & \frac{1}{4}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F}|\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}} \\
& - \left[\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i|z\right)\right]^{\frac{1}{2}}\left[\operatorname{Re}\left(\sum_{i\in F}\Gamma_i e_i - y, y - \sum_{i\in F}\gamma_i e_i|z\right)\right]^{\frac{1}{2}} \\
& \left(\leq \frac{1}{4}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F}|\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{3.3}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Using Schwarz's inequality in the 2-inner product space $(X, (\cdot,\cdot|\cdot))$ one has

$$\begin{aligned}
&\left|\left(x - \sum_{i\in F}(x,e_i|z)e_i, y - \sum_{i\in F}(y,e_i|z)e_i|z\right)\right|^2 \\
&\leq \left\|x - \sum_{i\in F}(x,e_i|z)e_i|z\right\|^2\left\|y - \sum_{i\in F}(y,e_i|z)e_i|z\right\|^2
\end{aligned} \tag{3.4}$$

and since a simple calculation shows that

$$\left(x - \sum_{i\in F}(x,e_i|z)e_i, y - \sum_{i\in F}(y,e_i|z)e_i|z\right) = (x,y|z) - \sum_{i\in F}(x,e_i|z)(e_i,y|z)$$

and

$$\left\| x-\sum_{i\in F}(x,e_i|z)\,e_i|z\right\|^2=\|x|z\|^2-\sum_{i\in F}|(x,e_i|z)|^2$$

for any $x,y\in X$, then, by (3.4) and by the counterpart of Bessel's inequality in Theorem 1, we have

$$\begin{aligned}
&\left|(x,y|z)-\sum_{i\in F}(x,e_i|z)\,(e_i,y|z)\right|^2 \qquad (3.5)\\
&\leq \left(\|x|z\|^2-\sum_{i\in F}|(x,e_i|z)|^2\right)\left(\|y|z\|^2-\sum_{i\in F}|(y,e_i|z)|^2\right)\\
&\leq \left[\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2-\mathrm{Re}\left(\sum_{i\in F}\Phi_i e_i-x,x-\sum_{i\in F}\phi_i e_i|z\right)\right]\\
&\quad\times\left[\frac{1}{4}\sum_{i\in F}|\Gamma_i-\gamma_i|^2-\mathrm{Re}\left(\sum_{i\in F}\Gamma_i e_i-y,y-\sum_{i\in F}\gamma_i e_i|z\right)\right]\\
&\leq \left[\frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}\right.\\
&\quad-\left[\mathrm{Re}\left(\sum_{i\in F}\Phi_i e_i-x,x-\sum_{i\in F}\phi_i e_i|z\right)\right]^{\frac{1}{2}}\\
&\quad\left.\times\left[\mathrm{Re}\left(\sum_{i\in F}\Gamma_i e_i-y,y-\sum_{i\in F}\gamma_i e_i|z\right)\right]^{\frac{1}{2}}\right]
\end{aligned}$$

where, for the last inequality, we have made use of the elementary inequality

$$\left(m^2-n^2\right)\left(p^2-q^2\right)\leq(mp-nq)^2$$

valid for any $m,n,p,q>0$.

Taking the square root in (3.5) and observing that the quantity in the last square bracket is nonnegative (see for example (2.2)), we deduce the desired result (3.3).

The best constant follows by Theorem 1 and we omit the details. ∎

4. Some Companion Inequalities

The following companion of the Grüss inequality also holds.

Theorem 3. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in X, F a finite part of I, $\phi_i,\Phi_i\in\mathbb{K}$, $i\in F$ and $x,y\in X$ such that*

$$\mathrm{Re}\left(\sum_{i\in F}\Phi_i e_i-\frac{x+y}{2},\frac{x+y}{2}-\sum_{i\in F}\phi_i e_i|z\right)\geq 0, \qquad (4.1)$$

or equivalently,

$$\left\|\frac{x+y}{2}-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i|z\right\|\leq\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}. \qquad (4.2)$$

Then we have the inequality

$$\operatorname{Re}\left[(x,y|z)-\sum_{i\in F}(x,e_i|z)(e_i,y|z)\right]\le\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2. \tag{4.3}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. Start with the well known inequality

$$\operatorname{Re}\langle z,u|v\rangle\le\frac{1}{4}\|z+u|v\|^2,\quad z,u,v\in X. \tag{4.4}$$

