A COUNTERPART OF SCHWARZ'S INEQUALITY IN INNER PRODUCT SPACES

SEVER S. DRAGOMIR

Abstract. A new counterpart of Schwarz's inequality in inner product spaces and applications for isotonic functionals, integrals and sequences are provided.

1. Introduction

Let $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ be two positive $n-$tuples with

$$0 < m_1 \le a_i \le M_1 < \infty \text{ and } 0 < m_2 \le b_i \le M_2 < \infty; \tag{1.1}$$

for each $i \in \{1, \ldots, n\}$, and some constants m_1, m_2, M_1, M_2.

The following counterparts of the Cauchy-Bunyakowsy-Schwarz inequality are valid:

(1) Pólya-Szegö's inequality [8]

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2; \tag{1.2}$$

(2) Shisha-Mond's inequality [9]

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \le \left[\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right]^2; \tag{1.3}$$

(3) Ozeki's inequality [7]

$$\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left(\sum_{k=1}^n a_k b_k\right)^2 \le \frac{n^2}{4}\left(M_1 M_2 - m_1 m_2\right)^2; \tag{1.4}$$

(4) Diaz-Metcalf's inequality [1]

$$\sum_{k=1}^n b_k^2 + \frac{m_2 M_2}{m_1 M_1}\sum_{k=1}^n a_k^2 \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\sum_{k=1}^n a_k b_k. \tag{1.5}$$

If $\overline{\mathbf{w}} = (w_1, \ldots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

(1) Cassel's inequality [10]. If the positive real sequences $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ satisfy the condition

$$0 < m \le \frac{a_k}{b_k} \le M < \infty \text{ for each } k \in \{1, ..., n\} \tag{1.6}$$

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then

$$\frac{\left(\sum_{k=1}^n w_k a_k^2\right)\left(\sum_{k=1}^n w_k b_k^2\right)}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \leq \frac{(M+m)^2}{4mM};$$

(2) Greub-Reinboldt's inequality [4]

$$\left(\sum_{k=1}^n w_k a_k^2\right)\left(\sum_{k=1}^n w_k b_k^2\right) \leq \frac{(M_1M_2+m_1m_2)^2}{4m_1m_2M_1M_2}\left(\sum_{k=1}^n w_k a_k b_k\right)^2. \tag{1.7}$$

provided $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ satisfy the condition (1.1).

(3) Generalised Diaz-Metcalf inequality [1], see also [6, p. 123]. If $u, v \in [0,1]$ and $v \leq u$, $u + v = 1$ and (1.6) holds, then one has the inequality

$$u\sum_{k=1}^n w_k b_k^2 + vMm\sum_{k=1}^n w_k a_k^2 \leq (vm + uM)\sum_{k=1}^n w_k a_k b_k. \tag{1.8}$$

(4) Klamkin-McLenaghan's inequality [5]. If $\overline{a}, \overline{b}$ satisfy (1.6), then

$$\left(\sum_{i=1}^n w_i a_i^2\right)\left(\sum_{i=1}^n w_i b_i^2\right) - \left(\sum_{i=1}^n w_i a_i b_i\right)^2 \leq \left(M^{\frac{1}{2}} - m^{\frac{1}{2}}\right)^2 \sum_{i=1}^n w_i a_i b_i \sum_{i=1}^n w_i a_i^2. \tag{1.9}$$

For other results providing counterpart inequalities, see the recent monograph on line [3].

In this paper we point out a new counterpart of Schwarz's inequality in real or complex inner product spaces. Particular cases for isotonic linear functionals, integrals and sequences are also provided.

2. An Inequality in Inner Product Spaces

The following reverse of Schwarz's inequality in inner product spaces holds.

Theorem 1. *Let $A, a \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $x, y \in H$. If*

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0, \tag{2.1}$$

or, equivalently,

$$\left\|x - \frac{a+A}{2}\cdot y\right\| \leq \frac{1}{2}|A - a|\,\|y\|, \tag{2.2}$$

holds, then one has the inequality

$$0 \leq \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4}|A-a|^2\|y\|^4. \tag{2.3}$$

The constant $\frac{1}{4}$ is sharp in (2.3).

Proof. The equivalence between (2.1) and (2.2) can be easily proved, see for example [2].

