SOME BOMBIERI TYPE INEQUALITIES IN INNER PRODUCT SPACES

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Abstract. Companion results to the Bombieri generalisation of Bessel's inequality in inner product spaces are given.

1. Introduction

In 1971, E. Bombieri [1], has given the following generalisation of Bessel's inequality:

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\|^2 \max_{1\le i\le n} \left\{ \sum_{j=1}^{n} |(y_i, y_j)| \right\}, \tag{1.1}$$

where $x, y_1, \dots, y_n$ are vectors in the inner product space $(H; (\cdot, \cdot))$.

It is obvious that if $(y_i)_{1\le i\le n} = (e_i)_{1\le i\le n}$, where $(e_i)_{1\le i\le n}$ are orthornormal vectors in H, i.e., $(e_i, e_j) = \delta_{ij}$ $(i, j = 1, \dots, n)$, where δ_{ij} is the Kronecker delta, then (1.1) provides Bessel's inequality

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2, \quad x \in H. \tag{1.2}$$

In this paper we point out other Bombieri type inequalities and show that, some times, the new ones may provide better bounds for $\sum_{i=1}^{n} |(x, y_i)|^2$.

2. The Results

The following lemma which is of interest in itself holds.

Date: 11 June, 2003.
2000 *Mathematics Subject Classification.* 26D15, 46C05.
Key words and phrases. Bessel's inequality, Bombieri inequality.

Lemma 1. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \tag{2.1}$$

$$\leq \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i|^2 \sum_{i,j=1}^{n} |(z_i, z_j)|; \\ \\ \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{\frac{2}{p}} \left(\sum_{i,j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}}, \quad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left(\sum_{i=1}^{n} |\alpha_i|\right)^2 \max\limits_{1\leq i,j\leq n} |(z_i, z_j)|; \end{cases}$$

$$\leq \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i|^2 \left(\sum_{i=1}^{n} \|z_i\|\right)^2; \\ \\ \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{\frac{2}{p}} \left(\sum_{i=1}^{n} \|z_i\|^q\right)^{\frac{2}{q}}, \quad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left(\sum_{i=1}^{n} |\alpha_i|\right)^2 \max\limits_{1\leq i\leq n} \|z_i\|^2. \end{cases}$$

Proof. We observe that

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 = \left(\sum_{i=1}^{n} \alpha_i z_i, \sum_{j=1}^{n} \alpha_j z_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (z_i, z_j) \tag{2.2}$$

$$= \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} |(z_i, z_j)| \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| |(z_i, z_j)| =: M.$$

Firstly, we have

$$M \leq \max_{1\leq i,j\leq n} \{|\alpha_i| |\alpha_j|\} \sum_{i,j=1}^{n} |(z_i, z_j)|$$

$$= \max_{1\leq i\leq n} |\alpha_i|^2 \sum_{i,j=1}^{n} |(z_i, z_j)|.$$

Secondly, by the Hölder inequality for double sums, we have

$$M \leq \left[\sum_{i,j=1}^{n} (|\alpha_i| |\alpha_j|)^p \right]^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} |(z_i, z_j)|^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{i=1}^{n} |\alpha_i|^p \sum_{j=1}^{n} |\alpha_j|^p \right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} |(z_i, z_j)|^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{i=1}^{n} |\alpha_i|^p \right)^{\frac{2}{p}} \left(\sum_{i,j=1}^{n} |(z_i, z_j)|^q \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Finally, we have

$$M \leq \max_{1\leq i,j\leq n} |(z_i, z_j)| \sum_{i,j=1}^{n} |\alpha_i| |\alpha_j| = \left(\sum_{i=1}^{n} |\alpha_i|\right)^2 \max_{1\leq i,j\leq n} |(z_i, z_j)|$$

and the first part of the lemma is proved.

