NEW REVERSES OF SCHWARZ, TRIANGLE AND BESSEL INEQUALITIES IN INNER PRODUCT SPACES

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Abstract. New reverses of the Schwarz, triangle and Bessel inequalities in inner product spaces are pointed out. These results complement the recent ones obtained by the author in the earlier paper [13]. Further, they are employed to establish new Grüss type inequalities. Finally, some natural integral inequalities are stated as well.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$.

In the earlier paper [13], we have obtained the following simple reverse of Schwarz's inequality

$$\begin{aligned} 0 &\leq \|x\|^2\|a\|^2 - |\langle x,a\rangle|^2 \\ &\leq \|x\|^2\|a\|^2 - [\operatorname{Re}\langle x,a\rangle]^2 \leq r^2\|x\|^2, \end{aligned} \tag{1.1}$$

provided

$$\|x-a\| \leq r < \|a\|, \tag{1.2}$$

where $a, x \in H$ and $r > 0$. The constant $c = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller one.

This result has then been employed to prove (see [13]) that

$$\begin{aligned} \|x\|^2\|y\|^2 &\leq \frac{1}{4}\cdot\frac{\left\{\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\langle x,y\rangle\right]\right\}^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \\ &\leq \frac{1}{4}\cdot\frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}|\langle x,y\rangle|^2, \end{aligned} \tag{1.3}$$

provided, for $x, y \in H$ and $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, either

$$\operatorname{Re}\langle\Gamma y - x, x - \gamma y\rangle \geq 0, \tag{1.4}$$

or, equivalently,

$$\left\|x - \frac{\gamma+\Gamma}{2}y\right\| \leq \frac{1}{2}|\Gamma-\gamma|\,\|y\|, \tag{1.5}$$

holds. In both inequalities (1.3), $\frac{1}{4}$ is the best possible constant.

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The inequality (1.3) implies the following additive version of reverse Schwarz's inequality

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} |\langle x, y\rangle|^2 . \tag{1.6}$$

Here the constant $\frac{1}{4}$ is also the best.

If the condition (1.2) is satisfied, one may deduce the following reverse of the triangle inequality [13]

$$0 \leq \|x\| + \|a\| - \|x + a\| \leq \sqrt{2}r \sqrt{\frac{\operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}\left(\sqrt{\|a\|^2 - r^2} + \|a\|\right)}}. \tag{1.7}$$

If $M > m > 0$, $x, y \in H$ and either (1.4) or, equivalently, (1.5) holds for M, m instead of Γ, γ, then the following simpler reverse of the triangle inequality may be stated as well

$$0 \leq \|x\| + \|y\| - \|x + y\| \leq \frac{\sqrt{M} - \sqrt{m}}{\sqrt[4]{Mm}} \sqrt{\operatorname{Re}\langle x, y\rangle}. \tag{1.8}$$

Moving now onto Grüss type inequalities, we note that if $x, y, e \in H$, with $\|e\| = 1$ and $r_1, r_2 \in (0, 1)$ are such that

$$\|x - e\| \leq r_1, \quad \text{and} \quad \|y - e\| \leq r_2, \tag{1.9}$$

then one has the inequality [13]

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq r_1 r_2 \|x\| \|y\| . \tag{1.10}$$

The inequality (1.10) is sharp in the sense that the constant $c = 1$ in front of $r_1 r_2$ cannot be replaced by a smaller constant.

If we assumed that, for $x, y, e \in H$ with $\|e\| = 1$ and $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma})$, $\operatorname{Re}\left(\Phi\bar{\phi}\right) > 0$, either the condition

$$\operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle, \ \operatorname{Re}\langle \Phi e - y, y - \gamma\phi\rangle \geq 0 \tag{1.11}$$

or, equivalently,

$$\left\|x - \frac{\gamma + \Gamma}{2} e\right\| \leq \frac{1}{2} |\Gamma - \gamma| , \qquad \left\|y - \frac{\phi + \Phi}{2} e\right\| \leq \frac{1}{2} |\Phi - \phi| , \tag{1.12}$$

holds, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma| |\Phi - \phi|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma}) \operatorname{Re}\left(\Phi\bar{\phi}\right)}} |\langle x, e\rangle \langle e, y\rangle| . \tag{1.13}$$

Here the constant $\frac{1}{4}$ is also best possible.

In the case that both $\langle x, e\rangle$, $\langle e, y\rangle \neq 0$ (which is actually the interesting case), we have

$$\left|\frac{\langle x, y\rangle}{\langle x, e\rangle \langle e, y\rangle} - 1\right| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma| |\Phi - \phi|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma}) \operatorname{Re}\left(\Phi\bar{\phi}\right)}}. \tag{1.14}$$

Now, for an orthornormal family of vectors in H, i.e., we recall that $\langle e_i, e_j\rangle = 0$ if $i, j \in \mathbb{N}$, $i \neq j$ and $\|e_i\| = 1$ for $i \in \mathbb{N}$, the following inequality, called the *Bessel* *inequality*

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2, \qquad x \in H; \tag{1.15}$$

holds.

