SOME PEČARIĆ'S TYPE INEQUALITIES IN 2-INNER PRODUCT SPACES AND APPLICATIONS

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Abstract. Some results related to the Pečarić's type generalisation of Bessel's inequality in 2-inner product spaces are given. Applications for determinantal integral inequalities are also provided.

1. Introduction

In 1992, J.E. Pečarić [5] proved the following inequality for vectors in complex inner product spaces $(H;(\cdot,\cdot))$.

Theorem 1. *Suppose that $x, y_1, \dots, y_n$ are vectors in H and $c_1, \dots, c_n$ are complex numbers. Then the following inequalities*

$$\left|\sum_{i=1}^{n} c_i\,(x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \max_{1\le i\le n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \tag{1.1}$$

hold.

He also showed that for $c_i = \overline{(x, y_i)}, i \in \{1, ..., n\}$, one gets

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \leq \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \max_{1\le i\le n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right), \tag{1.2}$$

which improves *Bombieri's inequality* [1]

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \max_{1\le i\le n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right). \tag{1.3}$$

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Note that (1.3) is in its turn a natural generalization of *Bessel's inequality*

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2 \tag{1.4}$$

for any $x \in H$, which holds for the orthornormal vectors $(e_i)_{1 \le i \le n}$.

In this paper we point out some results of Pečarić's type for 2-inner products spaces. Some inequalities of Bombieri type holding in these spaces are also mentioned. Natural applications for determinantal integral inequalities are given as well.

2. Some Preliminary Results in 2-Inner Product Spaces

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [2]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2\mathrm{I}_1$) $(x, x|z) \ge 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,

($2\mathrm{I}_2$) $(x, x|z) = (z, z|x)$,

($2I_3$) $(y, x|z) = \overline{(x, y|z)}$,

($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,

($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-*inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product $(\cdot, \cdot|\cdot)$ can be immediately obtained as follows [3]:

(1) If $\mathbb{K} = \mathbb{R}$, then ($2\mathrm{I}_3$) reduces to

$$(y, x|z) = (x, y|z).$$

(2) From ($2\mathrm{I}_3$) and ($2\mathrm{I}_4$), we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha}(x, y|z). \tag{2.1}$$

(3) Using ($2I_2$)–($2I_5$), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{2.2}$$

In the real case $\mathbb{K} = \mathbb{R}$, (2.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{2.3}$$

and, using this formula, it is easy to see , for any $\alpha \in \mathbb{R}$, that

$$(x, y|\alpha z) = \alpha^2 (x, y|z). \tag{2.4}$$

In the complex case, using (2.1) and (2.2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (2.2), yields

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] + \frac{i}{4}[(z, z|x + iy) - (z, z|x - iy)]. \tag{2.5}$$

Using the above formula and (2.1), we have, for any $\alpha \in \mathbb{C}$, that

$$(x, y|\alpha z) = |\alpha|^2 (x, y|z). \tag{2.6}$$

However, for $\alpha \in \mathbb{R}$, (2.6) reduces to (2.4). Also, from (2.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By $(2I_1)$, we know that $(u, u|z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{2.7}$$

For $x = z$, (2.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{2.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (2.8) holds too. Thus (2.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (2.7), it follows

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{2.9}$$

Using (2.8), it is easy to check that (2.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (2.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x-(x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (2.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot|\cdot))$, we can define a function $\|\cdot|\cdot\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{2.10}$$

for all $x, z \in X$.

It is easy to see that, this function satisfies the following conditions:

$(2\mathrm{N}_1)$ $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

(2N$_2$) $\|z|x\| = \|x|z\|$,
(2N$_3$) $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,
(2N$_4$) $\|x + x'|z\| \le \|x|z\| + \|x'|z\|$.

Any function $\|\cdot|\cdot\|$ defined on $X \times X$ and satisfying the conditions $(2N_1)$–$(2N_4)$ is called a 2*-norm* on X and $(X, \|\cdot|\cdot\|)$ is called a *linear* 2*-normed space*. For recent result devoted to the geometry of linear 2-normed spaces, see [4].

Whenever a 2-inner product space $(X, (\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot|\cdot\|)$ with the 2-norm defined by (2.10).

