ON PEČARIĆ'S INEQUALITY IN INNER PRODUCT SPACES

S.S. DRAGOMIR

Abstract. Some related results to Pečarić's inequality in inner product spaces that generalises Bombieri's inequality, are given.

1. Introduction

In 1992, J.E. Pečarić [3] proved the following inequality for vectors in complex inner product spaces $(H;(\cdot,\cdot))$.

Theorem 1. *Suppose that $x, y_1, \dots, y_n$ are vectors in H and $c_1, \dots, c_n$ are complex numbers. Then the following inequalities*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right), \tag{1.1}$$

hold.

He also showed that for $c_i = \overline{(x, y_i)}, i \in \{1, ..., n\}$, one gets

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \leq \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right), \tag{1.2}$$

which improves Bombieri's result [1] (see also [2, p. 394])

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right). \tag{1.3}$$

Note that (1.3) is in its turn a natural generalisation of *Bessel's inequality*

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2, \quad x \in H, \tag{1.4}$$

which holds for the orthornormal vectors $(e_i)_{1\leq i\leq n}$.

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In this paper we point out some related results to Pečarić's inequality (1.1). Some results of Bombieri type are also mentioned.

2. Preliminary Results

We start with the following lemma that is interesting in its own right.

Lemma 1. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \left(\sum_{i=1}^n |\alpha_i|^p \left(\sum_{j=1}^n |(z_i, z_j)| \right) \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |\alpha_i|^q \left(\sum_{j=1}^n |(z_i, z_j)| \right) \right)^{\frac{1}{q}} \tag{2.1}$$

$$\leq \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i|^2 \sum\limits_{i,j=1}^n |(z_i, z_j)|; \\ \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^n |\alpha_i|^{\gamma q} \right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}}, \\ \qquad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^n |\alpha_i|^{q} \right)^{\frac{1}{q}} \left(\sum\limits_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}}; \\ \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^n |\alpha_i|^{\alpha p} \right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\beta} \right)^{\frac{1}{\beta q}}, \\ \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left(\sum\limits_{i=1}^n |\alpha_i|^{\alpha p} \right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i=1}^n |\alpha_i|^{\gamma q} \right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}} \\ \quad \times \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}} \quad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ and } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum\limits_{i=1}^n |\alpha_i|^{q} \right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n |\alpha_i|^{\alpha p} \right)^{\frac{1}{\alpha p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}}, \\ \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^n |\alpha_i|^{p} \right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}} \left(\sum\limits_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^n |\alpha_i|^{p} \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n |\alpha_i|^{\gamma q} \right)^{\frac{1}{\gamma q}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}}, \\ \qquad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum\limits_{i=1}^n |\alpha_i|^{p} \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n |\alpha_i|^{q} \right)^{\frac{1}{q}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i, z_j)| \right), \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We observe that

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 = \left(\sum_{i=1}^{n}\alpha_i z_i, \sum_{j=1}^{n}\alpha_j z_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\left(z_i, z_j\right) \tag{2.2}$$
$$= \left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\left(z_i, z_j\right)\right| \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\alpha_i\right|\left|\alpha_j\right|\left|(z_i, z_j)\right| =: M.$$

If one uses the Hölder inequality for double sums, i.e., we recall it

$$\sum_{i,j=1}^{n} m_{ij}a_{ij}b_{ij} \leq \left(\sum_{i,j=1}^{n} m_{ij}a_{ij}^p\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n} m_{ij}b_{ij}^q\right)^{\frac{1}{q}}, \tag{2.3}$$

where $m_{ij}, a_{ij}, b_{ij} \geq 0$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$; then

$$M \leq \left(\sum_{i,j=1}^{n}\left|(z_i, z_j)\right|\left|\alpha_i\right|^p\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}\left|(z_i, z_j)\right|\left|\alpha_i\right|^q\right)^{\frac{1}{q}} \tag{2.4}$$
$$= \left(\sum_{i=1}^{n}\left|\alpha_i\right|^p\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right)\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\left|\alpha_i\right|^q\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right)\right)^{\frac{1}{q}},$$

and the first inequality in (2.1) is proved.

