ON THE BOAS-BELLMAN INEQUALITY IN INNER PRODUCT SPACES

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Abstract. New results related to the Boas-Bellman generalisation of Bessel's inequality in inner product spaces are given.

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1. Introduction

Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\leq i\leq n}$ are orthonormal vectors in the inner product space H, i.e., $(e_i,e_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well known in the literature as Bessel's inequality (see for example [6, p. 391]):

$$\sum_{i=1}^{n}|(x,e_i)|^2\leq\|x\|^2 \quad \text{for any} \quad x\in H. \tag{1.1}$$

For other results related to Bessel's inequality, see [3] – [5] and Chapter XV in the book [6].

In 1941, R.P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalisation of Bessel's inequality (see also [6, p. 392]).

Theorem 1. *If $x,y_1,\dots,y_n$ are elements of an inner product space $(H;(\cdot,\cdot))$, then the following inequality:*

$$\sum_{i=1}^{n}|(x,y_i)|^2\leq\|x\|^2\left[\max_{1\leq i\leq n}\|y_i\|^2+\left(\sum_{1\leq i\neq j\leq n}|(y_i,y_j)|^2\right)^{\frac{1}{2}}\right] \tag{1.2}$$

holds.

A recent generalisation of the Boas-Bellman result was given in Mitrinović-Pečarić-Fink [6, p. 392] where they proved the following.

Theorem 2. *If $x,y_1,\dots,y_n$ are as in Theorem 1 and $c_1,\dots,c_n\in\mathbb{K}$, then one has the inequality:*

$$\left|\sum_{i=1}^{n}c_i(x,y_i)\right|^2\leq\|x\|^2\sum_{i=1}^{n}|c_i|^2\left[\max_{1\leq i\leq n}\|y_i\|^2+\left(\sum_{1\leq i\neq j\leq n}|(y_i,y_j)|^2\right)^{\frac{1}{2}}\right]. \tag{1.3}$$

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They also noted that if in (1.3) one chooses $c_i = \overline{(x, y_i)}$, then this inequality becomes (1.2).

For other results related to the Boas-Bellman inequality, see [4].

In this paper we point out some new results that may be related to both the Mitrinović-Pečarić-Fink and Boas-Bellman inequalities.

2. Some Preliminary Results

We start with the following lemma which is also interesting in itself.

Lemma 1. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequality:*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \begin{cases} \max\limits_{1 \le i \le n} |\alpha_i|^2 \sum\limits_{i=1}^{n} \|z_i\|^2 ; \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \|z_i\|^{2\beta} \right)^{\frac{1}{\beta}}, \quad \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{1 \le i \le n} \|z_i\|^2 , \end{cases} + \begin{cases} \max\limits_{1 \le i \ne j \le n} \{|\alpha_i \alpha_j|\} \sum\limits_{1 \le i \ne j \le n} |(z_i, z_j)| ; \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|^{\gamma} \right)^2 - \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2\gamma} \right) \right]^{\frac{1}{\gamma}} \left(\sum\limits_{1 \le i \ne j \le n} |(z_i, z_j)|^{\delta} \right)^{\frac{1}{\delta}}, \\ \qquad \text{where } \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i| \right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^2 \right] \max\limits_{1 \le i \ne j \le n} |(z_i, z_j)| . \end{cases} \tag{2.1}$$

Proof. We observe that

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 &= \left(\sum_{i=1}^{n} \alpha_i z_i, \sum_{j=1}^{n} \alpha_j z_j \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (z_i, z_j) = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (z_i, z_j) \right| \\ &\le \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\overline{\alpha_j}| |(z_i, z_j)| \\ &= \sum_{i=1}^{n} |\alpha_i|^2 \|z_i\|^2 + \sum_{1 \le i \ne j \le n} |\alpha_i| |\alpha_j| |(z_i, z_j)| . \end{aligned} \tag{2.2}$$

Using Hölder's inequality, we may write that

$$\sum_{i=1}^{n}|\alpha_i|^2\|z_i\|^2 \le \begin{cases} \max\limits_{1\le i\le n}|\alpha_i|^2\sum\limits_{i=1}^{n}\|z_i\|^2; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^{n}\|z_i\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \text{where} \quad \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{1\le i\le n}\|z_i\|^2. \end{cases} \tag{2.3}$$

