REVERSES OF SCHWARZ, TRIANGLE AND BESSEL INEQUALITIES IN INNER PRODUCT SPACES

S.S. DRAGOMIR

Abstract. Reverses of the Schwarz, triangle and Bessel inequalities in inner product spaces that improve some earlier results are pointed out. They are applied to obtain new Grüss type inequalities in inner product spaces. Some natural applications for integral inequalities are also pointed out.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. The following inequality is known in the literature as *Schwarz's inequality*:

$$|\langle x,y\rangle|^2 \leq \|x\|^2 \|y\|^2, \quad x,y\in H; \tag{1.1}$$

where $\|z\|^2 = \langle z,z\rangle$, $z\in H$. The equality occurs in (1.1) if and only if x and y are linearly dependent.

In [7], the following *reverse* of Schwarz's inequality has been obtained:

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2 \leq \frac{1}{4}|A-a|^2 \|y\|^4, \tag{1.2}$$

provided $x,y\in H$ and $a,A\in\mathbb{K}$ are so that either

$$\operatorname{Re}\langle Ay-x, x-ay\rangle \geq 0, \tag{1.3}$$

or, equivalently,

$$\left\| x - \frac{a+A}{2}\cdot y\right\| \leq \frac{1}{2}|A-a|\,\|y\|, \tag{1.4}$$

holds. The constant $\frac{1}{4}$ is best possible in (1.2) in the sense that it cannot be replaced by a smaller constant.

If x,y,A,a satisfy either (1.3) or (1.4), then the following reverse of Schwarz's inequality also holds [8]

$$\begin{aligned} \|x\|\,\|y\| &\leq \frac{1}{2}\cdot\frac{\operatorname{Re}\left[A\overline{\langle x,y\rangle} + \overline{a}\langle x,y\rangle\right]}{\left[\operatorname{Re}(\overline{a}A)\right]^{\frac{1}{2}}} \\ &\leq \frac{1}{2}\cdot\frac{|A|+|a|}{\left[\operatorname{Re}(\overline{a}A)\right]^{\frac{1}{2}}}\,|\langle x,y\rangle|, \end{aligned} \tag{1.5}$$

provided that, the complex numbers a and A satisfy the condition $\operatorname{Re}(\overline{a}A) > 0$. In both inequalities in (1.5), the constant $\frac{1}{2}$ is best possible.

Date: August 04, 2003.

2000 *Mathematics Subject Classification.* Primary 46C05, 46C99. Secondary 26D15, 26D10.

Key words and phrases. Schwarz's inequality, Triangle inequality, Bessel's inequality, Grüss type inequalities, Integral inequalities.

An additive version of (1.5) may be stated as well (see also [9])

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot \frac{(|A| - |a|)^2 + 4\left[|Aa| - \operatorname{Re}(\overline{a}A)\right]}{\operatorname{Re}(\overline{a}A)} |\langle x, y\rangle|^2 . \tag{1.6}$$

In this inequality, $\frac{1}{4}$ is the best possible constant.

It has been proven in [10], that

$$0 \leq \|x\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} |\phi - \varphi|^2 - \left|\frac{\phi + \varphi}{2} - \langle x, e\rangle\right|^2 ; \tag{1.7}$$

provided, either

$$\operatorname{Re}\langle \phi e - x, x - \varphi e\rangle \geq 0, \tag{1.8}$$

or, equivalently,

$$\left\| x - \frac{\phi + \varphi}{2} e \right\| \leq \frac{1}{2} |\phi - \varphi| , \tag{1.9}$$

where $e = H$, $\|e\| = 1$. The constant $\frac{1}{4}$ in (1.7) is also best possible.

If we choose $e = \frac{y}{\|y\|}$, $\phi = \Gamma \|y\|$, $\varphi = \gamma \|y\|$ $(y \neq 0)$, $\Gamma, \gamma \in \mathbb{K}$, then by (1.8), (1.9) we have,

$$\operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0, \tag{1.10}$$

or, equivalently,

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\| , \tag{1.11}$$

imply the following reverse of Schwarz's inequality:

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^4 - \left|\frac{\Gamma + \gamma}{2} \|y\|^2 - \langle x, y\rangle\right|^2 . \tag{1.12}$$

The constant $\frac{1}{4}$ in (1.12) is sharp.

Note that this inequality is an improvement of (1.2), but it might not be very convenient for applications.

Now, let $\{e_i\}_{i \in I}$ be a finite or infinite family of orthornormal vectors in the inner product space $(H; \langle \cdot, \cdot\rangle)$, i.e., we recall that

$$\langle e_i, e_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}, \quad i, j \in I.$$

In [11], we proved that, if $\{e_i\}_{i \in I}$ is as above, $F \subset I$ is a finite part of I such that either

$$\operatorname{Re}\left\langle \sum_{i \in F} \phi_i e_i - x, x - \sum_{i \in F} \varphi_i e_i \right\rangle \geq 0, \tag{1.13}$$

or, equivalently,

$$\left\| x - \sum_{i \in F} \frac{\phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} , \tag{1.14}$$

holds, where $(\phi_i)_{i\in I}$, $(\varphi_i)_{i\in I}$ are real or complex numbers, then we have the following reverse of *Bessel's inequality:*

$$\begin{aligned}(1.15)\qquad 0 &\le \|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2 \\ &\le \frac{1}{4}\cdot\sum_{i\in F}|\phi_i - \varphi_i|^2 - \operatorname{Re}\left\langle \sum_{i\in F}\phi_i e_i - x, x - \sum_{i\in F}\varphi_i e_i\right\rangle \\ &\le \frac{1}{4}\cdot\sum_{i\in F}|\phi_i - \varphi_i|^2 .\end{aligned}$$

The constant $\frac{1}{4}$ in both inequalities is sharp. This result improves an earlier result by N. Ujević obtained only for real spaces [21].