If $(H; \langle\cdot,\cdot\rangle)$ is an infinite dimensional Hilbert space over the real or complex number field $\mathbb{K}$, $(e_i)_{i\in\mathbb{N}}$ is an orthornormal family in H, $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$ and $r > 0$ is with the property that

$$\sum_{i=1}^{\infty} |\lambda_i|^2 > r^2, \tag{1.16}$$

then, for $x \in H$ with

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \leq r, \tag{1.17}$$

one has the inequalities [13]

$$\|x\|^2 \leq \frac{\left(\sum_{i=1}^{\infty} \operatorname{Re}\left[\bar{\lambda}_i \langle x, e_i\rangle\right]\right)^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \leq \frac{\left|\sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i\rangle\right|^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \tag{1.18}$$

$$\leq \frac{\sum_{i=1}^{\infty} |\lambda_i|^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 .$$

An additive version of interest is [13]

$$0 \leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \frac{r^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 . \tag{1.19}$$

Finally, if $\boldsymbol{\Gamma} = (\Gamma_i)_{i\in\mathbb{N}}$, $\boldsymbol{\gamma} = (\gamma_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$ are such that $\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \overline{\gamma_i}) > 0$ and for $x \in H$, either

$$\left\| x - \sum_{i=1}^{\infty} \frac{\gamma_i + \Gamma_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \tag{1.20}$$

or, equivalently,

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0 \tag{1.21}$$

holds, then [13]

$$\|x\|^2 \leq \frac{1}{4} \cdot \frac{\left(\sum_{i=1}^{\infty} \operatorname{Re}\left[\left(\bar{\Gamma}_i + \bar{\gamma}_i\right) \langle x, e_i\rangle\right]\right)^2}{\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \bar{\gamma}_i)} \leq \frac{1}{4} \cdot \frac{\left|\sum_{i=1}^{\infty} \left(\bar{\Gamma}_i + \bar{\gamma}_i\right) \langle x, e_i\rangle\right|^2}{\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \bar{\gamma}_i)} \tag{1.22}$$

$$\leq \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2}{\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \bar{\gamma}_i)} \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 .$$

The constant $\frac{1}{4}$ is best possible in all inequalities (1.22).

The following additive version may be stated as well [13]

$$0 \leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \bar{\gamma}_i)} \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 . \tag{1.23}$$

Here the constant $\frac{1}{4}$ is also best possible.

The present paper is a continuation of [13]. Here we point out different reverses of the Schwarz, triangle and Bessel inequalities that are also sharp. Applications for Grüss type inequalities are provided. Some integral inequalities that are natural consequences of the above, are stated as well.

2. New Reverses of Schwarz's Inequality

The following result holds.

Theorem 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x, a \in H$ and $r > 0$. If*

$$x \in \bar{B}(a, r) := \{z \in H | \|z - a\| \leq r\}, \tag{2.1}$$

then we have the inequalities:

$$\begin{aligned} 0 &\leq \|x\| \|a\| - |\langle x, a\rangle| \leq \|x\| \|a\| - |\operatorname{Re}\langle x, a\rangle| \\ &\leq \|x\| \|a\| - \operatorname{Re}\langle x, a\rangle \leq \frac{1}{2} r^2. \end{aligned} \tag{2.2}$$

The constant $\frac{1}{2}$ is best possible in (2.2) in the sense that it cannot be replaced by a smaller constant.

Proof. The condition (2.1) is clearly equivalent to

$$\|x\|^2 + \|a\|^2 \leq 2 \operatorname{Re}\langle x, a\rangle + r^2. \tag{2.3}$$

Using the elementary inequality

$$2 \|x\| \|a\| \leq \|x\|^2 + \|a\|^2, \quad a, x \in H \tag{2.4}$$

and (2.3), we deduce

$$2 \|x\| \|a\| \leq 2 \operatorname{Re}\langle x, a\rangle + r^2, \tag{2.5}$$

giving the last inequality in (2.2). The other inequalities are obvious.

To prove the sharpness of the constant $\frac{1}{2}$, assume that

$$0 \leq \|x\| \|a\| - \operatorname{Re}\langle x, a\rangle \leq c r^2 \tag{2.6}$$

for any $x, a \in H$ and $r > 0$ satisfying (2.1).

Assume that $a, e \in H$, $\|a\| = \|e\| = 1$ and $e \perp a$. If $r = \sqrt{\varepsilon}$, $\varepsilon > 0$ and if we define $x = a + \sqrt{\varepsilon} e$, then $\|x - a\| = \sqrt{\varepsilon} = r$ showing that the condition (2.1) is fulfilled.

On the other hand,

$$\begin{aligned} \|x\| \|a\| - \operatorname{Re}\langle x, a\rangle &= \sqrt{\left\|a + \sqrt{\varepsilon} e\right\|^2} - \operatorname{Re}\left\langle a + \sqrt{\varepsilon} e, a\right\rangle \\ &= \sqrt{\|a\|^2 + \varepsilon \|e\|^2} - \|a\|^2 \\ &= \sqrt{1 + \varepsilon} - 1. \end{aligned}$$

Utilising (2.6), we conclude that

$$\sqrt{1 + \varepsilon} - 1 \leq c\varepsilon \quad \text{for any } \varepsilon > 0. \tag{2.7}$$

Multiplying (2.7) by $\sqrt{1+\varepsilon}+1>0$ and then dividing by $\varepsilon>0$, we get

$$\left(\sqrt{1+\varepsilon}+1\right)c\geq 1 \quad \text{for any} \quad \varepsilon>0. \tag{2.8}$$

Letting $\varepsilon\to 0+$ in (2.8), we deduce $c\geq\frac{1}{2}$, and the theorem is proved. ∎

The following result also holds.