By Hölder's inequality for double sums we also have

$$\begin{aligned} &\sum_{1\le i\ne j\le n}|\alpha_i||\alpha_j||(z_i,z_j)| \\ &\le \begin{cases} \max\limits_{1\le i\ne j\le n}|\alpha_i\alpha_j|\sum\limits_{1\le i\ne j\le n}|(z_i,z_j)|; \\ \left(\sum\limits_{1\le i\ne j\le n}|\alpha_i|^{\gamma}|\alpha_j|^{\gamma}\right)^{\frac{1}{\gamma}}\left(\sum\limits_{1\le i\ne j\le n}|(z_i,z_j)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \text{where} \quad \gamma>1, \quad \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \sum\limits_{1\le i\ne j\le n}|\alpha_i||\alpha_j|\max\limits_{1\le i\ne j\le n}|(z_i,z_j)|, \end{cases} \\ &= \begin{cases} \max\limits_{1\le i\ne j\le n}\{|\alpha_i\alpha_j|\}\sum\limits_{1\le i\ne j\le n}|(z_i,z_j)|; \\ \left[\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma}\right)^2-\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}}\left(\sum\limits_{1\le i\ne j\le n}|(z_i,z_j)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \text{where} \quad \gamma>1, \quad \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left[\left(\sum\limits_{i=1}^{n}|\alpha_i|\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^2\right]\max\limits_{1\le i\ne j\le n}|(z_i,z_j)|. \end{cases} \end{aligned} \tag{2.4}$$

Utilising (2.3) and (2.4) in (2.2), we may deduce the desired result (2.1). ∎

Remark 1. *Inequality (2.1) contains in fact 9 different inequalities which may be obtained combining the first 3 ones with the last 3 ones.*

A particular case that may be related to the Boas-Bellman result is embodied in the following inequality.

Corollary 1. *With the assumptions in Lemma 1, we have*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \tag{2.5}$$

$$\leq \sum_{i=1}^{n} |\alpha_i|^2 \left\{ \max_{1 \leq i \leq n} \|z_i\|^2 + \frac{\left[\left(\sum_{i=1}^{n} |\alpha_i|^2 \right)^2 - \sum_{i=1}^{n} |\alpha_i|^4 \right]^{\frac{1}{2}}}{\sum_{i=1}^{n} |\alpha_i|^2} \left(\sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^2 \right)^{\frac{1}{2}} \right\}$$

$$\leq \sum_{i=1}^{n} |\alpha_i|^2 \left\{ \max_{1 \leq i \leq n} \|z_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^2 \right)^{\frac{1}{2}} \right\}.$$

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for $\gamma = \delta = 2$.

The second inequality in (2.5) follows by the fact that

$$\left[\left(\sum_{i=1}^{n} |\alpha_i|^2 \right)^2 - \sum_{i=1}^{n} |\alpha_i|^4 \right]^{\frac{1}{2}} \leq \sum_{i=1}^{n} |\alpha_i|^2 .$$

Applying the following Cauchy-Bunyakovsky-Schwarz type inequality

$$\left(\sum_{i=1}^{n} a_i \right)^2 \leq n \sum_{i=1}^{n} a_i^2, \quad a_i \in \mathbb{R}_+, \quad 1 \leq i \leq n, \tag{2.6}$$

we may write that

$$\left(\sum_{i=1}^{n} |\alpha_i|^{\gamma} \right)^2 - \sum_{i=1}^{n} |\alpha_i|^{2\gamma} \leq (n-1) \sum_{i=1}^{n} |\alpha_i|^{2\gamma} \qquad (n \geq 1) \tag{2.7}$$

and

$$\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \leq (n-1) \sum_{i=1}^{n} |\alpha_i|^2 \qquad (n \geq 1). \tag{2.8}$$

Also, it is obvious that:

$$\max_{1 \leq i \neq j \leq n} \{|\alpha_i \alpha_j|\} \leq \max_{1 \leq i \leq n} |\alpha_i|^2 . \tag{2.9}$$

Consequently, we may state the following coarser upper bounds for $\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2$ that may be useful in applications.

Corollary 2. *With the assumptions in Lemma 1, we have the inequalities:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \tag{2.10}$$
$$\leq\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|^2\sum\limits_{i=1}^{n}\|z_i\|^2; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^{n}\|z_i\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \textit{where}\quad \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{1\leq i\leq n}\|z_i\|^2,\end{cases}$$
$$+\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|^2\sum\limits_{1\leq i\neq j\leq n}|(z_i,z_j)|; \\ (n-1)^{\frac{1}{\gamma}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{\gamma}}\left(\sum\limits_{1\leq i\neq j\leq n}|(z_i,z_j)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \textit{where}\quad \gamma>1,\quad \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ (n-1)\sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{1\leq i\neq j\leq n}|(z_i,z_j)|.\end{cases}$$

The proof is obvious by Lemma 1 in applying the inequalities (2.7) – (2.9).