Let us define

$$I_1 := \operatorname{Re}\left[\left(A\|y\|^2 - |\langle x, y\rangle|\right)\left(\overline{\langle x, y\rangle} - \overline{a}\|y\|^2\right)\right]$$

and

$$I_2 := \|y\|^2 \operatorname{Re}\langle Ay - x, x - ay\rangle.$$

Then

$$I_1 = \|y\|^2 \operatorname{Re}\left[A\overline{\langle x,y\rangle} + \overline{a}\langle x,y\rangle\right] - |\langle x,y\rangle|^2 - \|y\|^4 \operatorname{Re}(A\overline{a})$$

and

$$I_2 = \|y\|^2 \operatorname{Re}\left[A\overline{\langle x,y\rangle} + \overline{a}\langle x,y\rangle\right] - \|x\|^2\|y\|^2 - \|y\|^4 \operatorname{Re}(A\overline{a}),$$

giving

$$I_1 - I_2 = \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2; \tag{2.4}$$

for any $x, y \in H$ and $a, A \in \mathbb{K}$, which is an interesting equality in itself as well.

If (2.1) holds, then $I_2 \geq 0$ and thus

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leq \operatorname{Re}\left[\left(A\|y\|^2 - |\langle x,y\rangle|\right)\left(\overline{\langle x,y\rangle} - \overline{a}\|y\|^2\right)\right]. \tag{2.5}$$

If we use the elementary inequality for $u, v \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$)

$$\operatorname{Re}[u\overline{v}] \leq \frac{1}{4}|u+v|^2, \tag{2.6}$$

then we have for

$$u := A\|y\|^2 - \langle x,y\rangle, \quad v := \langle x,y\rangle - a\|y\|^2$$

that

$$\operatorname{Re}\left[\left(A\|y\|^2 - |\langle x,y\rangle|\right)\left(\overline{\langle x,y\rangle} - \overline{a}\|y\|^2\right)\right]^2 \leq \frac{1}{4}|A-a|^2\|y\|^4. \tag{2.7}$$

Making use of the inequalities (2.5) and (2.7), we deduce (2.3).

Now, assume that (2.3) holds with a constant $C > 0$, i.e.,

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leq C|A-a|^2\|y\|^4, \tag{2.8}$$

where x, y, a, A satisfy (2.1).

Consider $y \in H$, $\|y\| = 1$, $a \neq A$ and $m \in H$, $\|m\| = 1$ with $m \perp y$. Define

$$x := \frac{A+a}{2}y + \frac{A-a}{2}m.$$

Then

$$\langle Ay - x, x - ay\rangle = \left|\frac{A-a}{2}\right|^2 \langle y - m, y + m\rangle = 0,$$

and thus the condition (2.1) is fulfilled. From (2.8) we deduce

$$\left\|\frac{A+a}{2}y + \frac{A-a}{2}m\right\|^2 - \left|\left\langle\frac{A+a}{2}y + \frac{A-a}{2}m, y\right\rangle\right|^2 \leq C|A-a|^2, \tag{2.9}$$

and since

$$\left\|\frac{A+a}{2}y + \frac{A-a}{2}m\right\|^2 = \left|\frac{A+a}{2}\right|^2 + \left|\frac{A-a}{2}\right|^2$$

and

$$\left|\left\langle\frac{A+a}{2}y + \frac{A-a}{2}m, y\right\rangle\right|^2 = \left|\frac{A+a}{2}\right|^2$$

then by (2.9) we obtain

$$\frac{|A-a|^2}{4} \leq C|A-a|^2,$$

giving $C \geq \frac{1}{4}$, and the theorem is completely proved. ∎

3. Applications for Isotonic Linear Functionals

Let $F(T)$ be an algebra of real functions defined on T and L a subclass of $F(T)$ satisfying the conditions:

(i) $f, g \in L$ implies $f + g \in L$;
(ii) $f \in L, \in \mathbb{R}$ implies $\alpha f \in L$.

A functional A defined on L is an *isotonic linear functional* on L provided that

(a) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in L$;
(aa) $f \geq g$, that is, $f(t) \geq g(t)$ for all $t \in T$, implies $A(f) \geq A(g)$.

The functional A is *normalised* on L, provided that $\mathbf{1} \in L$, i.e., $\mathbf{1}(t) = 1$ for all $t \in T$, implies $A(\mathbf{1}) = 1$.

Usual examples of isotonic linear functionals are integrals, sums, etc.

Now, suppose that $h \in F(T)$, $h \geq 0$ is given and satisfies the properties that $fgh \in L$, $fh \in L$, $gh \in L$ for all $f, g \in L$. For a given isotonic linear functional $A : L \to \mathbb{R}$ with $A(h) > 0$, define the mapping $(\cdot, \cdot)_{A,h} : L \times L \to \mathbb{R}$ by

$$(f, g)_{A,h} := \frac{A(fgh)}{A(h)}. \tag{3.1}$$

This functional satisfies the following properties:

(s) $(f, f)_{A,h} \geq 0$ for all $f \in L$;
(ss) $(\alpha f + \beta g, k)_{A,h} = \alpha (f, k)_{A,h} + \beta (g, k)_{A,h}$ for all $f, g, k \in L$ and $\alpha, \beta \in \mathbb{R}$;
(sss) $(f, g)_{A,h} = (g, f)_{A,h}$ for all $f, g \in L$.