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x,y\in H$, $\gamma,\Gamma\in\mathbb{K}$ $(\Gamma\neq-\gamma)$ so that either*

$$\operatorname{Re}\langle\Gamma y-x,x-\gamma y\rangle\geq 0, \tag{2.9}$$

or, equivalently,

$$\left\|x-\frac{\gamma+\Gamma}{2}y\right\|\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left\|y\right\| \tag{2.10}$$

holds. Then we have the inequalities

$$\begin{aligned}0&\leq\|x\|\|y\|-|\langle x,y\rangle|\\&\leq\|x\|\|y\|-\left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x,y\rangle\right]\right|\\&\leq\|x\|\|y\|-\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x,y\rangle\right]\\&\leq\frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\|y\|^2.\end{aligned} \tag{2.11}$$

The constant $\frac{1}{4}$ in the last inequality is best possible.

Proof. The proof of the equivalence between the inequalities (2.9) and (2.10) follows by the fact that in an inner product space, $\operatorname{Re}\langle Z-x,x-z\rangle\geq 0$ for $x,z,Z\in H$ is equivalent to

$$\left\|x-\frac{z+Z}{2}\right\|\leq\frac{1}{2}\|Z-z\|,$$

(see for example [9]).

Consider for $a,y\neq 0$, $a=\frac{\Gamma+\gamma}{2}\cdot y$ and $r=\frac{1}{2}|\Gamma-\gamma|\|y\|$. Thus from (2.2), we get

$$\begin{aligned}0&\leq\|x\|\left|\frac{\Gamma+\gamma}{2}\right|\|y\|-\left|\frac{\Gamma+\gamma}{2}\right||\langle x,y\rangle|\\&\leq\|x\|\left|\frac{\Gamma+\gamma}{2}\right|\|y\|-\left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x,y\rangle\right]\right|\\&\leq\|x\|\left|\frac{\Gamma+\gamma}{2}\right|\|y\|-\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x,y\rangle\right]\\&\leq\frac{1}{8}\cdot|\Gamma-\gamma|^2\|y\|^2.\end{aligned}$$

Dividing by $\frac{1}{2}|\Gamma+\gamma|\geq 0$, we deduce the desired inequality (2.11).

To prove the sharpness of the constant $\frac{1}{4}$, assume that there exists a $c>0$ such that:

$$\|x\|\|y\|-\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x,y\rangle\right]\leq c\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\|y\|^2, \tag{2.12}$$

provided either (2.9) or (2.10) holds.

Consider the real inner product space $\left(\mathbb{R}^2, \langle\cdot,\cdot\rangle\right)$ with $\langle\bar{\mathbf{x}},\bar{\mathbf{y}}\rangle = x_1y_1 + x_2y_2$, $\bar{\mathbf{x}} = (x_1, x_2)$, $\bar{\mathbf{y}} = (y_1, y_2) \in \mathbb{R}^2$. Let $y = (1,1)$ and $\Gamma, \gamma > 0$ with $\Gamma > \gamma$. Then, by (2.12), we deduce

$$\sqrt{2}\sqrt{x_1^2 + x_2^2} - (x_1 + x_2) \leq 2c \cdot \frac{(\Gamma-\gamma)^2}{\Gamma+\gamma}. \tag{2.13}$$

If $x_1 = \Gamma$, $x_2 = \gamma$, then

$$\langle\Gamma\bar{\mathbf{y}} - \bar{\mathbf{x}}, \bar{\mathbf{x}} - \gamma\bar{\mathbf{y}}\rangle = (\Gamma - x_1)(x_1 - \gamma) + (\Gamma - x_2)(x_2 - \gamma) = 0,$$

showing that the condition (2.9) is valid. Replacing x_1 and x_2 in (2.13), we deduce

$$\sqrt{2}\sqrt{\Gamma^2 + \gamma^2} - (\Gamma + \gamma) \leq 2c\frac{(\Gamma-\gamma)^2}{\Gamma+\gamma}. \tag{2.14}$$

If in (2.14) we choose $\Gamma = 1 + \varepsilon$, $\gamma = 1 - \varepsilon$ with $\varepsilon \in (0,1)$, then we have

$$2\sqrt{1+\varepsilon^2} - 2 \leq 2c\frac{4\varepsilon^2}{2},$$

giving

$$\sqrt{1+\varepsilon^2} - 1 \leq 2c\varepsilon^2. \tag{2.15}$$

Finally, multiplying (2.15) with $\sqrt{1+\varepsilon^2}+1 > 0$ and thus dividing by ε^2, we deduce

$$1 \leq 2c\left(\sqrt{1+\varepsilon^2} + 1\right) \quad \text{for any} \quad \varepsilon \in (0,1). \tag{2.16}$$

Letting $\varepsilon \to 0+$ in (2.16) we get $c \geq \frac{1}{4}$, and the sharpness of the constant is proved. ∎

For some recent results in connection to Schwarz's inequality, see [2], [14] and [16].

3. Reverses of the Triangle Inequality

The following reverse of the triangle inequality in inner product spaces holds.

Proposition 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x, a \in H$ and $r > 0$. If $\|x - a\| \leq r$, then we have the inequality*

$$0 \leq \|x\| + \|a\| - \|x + a\| \leq r. \tag{3.1}$$

Proof. Since

$$(\|x\| + \|a\|)^2 - \|x + a\|^2 \leq 2(\|x\|\|a\| - \operatorname{Re}\langle x, a\rangle),$$

then by Theorem 1 we deduce

$$(\|x\| + \|a\|)^2 - \|x + a\|^2 \leq r^2, \tag{3.2}$$

from where we obtain

$$\|x\| + \|a\| \leq \sqrt{r^2 + \|x + a\|^2} \leq r + \|x + a\|, \tag{3.3}$$

giving the desired result (3.1). ∎

We may state the following result.