Since

$$(x,y|z)-\sum_{i\in F}(x,e_i|z)(e_i,y|z)=\left(x-\sum_{i\in F}(x,e_i|z)e_i,y-\sum_{i\in F}(y,e_i|z)e_i|z\right),$$

for any $x,y\in X$, then, by (4.4), we get

$$\begin{aligned}
&\operatorname{Re}\left[(x,y|z)-\sum_{i\in F}(x,e_i|z)(e_i,y|z)\right] \\
&=\operatorname{Re}\left[\left(x-\sum_{i\in F}(x,e_i|z)e_i,y-\sum_{i\in F}(y,e_i|z)e_i|z\right)\right] \\
&\le\frac{1}{4}\left\|x-\sum_{i\in F}(x,e_i|z)e_i+y-\sum_{i\in F}(y,e_i|z)e_i|z\right\|^2 \\
&=\left\|\frac{x+y}{2}-\sum_{i\in F}\left(\frac{x+y}{2},e_i|z\right)e_i|z\right\|^2 \\
&=\left\|\frac{x+y}{2}|z\right\|^2-\sum_{i\in F}\left|\left(\frac{x+y}{2},e_i|z\right)\right|^2.
\end{aligned} \tag{4.5}$$

If we apply the reverse of Bessel's inequality from Theorem 1 for $\frac{x+y}{2}$, we may state that

$$\left\|\frac{x+y}{2}|z\right\|^2-\sum_{i\in F}\left|\left(\frac{x+y}{2},e_i|z\right)\right|^2\le\frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}. \tag{4.6}$$

Now, by making use of (4.5) and (4.6), we deduce (4.3).

The fact that $\frac{1}{4}$ is the best constant in (4.3) follows by the fact that if in (4.1) we choose $x=y$, then it becomes (i) of Lemma 1, implying (2.2), for which, we have shown that $\frac{1}{4}$ is the best constant. ∎

The following corollary may be of interest if we wish to evaluate the absolute value of

$$\operatorname{Re}\left[(x,y|z)-\sum_{i\in F}(x,e_i|z)(e_i,y|z)\right].$$

Corollary 1. *With the assumptions of Theorem 3 and if*

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_ie_i-\frac{x\pm y}{2},\frac{x\pm y}{2}-\sum_{i\in F}\phi_ie_i|z\right)\ge 0, \tag{4.7}$$

or equivalently,

$$\left\| \frac{x \pm y}{2} - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i | z \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \tag{4.8}$$

then we have the inequality

$$\left| \operatorname{Re} \left[(x, y|z) - \sum_{i \in F} (x, e_i|z)(e_i, y|z) \right] \right| \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 . \tag{4.9}$$

Proof. We only remark that, if

$$\operatorname{Re} \left(\sum_{i \in F} \Phi_i e_i - \frac{x - y}{2}, \frac{x - y}{2} - \sum_{i \in F} \phi_i e_i | z \right) \geq 0$$

holds, then by Theorem 3 for $(-y)$ instead of y, we have

$$\operatorname{Re} \left[-(x, y|z) + \sum_{i \in F} (x, e_i|z)(e_i, y|z) \right] \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 ,$$

which shows that

$$\operatorname{Re} \left[(x, y|z) - \sum_{i \in F} (x, e_i|z)(e_i, y|z) \right] \geq -\frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 . \tag{4.10}$$

Making use of (4.3) and (4.10), we deduce the desired inequality (4.9). ∎

Remark 1. *If X is a real inner product space and $m_i, M_i \in \mathbb{R}$ with the property that*

$$\left(\sum_{i \in F} M_i e_i - \frac{x \pm y}{2}, \frac{x \pm y}{2} - \sum_{i \in F} m_i e_i | z \right) \geq 0 \tag{4.11}$$

or equivalently,

$$\left\| \frac{x \pm y}{2} - \sum_{i \in F} \frac{M_i + m_i}{2} \cdot e_i | z \right\| \leq \frac{1}{2} \left(\sum_{i \in F} (M_i - m_i)^2 \right)^{\frac{1}{2}}, \tag{4.12}$$

then we have the Grüss type inequality

$$\left| (x, y|z) - \sum_{i \in F} (x, e_i|z)(e_i, y|z) \right| \leq \frac{1}{4} \sum_{i \in F} (M_i - m_i)^2 . \tag{4.13}$$

5. Applications for Determinantal Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of subsets of Ω and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by formula

$$(f, g|h)_\rho := \frac{1}{2} \int_\Omega \int_\Omega \rho(s) \rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) d\mu(t), \tag{5.1}$$

where,

$$\left| \begin{array}{cc} f(s) & f(t) \\ h(s) & h(t) \end{array} \right|$$

denotes the determinant of the matrix

$$\left[\begin{array}{cc} f(s) & f(t) \\ h(s) & h(t) \end{array} \right],$$

generating the 2-norm on $L_\rho^2(\Omega)$ expressed by

$$\|f|h\|_\rho := \left(\frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) \left| \begin{array}{cc} f(s) & f(t) \\ h(s) & h(t) \end{array} \right|^2 d\mu(s)\, d\mu(t) \right)^{1/2}. \tag{5.2}$$

A simple calculation with integrals reveals that

$$(f,g|h)_\rho = \left| \begin{array}{cc} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right| \tag{5.3}$$

and

$$\|f|h\|_\rho = \left| \begin{array}{cc} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right|^{1/2}, \tag{5.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho f g d\mu$.