In [10], by the use of a different technique, another reverse of Bessel's inequality has been proven, namely:

$$\begin{aligned}(1.16)\qquad 0 &\le \|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2 \\ &\le \frac{1}{4}\cdot\sum_{i\in F}|\phi_i - \varphi_i|^2 - \sum_{i\in F}\left|\frac{\phi_i + \varphi_i}{2} - \langle x, e_i\rangle\right|^2 \\ &\le \frac{1}{4}\cdot\sum_{i\in F}|\phi_i - \varphi_i|^2 ,\end{aligned}$$

provided that $(e_i)_{i\in I}$, $(\phi_i)_{i\in I}$, $(\varphi_i)_{i\in I}$, x and F are as above.

Here the constant $\frac{1}{4}$ is sharp in both inequalities.

It has also been shown that the bounds provided by (1.15) and (1.16) for the Bessel's difference $\|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2$ cannot be compared in general, meaning that there are examples for which one is smaller than the other [10].

Finally, we recall another type of reverse for Bessel inequality that has been obtained in [12]:

$$(1.17)\qquad \|x\|^2 \le \frac{1}{4}\cdot\frac{\sum_{i\in F}(|\phi_i| + |\varphi_i|)^2}{\sum_{i\in F}\operatorname{Re}(\phi_i\overline{\varphi_i})}\sum_{i\in F}|\langle x, e_i\rangle|^2 ;$$

provided $(\phi_i)_{i\in I}$, $(\varphi_i)_{i\in I}$ satisfy (1.13) (or, equivalently (1.14)) and $\sum_{i\in F}\operatorname{Re}(\phi_i\overline{\varphi_i}) > 0$. Here the constant $\frac{1}{4}$ is also best possible.

An additive version of (1.17) is

$$\begin{aligned}(1.18)\qquad 0 &\le \|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2 \\ &\le \frac{1}{4}\cdot\frac{\sum_{i\in F}\left\{(|\phi_i| - |\varphi_i|)^2 + 4\left[|\phi_i\varphi_i| - \operatorname{Re}(\phi_i\overline{\varphi_i})\right]\right\}}{\sum_{i\in F}\operatorname{Re}(\phi_i\overline{\varphi_i})} .\end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

It is the main aim of the present paper to point out new reverse inequalities to Schwarz's, triangle and Bessel's inequalities.

Some results related to Grüss' inequality in inner product spaces are also pointed out. Natural applications for integrals are also provided.

2. Some Reverses of Schwarz's Inequality

The following result holds.

Theorem 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\ \mathbb{K}=\mathbb{C})$ and $x,a\in H$, $r>0$ are such that*

$$x\in\overline{B}(a,r):=\{z\in H|\,\|z-a\|\le r\}. \tag{2.1}$$

(i) *If $\|a\|>r$, then we have the inequalities*

$$0\le\|x\|^2\|a\|^2-|\langle x,a\rangle|^2\le\|x\|^2\|a\|^2-[\operatorname{Re}\langle x,a\rangle]^2\le r^2\|x\|^2. \tag{2.2}$$

The constant $C=1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller one.

(ii) *If $\|a\|=r$, then*

$$\|x\|^2\le 2\operatorname{Re}\langle x,a\rangle\le 2|\langle x,a\rangle|. \tag{2.3}$$

The constant 2 is best possible in both inequalities.

(iii) *If $\|a\|<r$, then*

$$\|x\|^2\le r^2-\|a\|^2+2\operatorname{Re}\langle x,a\rangle\le r^2-\|a\|^2+2|\langle x,a\rangle|. \tag{2.4}$$

Here the constant 2 is also best possible.

Proof. Since $x\in\overline{B}(a,r)$, then obviously $\|x-a\|^2\le r^2$, which is equivalent to

$$\|x\|^2+\|a\|^2-r^2\le 2\operatorname{Re}\langle x,a\rangle. \tag{2.5}$$

(i) If $\|a\|>r$, then we may divide (2.5) by $\sqrt{\|a\|^2-r^2}>0$ getting

$$\frac{\|x\|^2}{\sqrt{\|a\|^2-r^2}}+\sqrt{\|a\|^2-r^2}\le\frac{2\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^2-r^2}}. \tag{2.6}$$

Using the elementary inequality

$$\alpha p+\frac{1}{\alpha}q\ge 2\sqrt{pq},\quad \alpha>0,\quad p,q\ge 0,$$

we may state that

$$2\|x\|\le\frac{\|x\|^2}{\sqrt{\|a\|^2-r^2}}+\sqrt{\|a\|^2-r^2}. \tag{2.7}$$

Making use of (2.6) and (2.7), we deduce

$$\|x\|\sqrt{\|a\|^2-r^2}\le\operatorname{Re}\langle x,a\rangle. \tag{2.8}$$

Taking the square in (2.8) and re-arranging the terms, we deduce the third inequality in (2.2). The others are obvious.

To prove the sharpness of the constant, assume, under the hypothesis of the theorem, that, there exists a constant $c>0$ such that

$$\|x\|^2\|a\|^2-[\operatorname{Re}\langle x,a\rangle]^2\le cr^2\|x\|^2, \tag{2.9}$$

provided $x\in\overline{B}(a,r)$ and $\|a\|>r$.