Let $(X; (\cdot,\cdot|\cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1\le i\le n}$ are linearly independent vectors in the 2-inner product space X, and, for a given $z \in X, (f_i, f_j|z) = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(f_i)_{1\le i\le n}$ is $z-$*orthonormal*), then the following inequality is the corresponding Bessel's inequality (see for example [3]) for $z-$orthonormal family $(f_i)_{1\le i\le n}$ in the 2-inner product space $(X; (\cdot,\cdot|\cdot))$:

$$\sum_{i=1}^{n} |(x, f_i|z)|^2 \le \|x|z\|^2 \tag{2.11}$$

for any $x \in X$. For more details on this inequality, see the recent paper [3] and the references therein.

3. Some Inequalities for 2-Norms

We start with the following lemma that is interesting in its own right.

Lemma 1. *Let $(X, (\cdot,\cdot|\cdot))$ be a 2-inner product space on $\mathbb{K}$ and $z_1, \dots, z_n, z \in X$, $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\begin{aligned} &\left\|\sum_{i=1}^{n} \alpha_i z_i | z\right\|^2 \\ \le\ & \left(\sum_{i=1}^{n} |\alpha_i|^p \left(\sum_{j=1}^{n} |(z_i, z_j|z)|\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |\alpha_i|^q \left(\sum_{j=1}^{n} |(z_i, z_j|z)|\right)\right)^{\frac{1}{q}} \\ \le\ & \begin{cases} A; \\ B; \\ C; \end{cases} \end{aligned} \tag{3.1}$$

where

$$A:=\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|^2\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|;\\[2ex] \max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}},\\ \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1;\\[2ex] \max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{q}\right)^{\frac{1}{q}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}};\end{cases}$$

$$B:=\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}}\\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\beta}\right)^{\frac{1}{\beta q}}, \qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}}\\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1\\ \text{and } \ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1;\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^{q}\right)^{\frac{1}{q}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}}\\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \ \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\end{cases}$$

$$C:=\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{p}\right)^{\frac{1}{p}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}};\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^{p}\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}\\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \qquad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1;\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^{p}\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{q}\right)^{\frac{1}{q}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right),\end{cases}$$

and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We observe that

$$\left\| \sum_{i=1}^{n} \alpha_i z_i | z \right\|^2 = \left(\sum_{i=1}^{n} \alpha_i z_i, \sum_{j=1}^{n} \alpha_j z_j | z \right) \tag{3.2}$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \left(z_i, z_j | z \right) = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \left(z_i, z_j | z \right) \right|$$
$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| \, |\alpha_j| \, |(z_i, z_j | z)| =: M.$$

If one uses the Hölder inequality for double sums, i.e., we recall it

$$\sum_{i,j=1}^{n} m_{ij} a_{ij} b_{ij} \leq \left(\sum_{i,j=1}^{n} m_{ij} a_{ij}^{p} \right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} m_{ij} b_{ij}^{q} \right)^{\frac{1}{q}}, \tag{3.3}$$

where $m_{ij}, a_{ij}, b_{ij} \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$; then

$$M \leq \left(\sum_{i,j=1}^{n} |(z_i, z_j | z)| \, |\alpha_i|^p \right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} |(z_i, z_j | z)| \, |\alpha_i|^q \right)^{\frac{1}{q}} \tag{3.4}$$
$$= \left(\sum_{i=1}^{n} |\alpha_i|^p \left(\sum_{j=1}^{n} |(z_i, z_j | z)| \right) \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |\alpha_i|^q \left(\sum_{j=1}^{n} |(z_i, z_j | z)| \right) \right)^{\frac{1}{q}}$$

and the first inequality in (3.1) is proved.