Proposition 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x,y\in H$, $M>m>0$ such that either*

$$\operatorname{Re}\langle My-x,x-my\rangle\geq 0, \tag{3.4}$$

or, equivalently,

$$\left\|x-\frac{M+m}{2}y\right\|\leq\frac{1}{2}(M-m)\|y\|, \tag{3.5}$$

holds. Then we have the inequality

$$0\leq\|x\|+\|y\|-\|x+y\|\leq\frac{\sqrt{2}}{2}\cdot\frac{(M-m)}{\sqrt{M+m}}\|y\|. \tag{3.6}$$

Proof. By Theorem 2 for $\Gamma=M$, $\gamma=m$, we have the inequality

$$\|x\|\|y\|-\operatorname{Re}\langle x,y\rangle\leq\frac{1}{4}\cdot\frac{(M-m)^2}{(M+m)}\|y\|^2. \tag{3.7}$$

Then we may state that

$$\begin{aligned}(\|x\|+\|y\|)^2-\|x+y\|^2&=2(\|x\|\|y\|-\operatorname{Re}\langle x,y\rangle)\\&\leq\frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\|y\|^2,\end{aligned}$$

from where we get

$$\begin{aligned}\|x\|+\|y\|&\leq\sqrt{\frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\|y\|^2+\|x+y\|^2}\\&\leq\|x+y\|+\frac{(M-m)}{\sqrt{2(M+m)}}\|y\|,\end{aligned} \tag{3.8}$$

giving the desired inequality (3.6). ∎

For some results related to triangle inequality in inner product spaces, see [3], [18], [19] and [20].

4. Some Grüss Type Inequalities

We may state the following result.

Theorem 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x,y,e\in H$ with $\|e\|=1$. If $r_1,r_2>0$ and*

$$\|x-e\|\leq r_1,\qquad\|y-e\|\leq r_2, \tag{4.1}$$

then we have the inequalities

$$\begin{aligned}|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|&\leq\frac{1}{2}r_1r_2\sqrt{\|x\|+|\langle x,e\rangle|}\cdot\sqrt{\|y\|+|\langle y,e\rangle|}\\&\leq r_1r_2\|x\|\|y\|.\end{aligned} \tag{4.2}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Apply Schwarz's inequality for the vectors $x - \langle x, e\rangle e$, $y - \langle y, e\rangle e$ to get (see also [9]) that:

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \leq \left(\|x\|^2 - |\langle x, e\rangle|^2\right)\left(\|y\|^2 - |\langle y, e\rangle|^2\right). \tag{4.3}$$

Using Theorem 1 for $a = e$, we have

$$\begin{aligned} 0 &\leq \|x\|^2 - |\langle x, e\rangle|^2 \\ &= (\|x\| - |\langle x, e\rangle|)(\|x\| + |\langle x, e\rangle|) \\ &\leq \frac{1}{2} r_1^2 (\|x\| + |\langle x, e\rangle|) \leq r_1^2 \|x\|, \end{aligned} \tag{4.4}$$

and, in a similar way

$$\begin{aligned} 0 &\leq \|y\|^2 - |\langle y, e\rangle|^2 \\ &\leq \frac{1}{2} r_2^2 (\|y\| + |\langle y, e\rangle|) \leq r_2^2 \|y\|. \end{aligned} \tag{4.5}$$

Utilising (4.3) – (4.5), we may state that

$$\begin{aligned} |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 &\leq \frac{1}{4} r_1^2 r_2^2 (\|x\| + |\langle x, e\rangle|)(\|y\| + |\langle y, e\rangle|) \\ &\leq r_1^2 r_2^2 \|x\| \|y\|, \end{aligned} \tag{4.6}$$

giving the desired inequality (4.2).

To prove the sharpness of the constant $\frac{1}{2}$, let assume $x = y$ in (4.2), to get

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \frac{1}{2} r_1^2 (\|x\| + |\langle x, e\rangle|), \tag{4.7}$$

provided $\|x - e\| \leq r_1$. If $x \neq 0$, then dividing (4.7) with $\|x\| + |\langle x, e\rangle| > 0$ we get

$$\|x\| - |\langle x, e\rangle| \leq \frac{1}{2} r_1^2 \tag{4.8}$$

provided $\|x - e\| \leq r_1$, $\|e\| = 1$. However, (4.8) is in fact (2.2) for $a = e$, for which we have shown that $\frac{1}{2}$ is the best possible constant. ∎

The following result also holds.

Theorem 4. *With the assumptions of Theorem 3, we have the inequality*

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq r_1 r_2 \sqrt{\frac{1}{4} r_1^2 + |\langle x, e\rangle|} \cdot \sqrt{\frac{1}{4} r_2^2 + |\langle y, e\rangle|}. \tag{4.9}$$

Proof. Note that, from Theorem 2, we have

$$\|x\| \|a\| \leq |\langle x, a\rangle| + \frac{1}{2} r^2 \tag{4.10}$$

provided $\|x - a\| \leq r$.

Taking the square in (4.10) and arranging the terms, we obtain:

$$0 \leq \|x\|^2 \|a\|^2 - |\langle x, a\rangle|^2 \leq r^2 \left(\frac{1}{4} r^2 + |\langle x, a\rangle|\right), \tag{4.11}$$

provided $\|x - a\| \leq r$.