The following reverse of Schwarz's inequality for positive linear functionals holds.

Proposition 1. *Let $f, g, h \in F(T)$ be such that $fgh \in L$, $f^2 h \in L$, $g^2 h \in L$. If m, M are real numbers such that*

$$mg \leq f \leq Mg \text{ on } F(T), \tag{3.2}$$

then for any isotonic linear functional $A : L \to \mathbb{R}$ with $A(h) > 0$ we have the inequality

$$0 \leq A\left(hf^2\right) A\left(hg^2\right) - [A(hfg)]^2 \leq \frac{1}{4}(M - m)^2 A^2\left(hg^2\right). \tag{3.3}$$

The constant $\frac{1}{4}$ in (3.3) is sharp.

Proof. We observe that

$$(Mg - f, f - mg)_{A,h} = A[h(Mg - f)(f - mg)] \geq 0.$$

Applying Theorem 1 for $(\cdot, \cdot)_{A,h}$ we get

$$0 \leq (f, f)_{A,h} (g, g)_{A,h} - (f, g)_{A,h}^2 \leq \frac{1}{4}(M - m)^2 (g, g)_{A,h}^2,$$

which is clearly equivalent to (3.3). ∎

The following corollary holds.

Corollary 1. *Let $f, g \in F(T)$ such that fg, $f^2, g^2 \in F(T)$. If m, M are real numbers such that (3.2) holds, then*

$$0 \leq A\left(f^2\right) A\left(g^2\right) - A^2(fg) \leq \frac{1}{4}(M - m)^2 A^2\left(g^2\right). \tag{3.4}$$

The constant $\frac{1}{4}$ is sharp in (3.4).

Remark 1. *The condition (3.2) may be replaced with the weaker assumption*

$$(Mg-f,f-mg)_{A,h}\geq 0. \tag{3.5}$$

4. Applications for Integrals

Let (Ω,Σ,μ) be a measure space consisting of a set Ω, a $\sigma-$algebra Σ of subsets of Ω and a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$.

Denote by $L^2_\rho(\Omega,\mathbb{K})$ the Hilbert space of all $\mathbb{K}$-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(t)\left|f(s)\right|^2 d\mu(s)<\infty$, where $\rho:\Omega\to[0,\infty)$ is a measurable function on Ω.

The following proposition contains a counterpart of the weighted Cauchy-Buniakowsky-Schwarz's integral inequality.

Proposition 2. *Let $A,a\in\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $f,g\in L^2_\rho(\Omega,\mathbb{K})$. If*

$$\int_\Omega \operatorname{Re}\left[\left(Ag(s)-f(s)\right)\left(\overline{f(s)}-\overline{a}\,\overline{g}(s)\right)\right]\rho(s)\,d\mu(s)\geq 0 \tag{4.1}$$

or, equivalently,

$$\int_\Omega \rho(s)\left|f(s)-\frac{a+A}{2}g(s)\right|^2 d\mu(s)\leq\frac{1}{4}\left|A-a\right|^2\int_\Omega\rho(s)\left|g(s)\right|^2 d\mu(s),$$

then one has the inequality

$$\begin{aligned}0&\leq\int_\Omega\rho(s)\left|f(s)\right|^2 d\mu(s)\int_\Omega\rho(s)\left|g(s)\right|^2 d\mu(s)-\left|\int_\Omega\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^2 \\ &\leq\frac{1}{4}\left|A-a\right|^2\left(\int_\Omega\rho(s)\left|g(s)\right|^2 d\mu(s)\right)^2.\end{aligned}\tag{4.2}$$

Proof. Follows by Theorem 1 applied for the inner product $\langle\cdot,\cdot\rangle_\rho:=L^2_\rho(\Omega,\mathbb{K})\times L^2_\rho(\Omega,\mathbb{K})\to\mathbb{K}$,

$$\langle f,g\rangle_\rho:=\int_\Omega\rho(s)f(s)\overline{g(s)}d\mu(s).$$

∎

Remark 2. *A sufficient condition for (4.1) to hold is*

$$\operatorname{Re}\left[\left(Ag(s)-f(s)\right)\left(\overline{f(s)}-\overline{a}\,\overline{g}(s)\right)\right]\geq 0\quad for\quad \mu-a.e.\ s\in\Omega. \tag{4.3}$$

In the particular case $\rho=1$, we have the following counterpart of the Cauchy-Buniakowsky-Schwarz inequality.