Let $r=\sqrt{\varepsilon}>0$, $\varepsilon\in(0,1)$, $a,e\in H$ with $\|a\|=\|e\|=1$ and $a\perp e$. Put $x=a+\sqrt{\varepsilon}e$. Then obviously $x\in\overline{B}(a,r)$, $\|a\|>r$ and $\|x\|^2=\|a\|^2+$

$\varepsilon\|e\|^2 = 1+\varepsilon$, $\operatorname{Re}\langle x,a\rangle = \|a\|^2 = 1$, and thus $\|x\|^2\|a\|^2 - [\operatorname{Re}\langle x,a\rangle]^2 = \varepsilon$. Using (2.9), we may write that

$$\varepsilon \leq c\varepsilon(1+\varepsilon), \quad \varepsilon > 0$$

giving

$$c + c\varepsilon \geq 1 \text{ for any } \varepsilon > 0. \tag{2.10}$$

Letting $\varepsilon \to 0+$, we get from (2.10) that $c \geq 1$, and the sharpness of the constant is proved.

(ii) The inequality (2.3) is obvious by (2.5) since $\|a\| = r$. The best constant follows in a similar way to the above.

(iii) The inequality (2.3) is obvious. The best constant may be proved in a similar way to the above. We omit the details.

∎

The following reverse of Schwarz's inequality holds.

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x,y \in H$, $\gamma,\Gamma \in \mathbb{K}$ such that either*

$$\operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0, \tag{2.11}$$

or, equivalently,

$$\left\| x - \frac{\Gamma+\gamma}{2} y \right\| \leq \frac{1}{2}|\Gamma - \gamma|\,\|y\|, \tag{2.12}$$

holds.

(i) *If $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then we have the inequalities*

$$\begin{aligned} \|x\|^2\|y\|^2 &\leq \frac{1}{4}\cdot\frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\langle x,y\rangle\right]\right\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \\ &\leq \frac{1}{4}\cdot\frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}|\langle x,y\rangle|^2. \end{aligned} \tag{2.13}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

(ii) *If $\operatorname{Re}(\Gamma\overline{\gamma}) = 0$, then*

$$\|x\|^2 \leq \operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\langle x,y\rangle\right] \leq |\Gamma+\gamma|\,|\langle x,y\rangle|. \tag{2.14}$$

(iii) *If $\operatorname{Re}(\Gamma\overline{\gamma}) < 0$, then*

$$\begin{aligned} \|x\|^2 &\leq -\operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2 + \operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\langle x,y\rangle\right] \\ &\leq -\operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2 + |\Gamma+\gamma|\,|\langle x,y\rangle|. \end{aligned} \tag{2.15}$$

Proof. The proof of the equivalence between the inequalities (2.11) and (2.12) follows by the fact that in an inner product space $\operatorname{Re}\langle Z - x, x - z\rangle \geq 0$ for $x,z,Z \in H$ is equivalent with $\left\|x - \frac{z+Z}{2}\right\| \leq \frac{1}{2}\|Z - z\|$ (see for example [9]).

Consider, for $y \neq 0$, $a = \frac{\gamma+\Gamma}{2}y$ and $r = \frac{1}{2}|\Gamma-\gamma|\,\|y\|$. Then

$$\|a\|^2 - r^2 = \frac{|\Gamma+\gamma|^2 - |\Gamma-\gamma|^2}{4}\|y\|^2 = \operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2.$$

(i) If $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then the hypothesis of (i) in Theorem 1 is satisfied, and by the second inequality in (2.2) we have

$$\|x\|^2 \frac{|\Gamma+\gamma|^2}{4} \|y\|^2 - \frac{1}{4}\left\{\operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\langle x,y\rangle\right]\right\}^2 \le \frac{1}{4}|\Gamma-\gamma|^2 \|x\|^2 \|y\|^2$$

from where we derive

$$\frac{|\Gamma+\gamma|^2 - |\Gamma-\gamma|^2}{4} \|x\|^2 \|y\|^2 \le \frac{1}{4}\left\{\operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\langle x,y\rangle\right]\right\}^2,$$

giving the first inequality in (2.13).

The second inequality is obvious.

To prove the sharpness of the constant $\frac{1}{4}$, assume that the first inequality in (2.13) holds with a constant $c > 0$, i.e.,

$$\|x\|^2 \|y\|^2 \le c \cdot \frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\langle x,y\rangle\right]\right\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})}, \tag{2.16}$$

provided $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$ and either (2.11) or (2.12) holds.

Assume that $\Gamma, \gamma > 0$, and let $x = \gamma y$. Then (2.11) holds and by (2.16) we deduce

$$\gamma^2 \|y\|^4 \le c \cdot \frac{(\Gamma+\gamma)^2 \gamma^2 \|y\|^4}{\Gamma\gamma}$$

giving

$$\Gamma\gamma \le c(\Gamma+\gamma)^2 \quad \text{for any} \quad \Gamma,\gamma > 0. \tag{2.17}$$

Let $\varepsilon \in (0,1)$ and choose in (2.17), $\Gamma = 1+\varepsilon$, $\gamma = 1-\varepsilon > 0$ to get $1-\varepsilon^2 \le 4c$ for any $\varepsilon \in (0,1)$. Letting $\varepsilon \to 0+$, we deduce $c \ge \frac{1}{4}$, and the sharpness of the constant is proved.

(ii) and (iii) are obvious and we omit the details.

∎

Remark 1. *We observe that the second bound in (2.13) for* $\|x\|^2 \|y\|^2$ *is better than the second bound provided by (1.5).*

The following corollary provides a reverse inequality for the additive version of Schwarz's inequality.