Using the assumption of the theorem, we then have

$$0 \leq \|x\|^2 - |\langle x, e\rangle|^2 \leq r_1^2 \left(\frac{1}{4} r_1^2 + |\langle x, e\rangle|\right), \tag{4.12}$$

and

$$0 \leq \|y\|^2 - |\langle y, e\rangle|^2 \leq r_2^2 \left(\frac{1}{4} r_2^2 + |\langle y, e\rangle|\right). \tag{4.13}$$

Utilising (4.3), (4.12) and (4.13), we deduce the desired inequality (4.9). ∎

The following result may be stated as well.

Theorem 5. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x, y, e \in H$ with $\|e\| = 1$. Suppose also that $a, A, b, B \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) so that $A \neq -a, B \neq -b$. If either*

$$\operatorname{Re}\langle Ae - x, x - ae\rangle \geq 0, \quad \operatorname{Re}\langle Be - y, y - be\rangle \geq 0, \tag{4.14}$$

or, equivalently,

$$\left\|x - \frac{a+A}{2}e\right\| \leq \frac{1}{2}|A - a|, \quad \left\|y - \frac{b+B}{2}e\right\| \leq \frac{1}{2}|B - b|, \tag{4.15}$$

holds, then we have the inequality

$$\begin{aligned} &|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ &\leq \frac{1}{4} \cdot \frac{|A - a|\,|B - b|}{\sqrt{|A + a|\,|B + b|}} \sqrt{\|x\| + |\langle x, e\rangle|} \cdot \sqrt{\|y\| + |\langle y, e\rangle|} \\ &\leq \frac{1}{2} \cdot \frac{|A - a|\,|B - b|}{\sqrt{|A + a|\,|B + b|}} \sqrt{\|x\|\,\|y\|}. \end{aligned} \tag{4.16}$$

The constant $\frac{1}{4}$ is best possible in (4.16).

Proof. From Theorem 2, we may state that

$$\begin{aligned} 0 &\leq \|x\|^2 - |\langle x, e\rangle|^2 \\ &= (\|x\| - |\langle x, e\rangle|)(\|x\| + |\langle x, e\rangle|) \\ &\leq \frac{1}{4} \cdot \frac{|A - a|^2}{|A + a|}(\|x\| + |\langle x, e\rangle|), \end{aligned} \tag{4.17}$$

and

$$0 \leq \|y\|^2 - |\langle y, e\rangle|^2 \leq \frac{1}{4} \cdot \frac{|B - b|^2}{|B + b|}(\|y\| + |\langle y, e\rangle|). \tag{4.18}$$

Making use of (4.3) and (4.17), (4.18), we deduce the first inequality in (4.16).
The best constant follows by the use of Theorem 2, and we omit the details. ∎

Finally, we may state the following theorem as well.

Theorem 6. *With the assumptions of Theorem 5, we have the inequality*

$$\begin{aligned} &|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ &\leq \frac{1}{2} \cdot \frac{|A - a|\,|B - b|}{\sqrt{|A + a|\,|B + b|}} \sqrt{\frac{1}{8} \cdot \frac{|A - a|^2}{|A + a|} + |\langle x, e\rangle|} \cdot \sqrt{\frac{1}{8} \cdot \frac{|B - b|^2}{|B + b|} + |\langle y, e\rangle|}. \end{aligned} \tag{4.19}$$

Proof. Using Theorem 1, we may state that

$$0 \leq \|x\| - |\langle x, e\rangle| \leq \frac{1}{4} \cdot \frac{|A-a|^2}{|A+a|}.$$

This inequality implies that

$$\|x\|^2 \leq |\langle x, e\rangle|^2 + \frac{1}{2} |\langle x, e\rangle| \cdot \frac{|A-a|^2}{|A+a|} + \frac{1}{16} \cdot \frac{|A-a|^4}{|A+a|^2}$$

giving

$$0 \leq \|x\|^2 - |\langle x, e\rangle|^2 \leq \frac{1}{2} \cdot \frac{|A-a|^2}{|A+a|} \left[|\langle x, e\rangle| + \frac{1}{8} \cdot \frac{|A-a|^2}{|A+a|} \right]. \tag{4.20}$$

Similarly, we have

$$0 \leq \|y\|^2 - |\langle y, e\rangle|^2 \leq \frac{1}{2} \cdot \frac{|B-b|^2}{|B+b|} \left[|\langle y, e\rangle| + \frac{1}{8} \cdot \frac{|B-b|^2}{|B+b|} \right]. \tag{4.21}$$

By making use of (4.3) and (4.20), (4.21), we deduce the desired inequality (4.19). ∎

For some recent results on Grüss type inequalities in inner product spaces, see [4], [6] and [21].

5. Reverses of Bessel's Inequality

Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex infinite dimensional Hilbert space and $(e_i)_{i\in\mathbb{N}}$ an orthornormal family in H, i.e., we recall that $\langle e_i, e_j \rangle = 0$ if $i, j \in \mathbb{N}$, $i \neq j$ and $\|e_i\| = 1$ for $i \in \mathbb{N}$.

It is well known that, if $x \in H$, then the series $\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2$ is convergent and the following inequality, called *Bessel's inequality,*

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2, \tag{5.1}$$

holds.