Remark 2. *The following inequalities which are incorporated in (2.10) are of special interest:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2\leq\max_{1\leq i\leq n}|\alpha_i|^2\left[\sum_{i=1}^{n}\|z_i\|^2+\sum_{1\leq i\neq j\leq n}|(z_i,z_j)|\right]; \tag{2.11}$$

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \tag{2.12}$$
$$\leq\left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left[\left(\sum_{i=1}^{n}\|z_i\|^{2q}\right)^{\frac{1}{q}}+(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|(z_i,z_j)|^{q}\right)^{\frac{1}{q}}\right],$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$; *and*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2\leq\sum_{i=1}^{n}|\alpha_i|^2\left[\max_{1\leq i\leq n}\|z_i\|^2+(n-1)\max_{1\leq i\neq j\leq n}|(z_i,z_j)|\right]. \tag{2.13}$$

3. Some Mitrinović-Pečarić-Fink Type Inequalities

We are now able to point out the following result which complements the inequality (1.3) due to Mitrinović, Pečarić and Fink [6, p. 392].

Theorem 3. *Let $x, y_1, \ldots, y_n$ be vectors of an inner product space $(H; (\cdot, \cdot))$ and $c_1, \ldots, c_n \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$. Then one has the inequalities:*

$$\begin{aligned}
(3.1)\quad & \left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \\
& \leq \|x\|^2 \times \begin{cases} \max\limits_{1\leq i\leq n} |c_i|^2 \sum\limits_{i=1}^{n} \|y_i\|^2; \\[2ex] \left(\sum\limits_{i=1}^{n} |c_i|^{2\alpha}\right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \|y_i\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex] \sum\limits_{i=1}^{n} |c_i|^2 \max\limits_{1\leq i\leq n} \|y_i\|^2, \end{cases} \\
& \quad + \|x\|^2 \times \begin{cases} \max\limits_{1\leq i\neq j\leq n} \{|c_i c_j|\} \sum\limits_{1\leq i\neq j\leq n} |(y_i, y_j)|; \\[2ex] \left[\left(\sum\limits_{i=1}^{n} |c_i|^{\gamma}\right)^2 - \left(\sum\limits_{i=1}^{n} |c_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}} \left(\sum\limits_{1\leq i\neq j\leq n} |(y_i, y_j)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \hfill \text{where } \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex] \left[\left(\sum\limits_{i=1}^{n} |c_i|\right)^2 - \sum\limits_{i=1}^{n} |c_i|^2\right] \max\limits_{1\leq i\neq j\leq n} |(y_i, y_j)|. \end{cases}
\end{aligned}$$

Proof. We note that

$$\sum_{i=1}^{n} c_i (x, y_i) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i\right).$$

Using Schwarz's inequality in inner product spaces, we have:

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \left\|\sum_{i=1}^{n} \overline{c_i} y_i\right\|^2.$$

Now using Lemma 1 with $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \ldots, n)$, we deduce the desired inequality (3.1). ∎

The following particular inequalities that may be obtained by the Corollaries 1 and 2 and Remark 2 hold.

Corollary 3. *With the assumptions in Theorem 3, one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i\,(x,y_i)\right|^2 \tag{3.2}$$

$$\leq \times \begin{cases} \|x\|^2 \sum\limits_{i=1}^{n} |c_i|^2 \left\{ \max\limits_{1\leq i\leq n} \|y_i\|^2 + \left(\sum\limits_{1\leq i\neq j\leq n} |(y_i,y_j)|^2 \right)^{\frac{1}{2}} \right\}; \\[2ex] \|x\|^2 \max\limits_{1\leq i\leq n} |c_i|^2 \left\{ \sum\limits_{i=1}^{n} \|y_i\|^2 + \sum\limits_{1\leq i\neq j\leq n} |(y_i,y_j)| \right\} \\[2ex] \|x\|^2 \left(\sum\limits_{i=1}^{n} |c_i|^{2p} \right)^{\frac{1}{p}} \left\{ \left(\sum\limits_{i=1}^{n} \|y_i\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum\limits_{1\leq i\neq j\leq n} |(y_i,y_j)|^q \right)^{\frac{1}{q}} \right\}, \\ \hfill \textit{where}\ \ p>1, \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \|x\|^2 \sum\limits_{i=1}^{n} |c_i|^2 \left\{ \max\limits_{1\leq i\leq n} \|y_i\|^2 + (n-1) \max\limits_{1\leq i\neq j\leq n} |(y_i,y_j)| \right\}, \end{cases}$$

Remark 3. *Note that the first inequality in (3.2) is the result obtained by Mitrinović-Pečarić-Fink in* [6]*. The other 3 provide similar bounds in terms of the $p-$norms of the vector* $\left(|c_1|^2,\dots,|c_n|^2\right)$.

4. Some Boas-Bellman Type Inequalities

If one chooses $c_i = \overline{(x,y_i)}$ $(i=1,\dots,n)$ in (3.1), then it is possible to obtain 9 different inequalities between the Fourier coefficients (x,y_i) and the norms and inner products of the vectors y_i $(i=1,\dots,n)$. We restrict ourselves only to those inequalities that may be obtained from (3.2).