Observe that

$$\sum_{i=1}^{n}\left|\alpha_i\right|^p\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right) \leq \begin{cases} \max_{1\leq i\leq n}\left|\alpha_i\right|^p \sum_{i,j=1}^{n}\left|(z_i, z_j)\right|; \\ \left(\sum_{i=1}^{n}\left|\alpha_i\right|^{\alpha p}\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right)^{\beta}\right)^{\frac{1}{\beta}} \\ \qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum_{i=1}^{n}\left|\alpha_i\right|^p \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right); \end{cases}$$

giving

$$\left(\sum_{i=1}^{n}\left|\alpha_i\right|^p\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right)\right)^{\frac{1}{p}} \tag{2.5}$$
$$\leq \begin{cases} \max_{1\leq i\leq n}\left|\alpha_i\right|\left(\sum_{i,j=1}^{n}\left|(z_i, z_j)\right|\right)^{\frac{1}{p}}; \\ \left(\sum_{i=1}^{n}\left|\alpha_i\right|^{\alpha p}\right)^{\frac{1}{\alpha p}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right)^{\beta}\right)^{\frac{1}{\beta p}} \text{ if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left(\sum_{i=1}^{n}\left|\alpha_i\right|^p\right)^{\frac{1}{p}} \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right)^{\frac{1}{p}}. \end{cases}$$

Similarly, we have

$$\left(\sum_{i=1}^{n}\left|\alpha_i\right|^q\left(\sum_{j=1}^{n}\left|(z_i, z_j)\right|\right)\right)^{\frac{1}{q}} \tag{2.6}$$

$$\leq \begin{cases} \max_{1\leq i\leq n}|\alpha_i|\left(\sum_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{q}} \\ \left(\sum_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}} & \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum_{i=1}^{n}|\alpha_i|^{q}\right)^{\frac{1}{q}}\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{q}}. \end{cases}$$

Using (2.1) and (2.5) – (2.6), we deduce the 9 inequalities in the second part of (2.2). ∎

If we choose $p=q=2$, then the following result holds.

Corollary 1. *If $z_1,\dots,z_n\in H$ and $\alpha_1,\dots,\alpha_n\in\mathbb{K}$, then one has*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2\leq\sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{j=1}^{n}|(z_i,z_j)|\right) \tag{2.7}$$

$$\leq \begin{cases} \max_{1\leq i\leq n}|\alpha_i|^2\sum_{i,j=1}^{n}|(z_i,z_j)|; \\ \max_{1\leq i\leq n}|\alpha_i|\left(\sum_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \\ \qquad\qquad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \max_{1\leq i\leq n}|\alpha_i|\left(\sum_{i=1}^{n}|\alpha_i|^{2}\right)^{\frac{1}{2}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}; \\ \max_{1\leq i\leq n}|\alpha_i|\left(\sum_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left(\sum_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\left(\sum_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}} \\ \quad\times\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}} \quad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \text{ and } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum_{i=1}^{n}|\alpha_i|^{2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}} \\ \quad\times\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \max_{1\leq i\leq n}|\alpha_i|\left(\sum_{i=1}^{n}|\alpha_i|^{2}\right)^{\frac{1}{2}}\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\left(\sum_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}; \\ \left(\sum_{i=1}^{n}|\alpha_i|^{2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}} \\ \quad\times\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \qquad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \sum_{i=1}^{n}|\alpha_i|^2\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|(z_i,z_j)|\right). \end{cases}$$

3. Some Pečarić Type Inequalities

We are now able to point out the following result which complements and generalises the inequality (1.1) due to J. Pečarić.

Theorem 2. *Let $x, y_1, \ldots, y_n$ be vectors of an inner product space $(H; (\cdot, \cdot))$ and $c_1, \ldots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i(x, y_i)\right|^2 \leq \|x\|^2 \left(\sum_{i=1}^{n} |c_i|^p \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |c_i|^q \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right)^{\frac{1}{q}} \tag{3.1}$$