Observe, by Hölder inequality, that

$$\sum_{i=1}^{n} |\alpha_i|^p \left(\sum_{j=1}^{n} |(z_i, z_j | z)| \right) \leq \begin{cases} \max\limits_{1 \leq i \leq n} |\alpha_i|^p \sum\limits_{i,j=1}^{n} |(z_i, z_j | z)| \, ; \\[2ex] \left(\sum\limits_{i=1}^{n} |\alpha_i|^{\alpha p} \right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j | z)| \right)^{\beta} \right)^{\frac{1}{\beta}} \\ \text{if } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex] \sum\limits_{i=1}^{n} |\alpha_i|^p \max\limits_{1 \leq i \leq n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j | z)| \right); \end{cases}$$

which gives

$$\left(\sum_{i=1}^{n}|\alpha_i|^p\left(\sum_{j=1}^{n}|(z_i,z_j|z)|\right)\right)^{\frac{1}{p}} \tag{3.5}$$
$$\leq\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}};\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\beta}\right)^{\frac{1}{\beta p}} & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^p\right)^{\frac{1}{p}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}.\end{cases}$$

Similarly, we have

$$\left(\sum_{i=1}^{n}|\alpha_i|^q\left(\sum_{j=1}^{n}|(z_i,z_j|z)|\right)\right)^{\frac{1}{q}} \tag{3.6}$$
$$\leq\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}};\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}} & \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1;\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^q\right)^{\frac{1}{q}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}}.\end{cases}$$

Using (3.1) and (3.5)–(3.6), we deduce the 9 inequalities in the second part of (3.2). ∎

If we choose $p=q=2$, then the following result holds.

Corollary 1. *If $z_1,\dots,z_n,z\in X$ and $\alpha_1,\dots,\alpha_n\in\mathbb{K}$, then one has*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i|z\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{j=1}^{n}|(z_i,z_j|z)|\right)$$
$$\leq\begin{cases}D;\\ E;\\ F;\end{cases}$$

where

$$D := \begin{cases} \max\limits_{1\le i\le n} |\alpha_i|^2 \sum\limits_{i,j=1}^{n} |(z_i, z_j|z)|; \\[2ex] \max\limits_{1\le i\le n} |\alpha_i| \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}} \left(\sum\limits_{i,j=1}^{n} |(z_i, z_j|z)|\right)^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \\ \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex] \max\limits_{1\le i\le n} |\alpha_i| \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2}\right)^{\frac{1}{2}} \left(\sum\limits_{i,j=1}^{n} |(z_i, z_j|z)|\right)^{\frac{1}{2}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\frac{1}{2}}; \end{cases}$$

$$E := \begin{cases} \max\limits_{1\le i\le n} |\alpha_i| \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}} \left(\sum\limits_{i,j=1}^{n} |(z_i, z_j|z)|\right)^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \\ \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex] \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}} \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\beta}\right)^{\frac{1}{2\beta}} \\ \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \text{and } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex] \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2}\right)^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\frac{1}{2}}, \\ \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \ \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and

$$F := \begin{cases} \max\limits_{1\le i\le n} |\alpha_i| \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2}\right)^{\frac{1}{2}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\frac{1}{2}} \left(\sum\limits_{i,j=1}^{n} |(z_i, z_j|z)|\right)^{\frac{1}{2}}; \\[2ex] \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2}\right)^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\frac{1}{2}} \\ \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \qquad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex] \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right). \end{cases}$$

4. Some Pečarić Type Inequalities for 2-Inner Products

We are now able to point out the following result which complements and generalizes the Bessel inequality (2.11) in 2-inner product spaces.

Theorem 2. *Let $x, y_1, \dots, y_n, z$ be vectors of an inner product space $(X; (\cdot,\cdot))$ and $c_1, \dots, c_n \in \mathbb{K}$. Then we have*

$$\begin{aligned}
(4.1)\qquad & \left|\sum_{i=1}^{n} c_i\,(x, y_i|z)\right|^2 \\
\leq\ & \|x|z\|^2 \left(\sum_{i=1}^{n} |c_i|^p \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |c_i|^q \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)\right)^{\frac{1}{q}} \\
\leq\ & \|x|z\|^2 \times \begin{cases} G; \\ H; \\ L; \end{cases}
\end{aligned}$$

where

$$G := \begin{cases}
\max\limits_{1\leq i\leq n} |c_i|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\,; \\[2ex]
\max\limits_{1\leq i\leq n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \\
\text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex]
\max\limits_{1\leq i\leq n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^q\right)^{\frac{1}{q}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{q}};
\end{cases}$$

$$H := \begin{cases} \max\limits_{1\le i\le n}|c_i| \left(\sum\limits_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i,j=1}^n |(y_i,y_j|z)|\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \\ \textit{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left(\sum\limits_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i=1}^n |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}} \\ \times \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \ \textit{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \textit{and } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^n |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\frac{1}{q}} \\ \times \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \qquad \textit{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \end{cases}$$

and

$$L := \begin{cases} \max\limits_{1\le i\le n}|c_i| \left(\sum\limits_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\frac{1}{p}} \left(\sum\limits_{i,j=1}^n |(y_i,y_j|z)|\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\frac{1}{p}} \\ \times \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \ \textit{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n |c_i|^{q}\right)^{\frac{1}{q}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right); \end{cases}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1.$