We recall that the pair of functions $(q,p) \in L_\rho^2(\Omega) \times L_\rho^2(\Omega)$ is called *synchronous* if

$$(q(x) - q(y))(p(x) - p(y)) \geq 0$$

for *a.e.* $x, y \in \Omega$.

We note that, if $\Omega = [a,b]$, then a sufficient condition for synchronicity is that the functions are both monotonic increasing or decreasing. This condition is not necessary.

Now, suppose that $h \in L_\rho^2(\Omega)$ is such that $h(x) \neq 0$ for $\mu - a.e.$ $x \in \Omega$. Then, by the definition of 2-inner product $(f,g|h)_\rho$, we have

$$\begin{aligned} (5.5) \quad & (f,g|h)_\rho \\ = \ & \frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) h^2(s) h^2(t) \left(\frac{f(s)}{h(s)} - \frac{f(t)}{h(t)} \right) \left(\frac{g(s)}{h(s)} - \frac{g(t)}{h(t)} \right) d\mu(s)\, d\mu(t) \end{aligned}$$

and thus a *sufficient condition* for the inequality

$$(f,g|h)_\rho \geq 0 \tag{5.6}$$

to hold, is that, the functions $\left(\frac{f}{h}, \frac{g}{h}\right)$ are synchronous. It is obvious that, this condition is not necessary.

Using the representations (5.3), (5.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, we have some interesting determinantal integral inequalities.

Proposition 1. *Let $h \in L_{\rho}^{2}(\Omega)$ be such that $h(x) \neq 0$ for $\mu - a.e.$ $x \in \Omega$, and $(f_i)_{i\in I}$ a family of functions in $L_{\rho}^{2}(\Omega)$ with the property that*

$$\begin{vmatrix} \int_{\Omega} \rho f_i f_j d\mu & \int_{\Omega} \rho f_i h d\mu \\ \int_{\Omega} \rho f_j h d\mu & \int_{\Omega} \rho h^2 d\mu \end{vmatrix} = \delta_{i,j}$$

for any $i,j \in I$, where $\delta_{i,j}$ is the Kronecker delta.

If we assume that there exists the real numbers $M_i, m_i, i \in F$, where F is a given finite part of I, such that the functions

$$\sum_{i\in F} M_i \cdot \frac{f_i}{h} - \frac{f}{h}, \frac{f}{h} - \sum_{i\in F} m_i \cdot \frac{f_i}{h}$$

are synchronous on Ω, then we have the inequalities

$$\begin{aligned} 0 &\leq \begin{vmatrix} \int_{\Omega} \rho f^2 d\mu & \int_{\Omega} \rho f h d\mu \\ \int_{\Omega} \rho f h d\mu & \int_{\Omega} \rho h^2 d\mu \end{vmatrix} - \sum_{i\in F} \begin{vmatrix} \int_{\Omega} \rho f_i f d\mu & \int_{\Omega} \rho f_i h d\mu \\ \int_{\Omega} \rho f h d\mu & \int_{\Omega} \rho h^2 d\mu \end{vmatrix}^2 \\ &\leq \frac{1}{4}\sum_{i\in F}(M_i - m_i)^2 \\ &\quad - \begin{vmatrix} \int_{\Omega} \rho \left(\sum_{i\in F} M_i f_i - f\right)\left(f - \sum_{i\in F} m_i f_i\right) d\mu & \int_{\Omega} \rho \left(\sum_{i\in F} M_i f_i - f\right) h d\mu \\ \int_{\Omega} \rho \left(f - \sum_{i\in F} m_i f_i\right) h d\mu & \int_{\Omega} \rho h^2 d\mu \end{vmatrix} \\ &\leq \frac{1}{4}\sum_{i\in F}(M_i - m_i)^2 . \end{aligned}$$

The proof follows by Theorem 1 applied for the 2-inner product $(\cdot,\cdot|\cdot)_{\rho}$ and we omit the details.

Similar determinantal integral inequalities may be stated if one uses the other results for 2-inner products obtained above, but we do not present them here.

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