If

$$\ell^2(\mathbb{K}) := \left\{ \mathbf{a} = (a_i)_{i\in\mathbb{N}} \subset \mathbb{K} \left| \sum_{i=1}^{\infty} |a_i|^2 < \infty \right. \right\},$$

where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, is the Hilbert space of all real or complex sequences that are 2−summable and $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$, then the series $\sum_{i=1}^{\infty} \lambda_i e_i$ is convergent in H and if $y := \sum_{i=1}^{\infty} \lambda_i e_i \in H$, then $\|y\| = \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}$.

We may state the following result.

Theorem 7. *Let $(H; \langle \cdot, \cdot \rangle)$ be an infinite dimensional Hilbert space over the real or complex number field $\mathbb{K}$, $(e_i)_{i\in\mathbb{N}}$ is an orthornormal family in H, $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$, $\boldsymbol{\lambda} \neq 0$ and $r > 0$. If $x \in H$ is such that*

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \leq r, \tag{5.2}$$

then we have the inequality

$$0 \leq \|x\| - \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{r^2}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}}. \tag{5.3}$$

The constant $\frac{1}{2}$ is best possible in (5.3) in the sense that it cannot be replaced by a smaller constant.

Proof. Let $a := \sum_{i=1}^{\infty} \lambda_i e_i \in H$. Then by Theorem 1, we have

$$\|x\| \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\| - \left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right| \leq \frac{1}{2} r^2,$$

giving

$$\|x\| \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} r^2 + \left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right|, \tag{5.4}$$

since

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\| = \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}.$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, we may state that

$$\left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right| \leq \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}}, \tag{5.5}$$

and thus, by (5.4) and (5.5), we may state that

$$\|x\| \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} r^2 + \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}},$$

from where we get the desired inequality in (5.3).

The best constant, follows by Theorem 1 on choosing $(e_i)_{i \in \mathbb{N}} = \{e\}$, with $\|e\| = 1$ and we omit the details. ∎

Remark 1. *Under the assumptions of Theorem 7, and if we multiply by $\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} > 0$, we deduce from (5.3), that*

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \\ &\leq \frac{1}{2} \cdot \frac{r^2 \left(\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \right)}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}} \\ &\leq \frac{r^2 \|x\|}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}}, \end{aligned} \tag{5.6}$$

where for the last inequality, we have used Bessel's inequality

$$\left(\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\leq\|x\|,\qquad x\in H.$$

The following result also holds.

Theorem 8. *Assume that* $(H;\langle\cdot,\cdot\rangle)$ *and* $(e_i)_{i\in\mathbb{N}}$ *are as in Theorem 7. If* $\mathbf{\Gamma}=(\Gamma_i)_{i\in\mathbb{N}}$, $\boldsymbol{\gamma}=(\gamma_i)_{i\in\mathbb{N}}\in\ell^2(\mathbb{K})$, *with* $\mathbf{\Gamma}\neq-\boldsymbol{\gamma}$, *and* $x\in H$ *with the property that, either*

$$\left\|x-\sum_{i=1}^{\infty}\frac{\Gamma_i+\gamma_i}{2}\cdot e_i\right\|\leq\frac{1}{2}\left(\sum_{i=1}^{\infty}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}\tag{5.7}$$

or, equivalently,

$$\operatorname{Re}\left\langle\sum_{i=1}^{\infty}\Gamma_ie_i-x,x-\sum_{i=1}^{\infty}\gamma_ie_i\right\rangle\geq0\tag{5.8}$$

holds, then we have the inequality

$$0\leq\|x\|-\left(\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\leq\frac{1}{4}\cdot\frac{\sum_{i=1}^{\infty}|\Gamma_i-\gamma_i|^2}{\left(\sum_{i=1}^{\infty}|\Gamma_i+\gamma_i|^2\right)^{\frac{1}{2}}}.\tag{5.9}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. Since $\mathbf{\Gamma}$, $\boldsymbol{\gamma}\in\ell^2(\mathbb{K})$, then we have that $\frac{1}{2}(\mathbf{\Gamma}\pm\boldsymbol{\gamma})\in\ell^2(\mathbb{K})$, showing that the series

$$\sum_{i=1}^{\infty}\left|\frac{\Gamma_i+\gamma_i}{2}\right|^2,\quad\sum_{i=1}^{\infty}\left|\frac{\Gamma_i-\gamma_i}{2}\right|^2$$

are convergent. In addition, the series $\sum_{i=1}^{\infty}\Gamma_ie_i$, $\sum_{i=1}^{\infty}\gamma_ie_i$ and $\sum_{i=1}^{\infty}\frac{\Gamma_i+\gamma_i}{2}e_i$ are also convergent in the Hilbert space H.

The equivalence of the conditions (5.7) and (5.8) follows by the fact that, in an inner product space we have, for $x,z,Z\in H$, $\operatorname{Re}\langle Z-x,x-z\rangle\geq0$ is equivalent to $\left\|x-\frac{z+Z}{2}\right\|\leq\frac{1}{2}\|Z-z\|$, and we omit the details.

Now, we observe that the inequality (5.9) follows from Theorem 7 on choosing $\lambda_i=\frac{\Gamma_i+\gamma_i}{2}$, $i\in\mathbb{N}$ and $r=\frac{1}{2}\left(\sum_{i=1}^{\infty}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}$.

The fact that $\frac{1}{4}$ is the best possible constant in (5.9) follows from Theorem 2, and we omit the details. ∎

Remark 2. *With the assumptions of Theorem 8, we have*

$$0 \le \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \tag{5.10}$$
$$\le \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\left(\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2\right)^{\frac{1}{2}}} \left[\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}}\right]$$
$$\le \frac{1}{2} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\left(\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2\right)^{\frac{1}{2}}} \|x\| .$$

For some recent results related to Bessel inequality, see [1], [5], [15], and [17].