As Mitrinović, Pečarić and Fink noted in [6, p. 392], the first inequality in (3.2) for the above selection of c_i will produce the Boas-Bellman inequality (1.2).

From the second inequality in (3.2) for $c_i = \overline{(x,y_i)}$ we get

$$\left(\sum_{i=1}^{n} |(x,y_i)|^2\right)^2 \leq \|x\|^2 \max_{1\leq i\leq n} |(x,y_i)|^2 \left\{ \sum_{i=1}^{n} \|y_i\|^2 + \sum_{1\leq i\neq j\leq n} |(y_i,y_j)| \right\}.$$

Taking the square root in this inequality we obtain:

$$\sum_{i=1}^{n} |(x,y_i)|^2 \leq \|x\| \max_{1\leq i\leq n} |(x,y_i)| \left\{ \sum_{i=1}^{n} \|y_i\|^2 + \sum_{1\leq i\neq j\leq n} |(y_i,y_j)| \right\}^{\frac{1}{2}}, \tag{4.1}$$

for any $x, y_1, \dots, y_n$ vectors in the inner product space $(H;(\cdot,\cdot))$.

If we assume that $(e_i)_{1\leq i\leq n}$ is an orthonormal family in H, then by (4.1) we have

$$\sum_{i=1}^{n} |(x,e_i)|^2 \leq \sqrt{n}\,\|x\| \max_{1\leq i\leq n} |(x,e_i)|, \quad x\in H. \tag{4.2}$$

From the third inequality in (3.2) for $c_i = \overline{(x, y_i)}$ we deduce

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \left(\sum_{i=1}^{n} |(x, y_i)|^{2p}\right)^{\frac{1}{p}} \times \left\{\left(\sum_{i=1}^{n} \|y_i\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right\},$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Taking the square root in this inequality we get

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\| \left(\sum_{i=1}^{n} |(x, y_i)|^{2p}\right)^{\frac{1}{2p}} \times \left\{\left(\sum_{i=1}^{n} \|y_i\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right\}^{\frac{1}{2}}, \tag{4.3}$$

for any $x, y_1, \dots, y_n \in H$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The above inequality (4.3) becomes, for an orthornormal family $(e_i)_{1\leq i\leq n}$,

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq n^{\frac{1}{q}} \|x\| \left(\sum_{i=1}^{n} |(x, e_i)|^{2p}\right)^{\frac{1}{2p}}, \quad x \in H. \tag{4.4}$$

Finally, the choice $c_i = \overline{(x, y_i)}$ $(i = 1, \dots, n)$ will produce in the last inequality in (3.2)

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \left\{\max_{1\leq i\leq n} \|y_i\|^2 + (n-1) \max_{1\leq i\neq j\leq n} |(y_i, y_j)|\right\}$$

giving the following Boas-Bellman type inequality

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \left\{\max_{1\leq i\leq n} \|y_i\|^2 + (n-1) \max_{1\leq i\neq j\leq n} |(y_i, y_j)|\right\}, \tag{4.5}$$

for any $x, y_1, \dots, y_n \in H$.

It is obvious that (4.5) will give for orthonormal families the well known Bessel inequality.

Remark 4. *In order the compare the Boas-Bellman result with our result (4.5), it is enough to compare the quantities*

$$A := \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j)|^2\right)^{\frac{1}{2}}$$

and

$$B := (n-1) \max_{1\leq i\neq j\leq n} |(y_i, y_j)|.$$

Consider the inner product space $H = \mathbb{R}$ *with* $(x, y) = xy$*, and choose* $n = 3$*,* $y_1 = a > 0$*,* $y_2 = b > 0$*,* $y_3 = c > 0$*. Then*

$$A = \sqrt{2}\left(a^2b^2 + b^2c^2 + c^2a^2\right)^{\frac{1}{2}}, \qquad B = 2\max\left(ab, ac, bc\right).$$

Denote $ab = p$*,* $bc = q$*,* $ca = r$*. Then*

$$A = \sqrt{2}\left(p^2 + q^2 + r^2\right)^{\frac{1}{2}}, \qquad B = 2\max\left(p, q, r\right).$$

Firstly, if we assume that $p = q = r$*, then* $A = \sqrt{6}p$*,* $B = 2p$ *which shows that* $A > B$*.*

Now choose $r = 1$ *and* $p, q = \frac{1}{2}$*. Then* $A = \sqrt{3}$ *and* $B = 2$ *showing that* $B > A$*.*

Consequently, in general, the Boas-Bellman inequality and our inequality (4.5) cannot be compared.

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