Corollary 2. *Let $a,A\in\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $f,g\in L^2_\rho(\Omega,\mathbb{K})$. If*

$$\int_\Omega \operatorname{Re}\left[\left(Ag(s)-f(s)\right)\left(\overline{f(s)}-\overline{a}\,\overline{g}(s)\right)\right]d\mu(s)\geq 0, \tag{4.4}$$

or, equivalently

$$\int_\Omega\left|f(s)-\frac{a+A}{2}g(s)\right|^2 d\mu(s)\leq\frac{1}{4}\left|A-a\right|^2\int_\Omega\left|g(s)\right|^2 d\mu(s),$$

then one has the inequality

$$
\begin{aligned}
(4.5)\qquad 0 &\leq \int_{\Omega} |f(s)|^2 \, d\mu(s) \int_{\Omega} |g(s)|^2 \, d\mu(s) - \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) \right|^2 \\
&\leq \frac{1}{4} |A-a|^2 \left(\int_{\Omega} |g(s)|^2 \, d\mu(s) \right)^2 .
\end{aligned}
$$

Remark 3. *If $\mathbb{K} = \mathbb{R}$, then a sufficient condition for either (4.1) or (4.4) to hold is*

$$
ag(s) \leq f(s) \leq Ag(s) \quad \textit{for} \quad \mu - a.e. \ s \in \Omega, \tag{4.6}
$$

where, in this case, $a, A \in \mathbb{R}$ with $A > a$.

5. Applications for Sequences

For a given sequence $(w_i)_{i\in\mathbb{N}}$ of nonnegative real numbers, consider the Hilbert space $\ell_w^2(\mathbb{K})$, $(\mathbb{K} = \mathbb{C}, \mathbb{R})$, where

$$
\ell_w^2(\mathbb{K}) := \left\{ \overline{x} = (x_i)_{i\in\mathbb{N}} \subset \mathbb{K} \left| \sum_{i=0}^{\infty} w_i |x_i|^2 < \infty \right. \right\}. \tag{5.1}
$$

The following proposition that provides a counterpart of the weighted Cauchy-Bunyakowsky-Schwarz inequality for complex numbers holds.

Proposition 3. *Let $a, A \in \mathbb{K}$ and $\overline{x}, \overline{y} \in \ell_w^2(\mathbb{K})$. If*

$$
\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[(Ay_i - x_i)\left(\overline{x_i} - \overline{a}\ \overline{y_i}\right)\right] \geq 0, \tag{5.2}
$$

then one has the inequality

$$
0 \leq \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 - \left| \sum_{i=0}^{\infty} w_i x_i \overline{y_i} \right|^2 \leq \frac{1}{4} |A-a|^2 \left(\sum_{i=0}^{\infty} w_i |y_i|^2 \right)^2 . \tag{5.3}
$$

The constant $\frac{1}{4}$ is sharp.

Proof. Follows by Theorem 1 applied for the inner product $\langle \cdot, \cdot \rangle_w : \ell_w^2(\mathbb{K}) \times \ell_w^2(\mathbb{K}) \to \mathbb{K}$,

$$
\langle \overline{x}, \overline{y} \rangle_w := \sum_{i=0}^{\infty} w_i x_i \overline{y_i}.
$$

∎

Remark 4. *A sufficient condition for (5.2) to hold is*

$$
\operatorname{Re}\left[(Ay_i - x_i)\left(\overline{x_i} - \overline{a y_i}\right)\right] \geq 0 \quad \textit{for all } i \in \mathbb{N}. \tag{5.4}
$$

In the particular case $w_i = 1$, $i \in \mathbb{N}$, we have the following counterpart of the Cauchy-Bunyakowsky-Schwarz inequality.

Corollary 3. *Let $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $\overline{x}, \overline{y} \in \ell^2(\mathbb{K})$. If*

$$
\sum_{i=0}^{\infty} \operatorname{Re}\left[(Ay_i - x_i)\left(\overline{x_i} - \overline{a y_i}\right)\right] \geq 0, \tag{5.5}
$$

then one has the inequality

$$0 \leq \sum_{i=0}^{\infty} |x_i|^2 \sum_{i=0}^{\infty} |y_i|^2 - \left| \sum_{i=0}^{\infty} x_i \overline{y_i} \right|^2 \leq \frac{1}{4} |A - a|^2 \left(\sum_{i=0}^{\infty} |y_i|^2 \right)^2 . \tag{5.6}$$

Remark 5. *If $\mathbb{K} = \mathbb{R}$, then a sufficient condition for either (5.2) or (5.5) to hold is*

$$ay_i \leq x_i \leq Ay_i \quad \text{for each} \quad i \in \mathbb{N}, \tag{5.7}$$

with $A > a$.

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School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, MCMC 8001, Victoria, Australia.

E-mail address: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html