6. Some Grüss Type Inequalities for Orthonormal Families

The following result holds.

Theorem 9. *Let $(H; \langle \cdot, \cdot \rangle)$ be an infinite dimensional Hilbert space over the real or complex number field $\mathbb{K}$ and $(e_i)_{i\in\mathbb{N}}$ an orthornormal family in H. If $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}}$, $\boldsymbol{\mu} = (\mu_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \neq 0$, $r_1, r_2 > 0$ and $x, y \in H$ are such that*

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \le r_1, \quad \left\| y - \sum_{i=1}^{\infty} \mu_i e_i \right\| \le r_2, \tag{6.1}$$

then we have the inequality

$$\left| \langle x, y\rangle - \sum_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle \right| \tag{6.2}$$
$$\le \frac{1}{2} r_1 r_2 \frac{\left[\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \left[\|y\| + \left(\sum_{i=1}^{\infty} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2\right)^{\frac{1}{4}} \left(\sum_{i=1}^{\infty} |\mu_i|^2\right)^{\frac{1}{4}}}$$
$$\le r_1 r_2 \frac{\|x\|^{\frac{1}{2}} \|y\|^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2\right)^{\frac{1}{4}} \left(\sum_{i=1}^{\infty} |\mu_i|^2\right)^{\frac{1}{4}}}.$$

Proof. Apply Schwarz's inequality for the vectors $x - \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$, $y - \sum_{i=1}^{\infty} \langle y, e_i\rangle e_i$, to get

$$\left| \left\langle x - \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i, y - \sum_{i=1}^{\infty} \langle y, e_i\rangle e_i \right\rangle \right|^2 \tag{6.3}$$
$$\le \left\| x - \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i \right\|^2 \left\| y - \sum_{i=1}^{\infty} \langle y, e_i\rangle e_i \right\|^2 .$$

Since

$$\left\langle x - \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i, y - \sum_{i=1}^{\infty} \langle y, e_i\rangle e_i \right\rangle = \langle x, y\rangle - \sum_{i=1}^{\infty} \langle x, e_i\rangle \langle e_i, y\rangle$$

and

$$\left\| x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 ,$$

then, by (6.3) and (5.6) applied for x and y, we deduce the desired inequality (6.2). ∎

Finally we may state the following theorem.

Theorem 10. *Assume that $(H; \langle \cdot, \cdot \rangle)$ and $(e_i)_{i \in \mathbb{N}}$ are as in Theorem 9. If* $\mathbf{\Gamma} = (\Gamma_i)_{i \in \mathbb{N}}$, $\mathbf{\Gamma} = (\Gamma_i)_{i \in \mathbb{N}}$, $\phi = (\boldsymbol{\phi}_i)_{i \in \mathbb{N}}$, $\mathbf{\Phi} = (\Phi_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$, *with* $\mathbf{\Gamma} \neq -\boldsymbol{\gamma}$, $\mathbf{\Phi} \neq -\boldsymbol{\phi}$, *and $x, y \in H$ are such that, either*

$$\left\| x - \sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}, \tag{6.4}$$

$$\left\| y - \sum_{i=1}^{\infty} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

or, equivalently,

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0, \tag{6.5}$$

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Phi_i e_i - y, y - \sum_{i=1}^{\infty} \phi_i e_i \right\rangle \geq 0, \tag{6.6}$$

holds, then we have the inequality

$$\begin{aligned} &\left| \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \right| \tag{6.7} \\ &\leq \frac{1}{4} \cdot \left(\sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\quad \times \frac{\left[\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[\|y\| + \left(\sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} |\Phi_i + \phi_i|^2 \right)^{\frac{1}{4}} \left(\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2 \right)^{\frac{1}{4}}} \\ &\leq \frac{1}{2} \cdot \frac{\left(\sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} |\Phi_i + \phi_i|^2 \right)^{\frac{1}{4}} \left(\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2 \right)^{\frac{1}{4}}} \|x\|^{\frac{1}{2}} \|y\|^{\frac{1}{2}} . \end{aligned}$$

The proof follows by (6.3) and by (5.10) applied for x and y. We omit the details.

7. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$. Let $\rho \geq 0$ be a $\mu-$measurable function on Ω with $\int_{\Omega} \rho(s) \, d\mu(s) = 1$. Denote by

$L_{\rho}^{2}(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions defined on Ω and $2-\rho-$integrable on Ω, i.e.,

$$\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)<\infty. \tag{7.1}$$

It is obvious that the following inner product

$$\left\langle f,g\right\rangle_{\rho}:=\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s), \tag{7.2}$$

generates the norm

$$\left\Vert f\right\Vert_{\rho}:=\left(\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\right)^{\frac{1}{2}}$$

of $L_{\rho}^{2}(\Omega,\mathbb{K})$, and all the above results may be stated for integrals.

It is important to observe that, if

$$\operatorname{Re}\left[f(s)\overline{g(s)}\right]\geq 0\quad\text{for}\quad\mu-\text{a.e.}\quad s\in\Omega, \tag{7.3}$$

then, obviously,

$$\begin{aligned}\operatorname{Re}\left\langle f,g\right\rangle_{\rho}&=\operatorname{Re}\left[\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right]\\&=\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\geq 0.\end{aligned} \tag{7.4}$$

The reverse is evidently not true in general.