Proof. We note that

$$\sum_{i=1}^n c_i\,(x,y_i|z) = \left(x, \sum_{i=1}^n \overline{c_i} y_i|z\right).$$

Using Schwarz's inequality in 2-inner product spaces, we have

$$\left|\sum_{i=1}^n c_i\,(x,y_i|z)\right|^2 \le \|x|z\|^2 \left\|\sum_{i=1}^n \overline{c_i} y_i|z\right\|^2. \tag{4.2}$$

Finally, using Lemma 1 with $\alpha_i = \overline{c_i},\ z_i = y_i\ (i=1,\dots,n)$, we deduce the desired inequality (4.1). ∎

Remark 1. *If in (4.1) we choose $p = q = 2$, we obtain amongst others, the following result*

$$\left|\sum_{i=1}^{n} c_i\left(x, y_i|z\right)\right|^2 \tag{4.3}$$

$$\leq \|x|z\|^2\left(\sum_{i=1}^{n}|c_i|^2\left(\sum_{j=1}^{n}|(y_i, y_j|z)|\right)\right)$$

$$\leq \|x|z\|^2 \times \begin{cases} M; \\ N; \\ P; \end{cases}$$

where

$$M := \begin{cases} \max\limits_{1\leq i\leq n}|c_i|^2 \sum\limits_{i,j=1}^{n}|(y_i, y_j|z)|; \\ \max\limits_{1\leq i\leq n}|c_i|\left(\sum\limits_{i=1}^{n}|c_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\left(\sum\limits_{i,j=1}^{n}|(y_i, y_j|z)|\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(y_i, y_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \\ \text{if } \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta} = 1; \\ \max\limits_{1\leq i\leq n}|c_i|\left(\sum\limits_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\left(\sum\limits_{i,j=1}^{n}|(y_i, y_j|z)|\right)^{\frac{1}{2}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(y_i, y_j|z)|\right)^{\frac{1}{2}}; \end{cases}$$

$$N := \begin{cases} \max\limits_{1\leq i\leq n}\{|c_i|\}\left(\sum\limits_{i=1}^{n}|c_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\left(\sum\limits_{i,j=1}^{n}|(y_i, y_j|z)|\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(y_i, y_j|z)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \\ \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta} = 1; \\ \left(\sum\limits_{i=1}^{n}|c_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\left(\sum\limits_{i=1}^{n}|c_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(y_i, y_j|z)|\right)^{\beta}\right)^{\frac{1}{2p}} \\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(y_i, y_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta}},\ \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta} = 1 \\ \text{and } \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta} = 1; \\ \left(\sum\limits_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}|c_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(y_i, y_j|z)|\right)^{\frac{1}{2}} \\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(y_i, y_j|z)|\right)^{\beta}\right)^{\frac{1}{2\beta}},\ \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta} = 1; \end{cases}$$

and

$$P := \begin{cases} \max\limits_{1\le i\le n}\{|c_i|\}\left(\sum\limits_{i=1}^n |c_i|^2\right)^{\frac{1}{2}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\frac{1}{2}}\left(\sum\limits_{i,j=1}^n |(y_i,y_j|z)|\right)^{\frac{1}{2}}; \\ \left(\sum\limits_{i=1}^n |c_i|^2\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^n |c_i|^{2\gamma}\right)^{\frac{1}{2\gamma}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\frac{1}{2}} \\ \times\left(\sum\limits_{i=1}^n\left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \ \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^n |c_i|^2\right)\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^n |(y_i,y_j|z)|\right); \end{cases}$$

which contains the version of Pečarić's inequality for 2-inner products, i.e., the inequality

$$\left|\sum_{i=1}^n c_i\,(x,y_i|z)\right|^2 \le \|x|z\|^2\left(\sum_{i=1}^n |c_i|^2\left(\sum_{j=1}^n |(y_i,y_j|z)|\right)\right) \le \left(\sum_{i=1}^n |c_i|^2\right)\max_{1\le i\le n}\left(\sum_{j=1}^n |(y_i,y_j|z)|\right). \tag{4.4}$$

5. Some Results of Bombieri Type for 2-Inner Products

The following results of Bombieri type hold.