The second part is obvious on taking into account, by Schwarz's inequality in H, that we have

$$|(z_i, z_j)| \leq \|z_i\| \|z_j\|,$$

for any $i, j \in \{1, ..., n\}$. We omit the details. ∎

Corollary 1. *With the assumptions in Lemma 1, one has*

$$\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 \left(\sum_{i,j=1}^{n} |(z_i, z_j)|^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} |\alpha_i|^2 \sum_{i=1}^{n} \|z_i\|^2. \tag{2.3}$$

The proof follows by Lemma 1 on choosing $p = q = 2$.

Note also that (2.3) provides a refinement of the well known Cauchy-Bunyakovsky-Schwarz inequality for sequences of vectors in inner product spaces, namely

$$\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 \sum_{i=1}^{n} \|z_i\|^2.$$

The following lemma also holds.

Lemma 2. *Let $x, y_1, \ldots, y_n \in H$ and $c_1, \ldots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \tag{2.4}$$

$$\leq \|x\|^2 \times \begin{cases} \max\limits_{1\leq i\leq n} |c_i|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j)|; \\[2ex] \left(\sum\limits_{i=1}^{n} |c_i|^p\right)^{\frac{2}{p}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}, \quad \textit{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \left(\sum\limits_{i=1}^{n} |c_i|\right)^2 \max\limits_{1\leq i,j\leq n} |(y_i, y_j)|; \end{cases}$$

$$\leq \|x\|^2 \times \begin{cases} \max\limits_{1\leq i\leq n} |c_i|^2 \left(\sum\limits_{i=1}^{n} \|y_i\|\right)^2; \\[2ex] \left(\sum\limits_{i=1}^{n} |c_i|^p\right)^{\frac{2}{p}} \left(\sum\limits_{i=1}^{n} \|y_i\|^q\right)^{\frac{2}{q}}, \quad \textit{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \left(\sum\limits_{i=1}^{n} |c_i|\right)^2 \max\limits_{1\leq i\leq n} \|y_i\|^2. \end{cases}$$

Proof. We have, by Schwarz's inequality in the inner product $(H;(\cdot,\cdot))$, that

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 = \left|\left(x, \sum_{i=1}^{n} \overline{c_i} y_i\right)\right|^2 \leq \|x\|^2 \left\|\sum_{i=1}^{n} \overline{c_i} y_i\right\|^2 .$$

Now, applying Lemma 1 for $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \ldots, n)$, the inequality (2.4) is proved. ∎

Corollary 2. *With the assumptions in Lemma 2, one has*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right)^{\frac{1}{2}} \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \sum_{i=1}^{n} \|y_i\|^2 . \tag{2.5}$$

The proof follows by Lemma 2, on choosing $p = q = 2$.

Remark 1. *The inequality (2.5) was firstly obtained in* [2] *(see inequality (7)).*

The following theorem incorporating three Bombieri type inequalities holds.

Theorem 1. *Let $x, y_1, \ldots, y_n \in H$. Then one has the inequalities:*

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\| \times \begin{cases} \max\limits_{1\leq i\leq n} |(x, y_i)| \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2}} ; \\ \left(\sum\limits_{i=1}^{n} |(x, y_i)|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{2q}}, \quad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum\limits_{i=1}^{n} |(x, y_i)| \max\limits_{1\leq i,j\leq n} |(y_i, y_j)|^{\frac{1}{2}} . \end{cases} \tag{2.6}$$

Proof. Choosing $c_i = \overline{(x, y_i)}$ $(i = 1, \ldots, n)$ in (2.4) we deduce

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \times \begin{cases} \max\limits_{1\leq i\leq n} |(x, y_i)|^2 \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|\right) ; \\ \left(\sum\limits_{i=1}^{n} |(x, y_i)|^p\right)^{\frac{2}{p}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}, \quad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum\limits_{i=1}^{n} |(x, y_i)|\right)^2 \max\limits_{1\leq i,j\leq n} |(y_i, y_j)| ; \end{cases} \tag{2.7}$$

which, by taking the square root, is clearly equivalent to (2.6). ∎

Remark 2. *If $(y_i)_{1\leq i\leq n} = (e_i)_{1\leq i\leq n}$ where $(e_i)_{1\leq i\leq n}$ are orthornormal vectors in H, then by (2.6) we deduce*

$$\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\| \times \begin{cases} \sqrt{n} \max\limits_{1\leq i\leq n} |(x,e_i)|\,; \\ \\ n^{\frac{1}{2q}} \left(\sum\limits_{i=1}^{n} |(x,e_i)|^p\right)^{\frac{1}{p}}, \qquad \text{where } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum\limits_{i=1}^{n} |(x,e_i)|\,. \end{cases} \tag{2.8}$$

If in (2.7) we take $p = q = 2$, then we obtain the following inequality that was formulated in [2, p. 81].