Moreover, if the space is real, i.e., $\mathbb{K}=\mathbb{R}$, then a sufficient condition for (7.4) to hold is:

$$f(s)\geq 0,\quad g(s)\geq 0\quad\text{for}\quad\mu-\text{a.e.}\quad s\in\Omega. \tag{7.5}$$

We provide now, by the use of certain results obtained in Section 2, some integral inequalities that may be used in practical applications.

Proposition 3. *Let $f,g\in L_{\rho}^{2}(\Omega,\mathbb{K})$ and $r>0$ with the property that*

$$\left|f(s)-g(s)\right|\leq r\quad\textit{for}\quad\mu-\textit{a.e.}\quad s\in\Omega. \tag{7.6}$$

Then we have the inequalities

$$0 \leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right| \tag{7.7}$$
$$\leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_\Omega \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s)\right|$$
$$\leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \int_\Omega \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s)$$
$$\leq \frac{1}{2} r^2.$$

The constant $\frac{1}{2}$ is best possible in (7.7).

The proof follows by Theorem 1, and we omit the details.

Proposition 4. *Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and $\gamma, \Gamma \in \mathbb{K}$ so that $\Gamma \neq -\gamma$, and*

$$\operatorname{Re}\left[(\Gamma g(s) - f(s))\left(\overline{f(s)} - \overline{\gamma}\overline{g(s)}\right)\right] \geq 0, \quad \text{for } \mu - a.e. \quad s \in \Omega. \tag{7.8}$$

Then we have the inequalities

$$0 \leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right| \tag{7.9}$$
$$\leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right]\right|$$
$$\leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right]$$
$$\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \int_\Omega \rho(s)|g(s)|^2 d\mu(s).$$

The constant $\frac{1}{4}$ is best possible.

Remark 3. *If the space is real and we assume, for $M > m > 0$, that*

$$m g(s) \leq f(s) \leq M g(s) \quad \text{for } \mu - a.e. \quad s \in \Omega, \tag{7.10}$$

then by (7.9) we deduce the inequality:

$$
\begin{aligned}
(7.11) \qquad 0 &\leq \left[\int_\Omega \rho(s)\left|f(s)\right|^2 d\mu(s) \int_\Omega \rho(s)\left|g(s)\right|^2 d\mu(s)\right]^{\frac{1}{2}} \\
&\quad - \left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right| \\
&\leq \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\int_\Omega \rho(s)\left|g(s)\right|^2 d\mu(s).
\end{aligned}
$$

The constant $\frac{1}{4}$ is best possible.

The following reverse of the triangle inequality for integrals holds.

Proposition 5. *Assume that the functions $f, g \in L^2_\rho(\Omega, \mathbb{K})$ satisfy (7.10). Then we have the inequality*

$$
\begin{aligned}
(7.12) \qquad 0 &\leq \left(\int_\Omega \rho(s)\left|f(s)\right|^2 d\mu(s)\right)^{1/2} + \left(\int_\Omega \rho(s)\left|g(s)\right|^2 d\mu(s)\right)^{1/2} \\
&\quad - \left(\int_\Omega \rho(s)\left|f(s)+g(s)\right|^2 d\mu(s)\right)^{1/2} \\
&\leq \frac{\sqrt{2}}{2}\cdot\frac{(M-m)}{\sqrt{M+m}}\left(\int_\Omega \rho(s)\left|g(s)\right|^2 d\mu(s)\right)^{1/2}.
\end{aligned}
$$

The proof follows by Proposition 2.

By making use of Theorem 5, we may also state

Proposition 6. *Let $f, g, h \in L^2_\rho(\Omega, \mathbb{K})$ be so that $\int_\Omega \rho(s)\left|h(s)\right|^2 d\mu(s) = 1$. Suppose also that $a, A, b, B \in \mathbb{K}$ with $A \neq -a, B \neq -b$ and*

$$
\begin{aligned}
\operatorname{Re}\left[(Ah(s) - f(s))\left(\overline{f(s)} - \overline{a}\overline{h(s)}\right)\right] &\geq 0, \\
\operatorname{Re}\left[(Bh(s) - g(s))\left(\overline{g(s)} - \overline{b}\overline{h(s)}\right)\right] &\geq 0 \quad \text{for } \mu-a.e. \quad s \in \Omega.
\end{aligned}
$$

Then we have the inequality

$$
\begin{aligned}
&\left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s) - \int_\Omega \rho(s) f(s)\overline{h(s)} d\mu(s)\int_\Omega \rho(s) h(s)\overline{g(s)} d\mu(s)\right| \\
\leq\ & \frac{1}{4}\cdot\frac{|A-a|\,|B-b|}{\sqrt{|A+a|\,|B+b|}} \\
&\times\sqrt{\left(\int_\Omega \rho(s)\left|f(s)\right|^2 d\mu(s)\right)^{1/2} + \left|\int_\Omega \rho(s) f(s)\overline{h(s)} d\mu(s)\right|} \\
&\times\sqrt{\left(\int_\Omega \rho(s)\left|g(s)\right|^2 d\mu(s)\right)^{1/2} + \left|\int_\Omega \rho(s) g(s)\overline{h(s)} d\mu(s)\right|}.
\end{aligned}
$$

The constant $\frac{1}{4}$ is best possible.

Remark 4. *All the other inequalities in Sections 3 – 6 may be used in a similar manner to obtain the corresponding integral inequalities. We omit the details.*

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