Theorem 3. *Let $x, y_1, \dots, y_n, z \in X$. Then one has the inequalities:*

$$\begin{aligned}\sum_{i=1}^n |(x,y_i|z)|^2 &\le \|x|z\|\left[\sum_{i=1}^n |(x,y_i|z)|^p\left(\sum_{j=1}^n |(y_i,y_j|z)|\right)\right]^{\frac{1}{2p}} \\ &\quad\times\left[\sum_{i=1}^n |(x,y_i|z)|^q\left(\sum_{j=1}^n |(y_i,y_j|z)|\right)\right]^{\frac{1}{2q}} \\ &\le \|x|z\|\times\begin{cases} Q; \\ R; \\ S; \end{cases}\end{aligned} \tag{5.1}$$

where

$$Q := \begin{cases} \max\limits_{1\le i\le n} |(x,y_i|z)| \left(\sum\limits_{i,j=1}^{n} |(y_i,y_j|z)| \right)^{\frac{1}{2}} ; \\ \\ \max\limits_{1\le i\le n} |(x,y_i|z)|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\gamma q} \right)^{\frac{1}{2\gamma q}} \left(\sum\limits_{i,j=1}^{n} |(y_i,y_j|z)| \right)^{\frac{1}{2p}} \\ \times \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)| \right)^{\delta} \right)^{\frac{1}{2\delta q}}, \ \textit{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \\ \max\limits_{1\le i\le n} |(x,y_i|z)|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{q} \right)^{\frac{1}{2q}} \left(\sum\limits_{i,j=1}^{n} |(y_i,y_j|z)| \right)^{\frac{1}{2p}} \\ \times \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)| \right)^{\frac{1}{2q}} ; \end{cases}$$

$$R := \begin{cases} \max\limits_{1\le i\le n} |(x,y_i|z)|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\alpha p} \right)^{\frac{1}{2\alpha\beta}} \left(\sum\limits_{i,j=1}^{n} |(y_i,y_j|z)| \right)^{\frac{1}{2q}} \\ \times \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)| \right)^{\beta} \right)^{\frac{1}{p\beta}}, \ \textit{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\alpha p} \right)^{\frac{1}{2\alpha p}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\gamma q} \right)^{\frac{1}{2\gamma q}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)| \right)^{\beta} \right)^{\frac{1}{2p\beta}} \\ \times \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)| \right)^{\delta} \right)^{\frac{1}{2\delta q}}, \qquad \textit{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \textit{and } \ \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \\ \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{q} \right)^{\frac{1}{2q}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\alpha p} \right)^{\frac{1}{2\alpha p}} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)| \right)^{\frac{1}{2p}} \\ \quad \times \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)| \right)^{\beta} \right)^{\frac{1}{2p\beta}}, \ \textit{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and

$$S := \begin{cases} \max\limits_{1\le i\le n} \left|(x,y_i|z)\right|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} \left|(x,y_i|z)\right|^{p}\right)^{\frac{1}{2p}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}\left|(y_i,y_j|z)\right|\right)^{\frac{1}{2p}} \\ \times\left(\sum\limits_{i,j=1}^{n}\left|(y_i,y_j|z)\right|\right)^{\frac{1}{2q}}; \\ \\ \left(\sum\limits_{i=1}^{n} \left|(x,y_i|z)\right|^{p}\right)^{\frac{1}{2p}} \left(\sum\limits_{i=1}^{n} \left|(x,y_i|z)\right|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}\left|(y_i,y_j|z)\right|\right)^{\frac{1}{2p}} \\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left|(y_i,y_j|z)\right|\right)^{\delta}\right)^{\frac{1}{2\delta q}}, \qquad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \\ \left(\sum\limits_{i=1}^{n} \left|(x,y_i|z)\right|^{p}\right)^{\frac{1}{2p}} \left(\sum\limits_{i=1}^{n} \left|(x,y_i|z)\right|^{q}\right)^{\frac{1}{2q}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}\left|(y_i,y_j|z)\right|\right)^{\frac{1}{2}}, \end{cases}$$

where $p>1, \frac{1}{p}+\frac{1}{q}=1.$

Proof. The proof follows by Theorem 2 on choosing $c_i=\overline{(x,y_i|z)}$, for $i\in\{1,...,n\}$ and taking the square root in both sides of the inequalities involved. We omit the details. ∎