Corollary 1. *With the assumptions of Theorem 2 and if* $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, *then we have the inequality:*

$$0 \le \|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2 \le \frac{1}{4} \cdot \frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} |\langle x,y\rangle|^2. \tag{2.18}$$

The constant $\frac{1}{4}$ *is best possible in (2.18).*

The proof is obvious from (2.13) on subtracting in both sides the same quantity $|\langle x,y\rangle|^2$. The sharpness of the constant may be proven in a similar manner to the one incorporated in the proof of (i), Theorem 2. We omit the details.

Remark 2. *It is obvious that the inequality (2.18) is better than (1.6) obtained in* [9].

For some recent results in connection to Schwarz's inequality, see [2], [13] and [15].

3. Reverses of the Triangle Inequality

The following reverse of the triangle inequality holds.

Proposition 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ and $x,a\in H$, $r>0$ are such that*

$$\|x-a\| \leq r < \|a\| . \tag{3.1}$$

Then we have the inequality

$$0 \leq \|x\|+\|a\|-\|x+a\| \leq \sqrt{2}r\cdot\sqrt{\frac{\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^2-r^2}\left(\sqrt{\|a\|^2-r^2}+\|a\|\right)}}. \tag{3.2}$$

Proof. Using the inequality (2.8), we may write that

$$\|x\|\,\|a\| \leq \frac{\|a\|\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^2-r^2}},$$

giving

$$\begin{aligned} 0 &\leq \|x\|\,\|a\| - \operatorname{Re}\langle x,a\rangle \\ &\leq \frac{\|a\|-\sqrt{\|a\|^2-r^2}}{\sqrt{\|a\|^2-r^2}}\operatorname{Re}\langle x,a\rangle \\ &= \frac{r^2\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^2-r^2}\left(\sqrt{\|a\|^2-r^2}+\|a\|\right)}. \end{aligned} \tag{3.3}$$

Since

$$(\|x\|+\|a\|)^2-\|x+a\|^2 = 2\left(\|x\|\,\|a\|-\operatorname{Re}\langle x,a\rangle\right),$$

then by (3.3), we have

$$\begin{aligned} \|x\|+\|a\| &\leq \sqrt{\|x+a\|^2+\frac{2r^2\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^2-r^2}\left(\sqrt{\|a\|^2-r^2}+\|a\|\right)}} \\ &\leq \|x+a\|+\sqrt{2}r\cdot\sqrt{\frac{\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^2-r^2}\left(\sqrt{\|a\|^2-r^2}+\|a\|\right)}}, \end{aligned}$$

giving the desired inequality (3.2). ∎

The following proposition providing a simpler reverse for the triangle inequality also holds.

Proposition 2. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x,y\in H$, $M>m>0$ such that either*

$$\operatorname{Re}\langle My-x, x-my\rangle \geq 0, \tag{3.4}$$

or, equivalently,

$$\left\| x-\frac{M+m}{2}\cdot y\right\| \leq \frac{1}{2}(M-m)\|y\|, \tag{3.5}$$

holds. Then we have the inequality

$$0 \leq \|x\| + \|y\| - \|x+y\| \leq \frac{\sqrt{M}-\sqrt{m}}{\sqrt[4]{mM}} \sqrt{\operatorname{Re}\langle x,y\rangle}. \tag{3.6}$$

Proof. Choosing in (2.8), $a = \frac{M+m}{2} y$, $r = \frac{1}{2}(M-m)\|y\|$ we get

$$\|x\| \|y\| \sqrt{Mm} \leq \frac{M+m}{2} \operatorname{Re}\langle x,y\rangle$$

giving

$$0 \leq \|x\| \|y\| - \operatorname{Re}\langle x,y\rangle \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x,y\rangle .$$

Following the same arguments as in the proof of Proposition 1, we deduce the desired inequality (3.6). ∎

For some results related to triangle inequality in inner product spaces, see [3], [17], [18] and [19].

4. Some Grüss Type Inequalities

We may state the following result.

Theorem 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ ($\mathbb{K}=\mathbb{R},\mathbb{K}=\mathbb{C}$) and $x,y,e\in H$ with $\|e\|=1$. If $r_1,r_2\in(0,1)$ and*

$$\|x-e\| \leq r_1, \qquad \|y-e\| \leq r_2, \tag{4.1}$$

then we have the inequality

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leq r_1 r_2 \|x\| \|y\| . \tag{4.2}$$

The inequality (4.2) is sharp in the sense that the constant $c=1$ in front of r_1r_2 cannot be replaced by a smaller constant.

Proof. Apply Schwarz's inequality in $(H;\langle\cdot,\cdot\rangle)$ for the vectors $x-\langle x,e\rangle e$, $y-\langle y,e\rangle e$, to get (see also [9])

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|^2 \leq \left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,e\rangle|^2\right). \tag{4.3}$$

Using Theorem 1 for $a=e$, we may state that

$$\|x\|^2 - |\langle x,e\rangle|^2 \leq r_1^2 \|x\|^2 , \qquad \|y\|^2 - |\langle y,e\rangle|^2 \leq r_2^2 \|y\|^2 . \tag{4.4}$$

Utilizing (4.3) and (4.4), we deduce

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|^2 \leq r_1^2 r_2^2 \|x\|^2 \|y\|^2 , \tag{4.5}$$

which is clearly equivalent to the desired inequality (4.2).

The sharpness of the constant follows by the fact that for $x=y$, $r_1=r_2=r$, we get from (4.2) that

$$\|x\|^2 - |\langle x,e\rangle|^2 \leq r^2 \|x\|^2 , \tag{4.6}$$

provided $\|e\|=1$ and $\|x-e\| \leq r < 1$. The inequality (4.6) is sharp, as shown in Theorem 1, and the proof is completed. ∎

Another companion of the Grüss inequality may be stated as well.