Corollary 3. *With the assumptions in Theorem 1, we have:*

$$\sum_{i=1}^{n} |(x,y_i)|^2 \leq \|x\|^2 \left(\sum_{i,j=1}^{n} |(y_i,y_j)|^2\right)^{\frac{1}{2}}. \tag{2.9}$$

Remark 3. *Observe, that by the monotonicity of power means, we may write*

$$\left(\frac{\sum_{i=1}^{n} |(x,y_i)|^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{\sum_{i=1}^{n} |(x,y_i)|^2}{n}\right)^{\frac{1}{2}}, \quad 1 < p \leq 2. \tag{2.10}$$

Taking the square in both sides, one has

$$\left(\frac{\sum_{i=1}^{n} |(x,y_i)|^p}{n}\right)^{\frac{2}{p}} \leq \frac{\sum_{i=1}^{n} |(x,y_i)|^2}{n},$$

giving

$$\left(\sum_{i=1}^{n} |(x,y_i)|^p\right)^{\frac{2}{p}} \leq n^{\frac{2}{p}-1} \sum_{i=1}^{n} |(x,y_i)|^2. \tag{2.11}$$

Using (2.11) and the second inequality in (2.7) we may deduce the following result

$$\sum_{i=1}^{n} |(x,y_i)|^2 \leq n^{\frac{2}{p}-1} \|x\|^2 \left(\sum_{i,j=1}^{n} |(y_i,y_j)|^q\right)^{\frac{1}{q}}, \tag{2.12}$$

for $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$.

Note that for $p = 2$ $(q = 2)$ we recapture (2.9).

Remark 4. *Let us compare Bombieri's result*

$$\sum_{i=1}^{n} |(x,y_i)|^2 \leq \|x\|^2 \max_{1\leq i\leq n} \left\{\sum_{j=1}^{n} |(y_i,y_j)|\right\} \tag{2.13}$$

with our general result (2.12).

To do that, denote

$$M_1 := \max_{1\leq i\leq n} \left\{\sum_{j=1}^{n} |(y_i,y_j)|\right\}$$

and

$$M_2 := n^{\frac{2}{p}-1}\left(\sum_{i,j=1}^{n}|(y_i,y_j)|^q\right)^{\frac{1}{q}}, \quad 1<p\le 2,\ \frac{1}{p}+\frac{1}{q}=1.$$

Consider the inner product space $H=\mathbb{R}$, $(x,y)=x\cdot y$, $n=2$ *and* $y_1=a>0$, $y_2=b>0$. *Then*

$$M_1 = \max\left\{a^2+ab, ab+b^2\right\} = (a+b)\max\{a,b\},$$

$$M_2 = 2^{\frac{2}{p}-1}\left(a^q+b^q\right)^{\frac{2}{q}} = 2^{\frac{2}{p}-1}\left(a^{\frac{p}{p-1}}+b^{\frac{p}{p-1}}\right)^{\frac{2(p-1)}{p}},\ 1<p\le 2.$$

Assume $a=1$, $b\in[0,1]$, $p\in(1,2]$. *Utilizing Maple 6, one may easily see by plotting the function*

$$f(b,p) := M_2 - M_1 = 2^{\frac{2}{p}-1}\left(1+b^{\frac{p}{p-1}}\right)^{\frac{2(p-1)}{p}} - 1 - b$$

that it has positive and negative values in the box $[0,1]\times[1,2]$, *showing that the inequalities (2.12) and (2.13) cannot be compared. This means that one is not always better than the other.*

References

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