Remark 2. *We observe, by the last inequality in* (5.1)*, we get*

$$\frac{\left(\sum\limits_{i=1}^{n}\left|(x,y_i|z)\right|^2\right)^2}{\left(\sum\limits_{i=1}^{n}\left|(x,y_i|z)\right|^p\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}\left|(x,y_i|z)\right|^q\right)^{\frac{1}{q}}} \le \|x|z\|^2 \max_{1\le i\le n}\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|\right), \tag{5.2}$$

where $p>1, \frac{1}{p}+\frac{1}{q}=1$, *and provided that not all* $(x,y_i|z)$, *for* $i\in\{1,...,n\}$ *are zero.*

Remark 3. *If in this inequality we choose* $p=q=2$, *then we obtain the following Bombieri's type result for 2-inner products*

$$\sum_{i=1}^{n}\left|(x,y_i|z)\right|^2 \le \|x|z\|^2 \max_{1\le i\le n}\left(\sum_{j=1}^{n}\left|(y_i,y_j|z)\right|\right). \tag{5.3}$$

6. Applications for Determinantal Integral Inequalities

Let (Ω,Σ,μ) be a measure space consisting of a set Ω, a $\sigma-$algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R}\cup\{\infty\}$.

Denote by $L_\rho^2(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(s)\left|f(s)\right|^2 d\mu(s)<\infty$, where $\rho:\Omega\to[0,\infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L_\rho^2(\Omega)$ by formula

$$(f,g|h)_\rho := \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t)\end{vmatrix}\begin{vmatrix} g(s) & g(t) \\ h(s) & h(t)\end{vmatrix} d\mu(s)\,d\mu(t), \tag{6.1}$$

where

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}$$

denotes the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$

generating the 2-norm on $L_\rho^2(\Omega)$ expressed by

$$\|f|h\|_\rho := \left(\frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}^2 d\mu(s)\,d\mu(t)\right)^{1/2}. \tag{6.2}$$

A simple calculation with integrals reveals that

$$(f,g|h)_\rho = \begin{vmatrix} \int_\Omega \rho fg d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \tag{6.3}$$

and

$$\|f|h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho fh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{1/2} \tag{6.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho fg d\mu$.

Using the representations (6.3), (6.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, we can get some interesting determinantal integral inequalities.

We give here only two examples.

Proposition 1. *Let $f, g_1, ..., g_n, h \in L_\rho^2(\Omega)$, where $\rho : \Omega \to [0,\infty)$ is a measurable function on Ω, then we have the inequality*

$$\left(\sum_{i=1}^n \begin{vmatrix} \int_\Omega \rho fg_i d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^2\right)^2$$

$$\begin{aligned} &\le \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho fh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \\ &\quad\times \max_{1\le i\le n}\left\{\sum_{j=1}^n \left|\det\begin{bmatrix} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right|\right\} \\ &\quad\times \left(\sum_{i=1}^n \left|\det\begin{bmatrix} \int_\Omega \rho fg_i d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right|^p\right)^{1/p} \\ &\quad\times \left(\sum_{i=1}^n \left|\det\begin{bmatrix} \int_\Omega \rho fg_i d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right|^q\right)^{1/q}, \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The proof follows by the inequality for 2-inner products incorporated in (5.2).

Proposition 2. *Let $f, g_1, ..., g_n, h \in L^2_\rho(\Omega)$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω, then we have the inequality*

$$\sum_{i=1}^{n} \begin{vmatrix} \int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^2$$

$$\leq \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}$$

$$\times \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \left| \det \begin{bmatrix} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix} \right| \right\}.$$

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