Theorem 4. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x,y,e\in H$ with $\|e\|=1$. Suppose also that $a,A,b,B\in\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ such that* $\operatorname{Re}(A\overline{a})$, $\operatorname{Re}\left(B\overline{b}\right)>0$. *If either*

$$\operatorname{Re}\langle Ae-x,x-ae\rangle\geq 0,\quad \operatorname{Re}\langle Be-y,y-be\rangle\geq 0, \tag{4.7}$$

or, equivalently,

$$\left\|x-\frac{a+A}{2}e\right\|\leq\frac{1}{2}|A-a|,\quad \left\|y-\frac{b+B}{2}e\right\|\leq\frac{1}{2}|B-b|, \tag{4.8}$$

holds, then we have the inequality

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}\cdot\frac{|A-a||B-b|}{\sqrt{\operatorname{Re}(A\overline{a})\operatorname{Re}\left(B\overline{b}\right)}}|\langle x,e\rangle\langle e,y\rangle|. \tag{4.9}$$

The constant $\frac{1}{4}$ is best possible.

Proof. We know, by (4.3), that

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|^2\leq\left(\|x\|^2-|\langle x,e\rangle|^2\right)\left(\|y\|^2-|\langle y,e\rangle|^2\right). \tag{4.10}$$

If we use Corollary 1, then we may state that

$$\|x\|^2-|\langle x,e\rangle|^2\leq\frac{1}{4}\cdot\frac{|A-a|^2}{\operatorname{Re}(A\overline{a})}|\langle x,e\rangle|^2 \tag{4.11}$$

and

$$\|y\|^2-|\langle y,e\rangle|^2\leq\frac{1}{4}\cdot\frac{|B-b|^2}{\operatorname{Re}\left(B\overline{b}\right)}|\langle y,e\rangle|^2. \tag{4.12}$$

Utilizing (4.10) – (4.12), we deduce

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|^2\leq\frac{1}{16}\cdot\frac{|A-a|^2|B-b|^2}{\operatorname{Re}(A\overline{a})\operatorname{Re}\left(B\overline{b}\right)}|\langle x,e\rangle\langle e,y\rangle|^2,$$

which is clearly equivalent to the desired inequality (4.9).

The sharpness of the constant follows from Corollary 1, and we omit the details. ∎

Remark 3. *With the assumptions of Theorem 4 and if $\langle x,e\rangle,\langle y,e\rangle\neq 0$ (that is actually the interesting case), one has the inequality*

$$\left|\frac{\langle x,y\rangle}{\langle x,e\rangle\langle e,y\rangle}-1\right|\leq\frac{1}{4}\cdot\frac{|A-a||B-b|}{\sqrt{\operatorname{Re}(A\overline{a})\operatorname{Re}\left(B\overline{b}\right)}}. \tag{4.13}$$

The constant $\frac{1}{4}$ is best possible.

Remark 4. *The inequality (4.9) provides a better bound for the quantity*

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|$$

than (2.3) of [9].

For some recent results on Grüss type inequalities in inner product spaces, see [4], [6] and [20].

5. Reverses of Bessel's Inequality

Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex infinite dimensional Hilbert space and $(e_i)_{i\in\mathbb{N}}$ an orthornormal family in H, i.e., we recall that $\langle e_i,e_j\rangle=0$ if $i,j\in\mathbb{N}$, $i\neq j$ and $\|e_i\|=1$ for $i\in\mathbb{N}$.

It is well known that, if $x\in H$, then the sum $\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2$ is convergent and the following inequality, called *Bessel's inequality*

$$\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2\leq\|x\|^2, \tag{5.1}$$

holds.

If $\ell^2(\mathbb{K}):=\left\{\mathbf{a}=(a_i)_{i\in\mathbb{N}}\subset\mathbb{K}\,\middle|\,\sum_{i=1}^{\infty}|a_i|^2<\infty\right\}$, where $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$, is the Hilbert space of all complex or real sequences that are 2-summable and $\boldsymbol{\lambda}=(\lambda_i)_{i\in\mathbb{N}}\in\ell^2(\mathbb{K})$, then the sum $\sum_{i=1}^{\infty}\lambda_ie_i$ is convergent in H and if $y:=\sum_{i=1}^{\infty}\lambda_ie_i\in H$, then $\|y\|=\left(\sum_{i=1}^{\infty}|\lambda_i|^2\right)^{\frac{1}{2}}$.

We may state the following result.

Theorem 5. *Let $(H;\langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space over the real or complex number field $\mathbb{K}$, $(e_i)_{i\in\mathbb{N}}$ an orthornormal family in H, $\boldsymbol{\lambda}=(\lambda_i)_{i\in\mathbb{N}}\in\ell^2(\mathbb{K})$ and $r>0$ with the property that*

$$\sum_{i=1}^{\infty}|\lambda_i|^2>r^2. \tag{5.2}$$

If $x\in H$ is such that

$$\left\|x-\sum_{i=1}^{\infty}\lambda_ie_i\right\|\leq r, \tag{5.3}$$

then we have the inequality

$$\begin{aligned}\|x\|^2&\leq\frac{\left(\sum_{i=1}^{\infty}\operatorname{Re}\left[\overline{\lambda_i}\langle x,e_i\rangle\right]\right)^2}{\sum_{i=1}^{\infty}|\lambda_i|^2-r^2}\\&\leq\frac{\left|\sum_{i=1}^{\infty}\overline{\lambda_i}\langle x,e_i\rangle\right|^2}{\sum_{i=1}^{\infty}|\lambda_i|^2-r^2}\\&\leq\frac{\sum_{i=1}^{\infty}|\lambda_i|^2}{\sum_{i=1}^{\infty}|\lambda_i|^2-r^2}\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2;\end{aligned} \tag{5.4}$$

and

$$0\leq\|x\|^2-\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2 \tag{5.5}$$

$$\leq\frac{r^2}{\sum_{i=1}^{\infty}|\lambda_i|^2-r^2}\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2. \tag{5.6}$$