$$\leq \|x\|^2 \times \begin{cases} \max_{1\leq i\leq n} |c_i|^2 \sum_{i,j=1}^{n} |(y_i, y_j)|; \\ \max_{1\leq i\leq n} |c_i| \left(\sum_{i=1}^{n} |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{p}} \\ \quad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \qquad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max_{1\leq i\leq n} |c_i| \left(\sum_{i=1}^{n} |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{p}} \\ \quad \times \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{q}}; \\ \max_{1\leq i\leq n} |c_i| \left(\sum_{i=1}^{n} |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{q}} \\ \quad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \text{ if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left(\sum_{i=1}^{n} |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum_{i=1}^{n} |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}} \\ \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}} \quad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \qquad\qquad \text{and } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum_{i=1}^{n} |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{q}} \\ \quad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \max_{1\leq i\leq n} |c_i| \left(\sum_{i=1}^{n} |c_i|^{p}\right)^{\frac{1}{p}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{p}} \\ \quad \times \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{q}}; \\ \left(\sum_{i=1}^{n} |c_i|^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{p}} \\ \quad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \qquad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum_{i=1}^{n} |c_i|^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |c_i|^{q}\right)^{\frac{1}{q}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right); \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We note that

$$\sum_{i=1}^{n} c_i (x, y_i) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i\right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \left\|\sum_{i=1}^{n} \overline{c_i} y_i\right\|^2. \tag{3.2}$$

Finally, using Lemma 1 with $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \dots, n)$, we deduce the desired inequality (3.1). ∎

Remark 1. *If in (3.1) we choose $p = q = 2$, we obtain amongst others, the result* (1.1) *due to J. Pečarić.*

4. Some Results of Bombieri Type

The following results of Bombieri type hold.

Theorem 3. *Let $x, y_1, \dots, y_n \in H$. Then one has the inequality:*

$$
\begin{aligned}
(4.1)\quad \sum_{i=1}^{n} |(x, y_i)|^2 &\le \|x\| \left[\sum_{i=1}^{n} |(x, y_i)|^p \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right]^{\frac{1}{2p}} \left[\sum_{i=1}^{n} |(x, y_i)|^q \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right]^{\frac{1}{2q}} \\
&\le \|x\| \times \begin{cases}
\max_{1\le i\le n} |(x, y_i)| \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2}}; \\
\max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2p}} \\
\quad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{2\delta q}}, & \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\
\max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^{q}\right)^{\frac{1}{2q}} \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2p}} \\
\quad \times \max_{1\le i\le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2q}}; \\
\max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^{\alpha p}\right)^{\frac{1}{2\alpha\beta}} \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2q}} \\
\quad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, & \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\
\left(\sum_{i=1}^{n} |(x, y_i)|^{\alpha p}\right)^{\frac{1}{2\alpha p}} \left(\sum_{i=1}^{n} |(x, y_i)|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{2p\beta}} \\
\times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{2\delta q}} & \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\
& \text{and } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\
\left(\sum_{i=1}^{n} |(x, y_i)|^{q}\right)^{\frac{1}{2q}} \left(\sum_{i=1}^{n} |(x, y_i)|^{\alpha p}\right)^{\frac{1}{2\alpha p}} \max_{1\le i\le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2p}} \\
\quad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{\frac{1}{2p\beta}}, & \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\
\max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^{p}\right)^{\frac{1}{2p}} \max_{1\le i\le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2p}} \\
\quad \times \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2q}}; \\
\left(\sum_{i=1}^{n} |(x, y_i)|^{p}\right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} |(x, y_i)|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \max_{1\le i\le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2p}} \\
\quad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{\frac{1}{2\delta q}}, & \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\
\left(\sum_{i=1}^{n} |(x, y_i)|^{p}\right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} |(x, y_i)|^{q}\right)^{\frac{1}{2q}} \max_{1\le i\le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2}},
\end{cases}
\end{aligned}
$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof follows by Theorem 2 on choosing $c_i = \overline{(x, y_i)}, i \in \{1, ..., n\}$ and taking the square root in both sides of the inequalities involved. We omit the details. ∎

Remark 2. *We observe, by the last inequality in* (4.1)*, we get*

$$\frac{\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2}{\left(\sum_{i=1}^{n} |(x, y_i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |(x, y_i)|^q\right)^{\frac{1}{q}}} \leq \|x\|^2 \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right),$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1.$

If in this inequality we choose $p = q = 2$*, then we recapture Bombieri's result* (1.3).

References

[1] E. BOMBIERI, A note on the large sieve, *Acta Arith.,* **18** (1971), 401-404.

[2] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, 1993.

[3] J.E. PEČARIĆ, On some classical inequalities in unitary spaces, *Mat. Bilten* (Skopje), **16** (1992), 63-72.

School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, MCMC, Victoria 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html