Proof. Applying the third inequality in (2.2) for $a=\sum_{i=1}^{\infty}\lambda_ie_i\in H$, we have

$$\|x\|^2\left\|\sum_{i=1}^{\infty}\lambda_ie_i\right\|^2-\left[\operatorname{Re}\left\langle x,\sum_{i=1}^{\infty}\lambda_ie_i\right\rangle\right]^2\leq r^2\|x\|^2 \tag{5.7}$$

and since

$$\left\|\sum_{i=1}^{\infty}\lambda_i e_i\right\|^2 = \sum_{i=1}^{\infty}|\lambda_i|^2,$$
$$\operatorname{Re}\left\langle x, \sum_{i=1}^{\infty}\lambda_i e_i\right\rangle = \sum_{i=1}^{\infty}\operatorname{Re}\left[\overline{\lambda_i}\left\langle x, e_i\right\rangle\right],$$

then by (5.7) we deduce

$$\|x\|^2\sum_{i=1}^{\infty}|\lambda_i|^2 - \left[\operatorname{Re}\left\langle x, \sum_{i=1}^{\infty}\lambda_i e_i\right\rangle\right]^2 \le r^2\|x\|^2,$$

giving the first inequality in (5.4).

The second inequality is obvious by the modulus property.

The last inequality follows by the Cauchy-Bunyakovsky-Schwarz inequality

$$\left|\sum_{i=1}^{\infty}\overline{\lambda_i}\left\langle x, e_i\right\rangle\right|^2 \le \sum_{i=1}^{\infty}|\lambda_i|^2\sum_{i=1}^{\infty}|\langle x, e_i\rangle|^2.$$

The inequality (5.5) follows by the last inequality in (5.4) on subtracting in both sides the quantity $\sum_{i=1}^{\infty}|\langle x, e_i\rangle|^2 < \infty$. ∎

The following result provides a generalization for the reverse of Bessel's inequality obtained in [12].

Theorem 6. *Let $(H;\langle\cdot,\cdot\rangle)$ and $(e_i)_{i\in\mathbb{N}}$ be as in Theorem 5. Suppose that $\mathbf{\Gamma} = (\Gamma_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$, $\boldsymbol{\gamma} = (\gamma_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{K})$ are sequences of real or complex numbers such that*

$$\sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right) > 0. \tag{5.8}$$

If $x\in H$ is such that either

$$\left\|x - \sum_{i=1}^{\infty}\frac{\Gamma_i+\gamma_i}{2}e_i\right\| \le \frac{1}{2}\left(\sum_{i=1}^{\infty}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}} \tag{5.9}$$

or, equivalently,

$$\operatorname{Re}\left\langle\sum_{i=1}^{\infty}\Gamma_i e_i - x, x - \sum_{i=1}^{\infty}\gamma_i e_i\right\rangle \ge 0 \tag{5.10}$$

holds, then we have the inequalities

$$\begin{aligned}\|x\|^2 &\le \frac{1}{4}\cdot\frac{\left(\sum_{i=1}^{\infty}\operatorname{Re}\left[\left(\overline{\Gamma_i}+\overline{\gamma_i}\right)\langle x, e_i\rangle\right]\right)^2}{\sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right)} \\ &\le \frac{1}{4}\cdot\frac{\left|\sum_{i=1}^{\infty}\left(\overline{\Gamma_i}+\overline{\gamma_i}\right)\langle x, e_i\rangle\right|^2}{\sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right)} \\ &\le \frac{1}{4}\cdot\frac{\sum_{i=1}^{\infty}|\Gamma_i+\gamma_i|^2}{\sum_{i=1}^{\infty}\operatorname{Re}\left(\Gamma_i\overline{\gamma_i}\right)}\sum_{i=1}^{\infty}|\langle x, e_i\rangle|^2.\end{aligned} \tag{5.11}$$

The constant $\frac{1}{4}$ is best possible in all inequalities in (5.11).

We also have the inequalities:

$$0 \leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \overline{\gamma_i})} \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 . \tag{5.12}$$

Here the constant $\frac{1}{4}$ *is also best possible.*

Proof. Since $\mathbf{\Gamma}, \boldsymbol{\gamma} \in \ell^2(\mathbb{K})$, then also $\frac{1}{2}(\mathbf{\Gamma} \pm \boldsymbol{\gamma}) \in \ell^2(\mathbb{K})$, showing that the series

$$\sum_{i=1}^{\infty} \left| \frac{\Gamma_i + \gamma_i}{2} \right|^2, \ \sum_{i=1}^{\infty} \left| \frac{\Gamma_i - \gamma_i}{2} \right|^2 \text{ and } \sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \overline{\gamma_i})$$

are convergent. Also, the series

$$\sum_{i=1}^{\infty} \Gamma_i e_i, \ \sum_{i=1}^{\infty} \gamma_i e_i \text{ and } \sum_{i=1}^{\infty} \frac{\gamma_i + \Gamma_i}{2} e_i$$

are convergent in the Hilbert space H.

The equivalence of the conditions (5.9) and (5.10) follows by the fact that in an inner product space we have, for $x, z, Z \in H$, $\operatorname{Re}\langle Z - x, x - z \rangle \geq 0$ is equivalent to $\left\| x - \frac{z+Z}{2} \right\| \leq \frac{1}{2} \|Z - z\|$, and we omit the details.

Now, we observe that the inequalities (5.11) and (5.12) follow from Theorem 5 on choosing $\lambda_i = \frac{\gamma_i + \Gamma_i}{2}$, $i \in \mathbb{N}$ and $r = \frac{1}{2} \left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}$.

The fact that $\frac{1}{4}$ is the best constant in both (5.11) and (5.12) follows from Theorem 2 and Corollary 1, and we omit the details. ∎

Remark 5. *Note that (5.11) improves (1.17) and (5.12) improves (1.18), that have been obtained in* [12].

For some recent results related to Bessel inequality, see [1], [5], [14], and [16].

6. Some Grüss Type Inequalities for Orthonormal Families

The following result related to Grüss inequality in inner product spaces, holds.

Theorem 7. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an infinite dimensional Hilbert space over the real or complex number field* $\mathbb{K}$, *and* $(e_i)_{i \in \mathbb{N}}$ *an orthornormal family in* H. *Assume that* $\boldsymbol{\lambda} = (\lambda_i)_{i \in \mathbb{N}}$, $\boldsymbol{\mu} = (\mu_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ *and* $r_1, r_2 > 0$ *with the properties that*

$$\sum_{i=1}^{\infty} |\lambda_i|^2 > r_1^2, \quad \sum_{i=1}^{\infty} |\mu_i|^2 > r_2^2. \tag{6.1}$$

If $x, y \in H$ *are such that*

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \leq r_1, \qquad \left\| y - \sum_{i=1}^{\infty} \mu_i e_i \right\| \leq r_2, \tag{6.2}$$

then we have the inequalities

$$\left|\langle x,y\rangle-\sum_{i=1}^{\infty}\langle x,e_i\rangle\langle e_i,y\rangle\right| \tag{6.3}$$
$$\leq \frac{r_1r_2}{\sqrt{\sum_{i=1}^{\infty}|\lambda_i|^2-r_1^2}\sqrt{\sum_{i=1}^{\infty}|\mu_i|^2-r_2^2}}\cdot\sqrt{\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2\sum_{i=1}^{\infty}|\langle y,e_i\rangle|^2}$$
$$\leq \frac{r_1r_2\|x\|\|y\|}{\sqrt{\sum_{i=1}^{\infty}|\lambda_i|^2-r_1^2}\sqrt{\sum_{i=1}^{\infty}|\mu_i|^2-r_2^2}}.$$

Proof. Applying Schwarz's inequality for the vectors $x-\sum_{i=1}^{\infty}\langle x,e_i\rangle e_i$, $y-\sum_{i=1}^{\infty}\langle y,e_i\rangle e_i$, we have

$$\left|\left\langle x-\sum_{i=1}^{\infty}\langle x,e_i\rangle e_i, y-\sum_{i=1}^{\infty}\langle y,e_i\rangle e_i\right\rangle\right|^2 \tag{6.4}$$
$$\leq \left\|x-\sum_{i=1}^{\infty}\langle x,e_i\rangle e_i\right\|^2\left\|y-\sum_{i=1}^{\infty}\langle y,e_i\rangle e_i\right\|^2.$$

Since

$$\left\langle x-\sum_{i=1}^{\infty}\langle x,e_i\rangle e_i, y-\sum_{i=1}^{\infty}\langle y,e_i\rangle e_i\right\rangle=\langle x,y\rangle-\sum_{i=1}^{\infty}\langle x,e_i\rangle\langle e_i,y\rangle$$

and

$$\left\|x-\sum_{i=1}^{\infty}\langle x,e_i\rangle e_i\right\|^2=\|x\|^2-\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2,$$

then by (5.5) applied for x and y, and from (6.4), we deduce the first part of (6.3).

The second part follows by Bessel's inequality. ∎

The following Grüss type inequality may be stated as well.

Theorem 8. *Let $(H;\langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space and $(e_i)_{i\in\mathbb{N}}$ an orthornormal family in H. Suppose that $(\Gamma_i)_{i\in\mathbb{N}}, (\gamma_i)_{i\in\mathbb{N}}, (\phi_i)_{i\in\mathbb{N}}, (\Phi_i)_{i\in\mathbb{N}}\in\ell^2(\mathbb{K})$ are sequences of real and complex numbers such that*

$$\sum_{i=1}^{\infty}\mathrm{Re}\left(\Gamma_i\overline{\gamma_i}\right)>0,\quad \sum_{i=1}^{\infty}\mathrm{Re}\left(\Phi_i\overline{\phi_i}\right)>0. \tag{6.5}$$

If $x,y\in H$ are such that either

$$\left\|x-\sum_{i=1}^{\infty}\frac{\Gamma_i+\gamma_i}{2}\cdot e_i\right\|\leq\frac{1}{2}\left(\sum_{i=1}^{\infty}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}} \tag{6.6}$$
$$\left\|y-\sum_{i=1}^{\infty}\frac{\Phi_i+\phi_i}{2}\cdot e_i\right\|\leq\frac{1}{2}\left(\sum_{i=1}^{\infty}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}$$

or, equivalently,

$$\operatorname{Re}\left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0, \tag{6.7}$$
$$\operatorname{Re}\left\langle \sum_{i=1}^{\infty} \Phi_i e_i - y, y - \sum_{i=1}^{\infty} \phi_i e_i \right\rangle \geq 0,$$

holds, then we have the inequality

$$\begin{aligned} &\left| \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\leq \frac{1}{4} \cdot \frac{\left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} \operatorname{Re}\left(\Gamma_i \overline{\gamma_i}\right)\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right)^{\frac{1}{2}}} \\ &\times \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \cdot \frac{\left(\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}}{\left[\sum_{i=1}^{\infty} \operatorname{Re}\left(\Gamma_i \overline{\gamma_i}\right)\right]^{\frac{1}{2}} \left[\sum_{i=1}^{\infty} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}} \|x\| \|y\| \end{aligned} \tag{6.8}$$

The constant $\frac{1}{4}$ is best possible in the first inequality.

Proof. Follows by (5.12) and (6.4).

The best constant follows from Theorem 4, and we omit the details. ∎

Remark 6. *We note that the inequality (6.8) is better than the inequality (3.3) in* [12]*. We omit the details.*

7. Integral Inequalities

Let (Ω, Σ, μ) be a measurable space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$. Let $\rho \geq 0$ be a $g-$measurable function on Ω with $\int_{\Omega} \rho(s) \, d\mu(s) = 1$. Denote by $L_{\rho}^2(\Omega, \mathbb{K})$ the Hilbert space of all real or complex valued functions defined on Ω and $2-\rho-$integrable on Ω, i.e.,

$$\int_{\Omega} \rho(s) |f(s)|^2 \, d\mu(s) < \infty. \tag{7.1}$$

It is obvious that the following inner product

$$\langle f, g \rangle_{\rho} := \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s), \tag{7.2}$$

generates the norm $\|f\|_{\rho} := \left(\int_{\Omega} \rho(s) |f(s)|^2 \, d\mu(s)\right)^{\frac{1}{2}}$ of $L_{\rho}^2(\Omega, \mathbb{K})$, and all the above results may be stated for integrals.

It is important to observe that, if

$$\operatorname{Re}\left[f(s) \overline{g(s)}\right] \geq 0 \text{ for } \mu-\text{a.e. } s \in \Omega, \tag{7.3}$$

then, obviously,

$$\operatorname{Re}\langle f,g\rangle_{\rho}=\operatorname{Re}\left[\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right]=\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\geq 0. \tag{7.4}$$

The reverse is evidently not true in general.

Moreover, if the space is real, i.e., $\mathbb{K}=\mathbb{R}$, then a sufficient condition for (7.4) to hold is:

$$f(s)\geq 0,\quad g(s)\geq 0\ \text{ for } \mu-\text{a.e. } s\in\Omega. \tag{7.5}$$

We provide now, by the use of certain results obtained in Section 2, some integral inequalities that may be used in practical applications.

Proposition 3. *Let $f,g\in L_{\rho}^{2}(\Omega,\mathbb{K})$ and $r>0$ with the properties that*

$$|f(s)-g(s)|\leq r\leq|g(s)|\quad \textit{for } \mu-a.e.\ s\in\Omega, \tag{7.6}$$

and $\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s)\neq r$. Then we have the inequalities

$$\begin{aligned}0&\leq\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s)-\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2}\\&\leq\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s)\\&\qquad-\left[\int_{\Omega}\rho(s)\operatorname{Re}\left(f(s)\overline{g(s)}\right)d\mu(s)\right]^{2}\\&\leq r^{2}\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s).\end{aligned} \tag{7.7}$$

The constant $c=1$ in front of r^{2} is best possible.

The proof follows by Theorem 1 and we omit the details.

Proposition 4. *Let $f,g\in L_{\rho}^{2}(\Omega,\mathbb{K})$ and $\gamma,\Gamma\in\mathbb{K}$ such that $\operatorname{Re}(\Gamma\overline{\gamma})>0$ and*

$$\operatorname{Re}\left[(\Gamma g(s)-f(s))\left(\overline{f(s)}-\overline{\gamma}\overline{g(s)}\right)\right]\geq 0\quad\textit{for } \mu-a.e.\ s\in\Omega. \tag{7.8}$$

Then we have the inequalities

$$\begin{aligned}&\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s)\\&\leq\frac{1}{4}\cdot\frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right]\right\}^{2}}{\operatorname{Re}(\Gamma\overline{\gamma})}\\&\leq\frac{1}{4}\cdot\frac{|\Gamma+\gamma|^{2}}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2}.\end{aligned} \tag{7.9}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

The proof follows by Theorem 2 and we omit the details.

Corollary 2. *With the assumptions of Proposition 4, we have the inequality*

$$0 \leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) - \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{\mathrm{Re}(\Gamma \overline{\gamma})} \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 . \tag{7.10}$$

The constant $\frac{1}{4}$ *is best possible.*

Remark 7. *If the space is real and we assume, for* $M > m > 0$, *that*

$$mg(s) \leq f(s) \leq Mg(s), \quad \text{for } \mu - \text{a.e. } s \in \Omega, \tag{7.11}$$

then, by (7.9) and (7.10), we deduce the inequalities

$$\int_{\Omega} \rho(s) [f(s)]^2 d\mu(s) \int_{\Omega} \rho(s) [g(s)]^2 d\mu(s) \leq \frac{1}{4} \cdot \frac{(M+m)^2}{mM} \left[\int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2 . \tag{7.12}$$

and

$$0 \leq \int_{\Omega} \rho(s) [f(s)]^2 d\mu(s) \int_{\Omega} \rho(s) [g(s)]^2 d\mu(s) - \left[\int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2 \leq \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \left[\int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2 . \tag{7.13}$$

The inequality (7.12) is known in the literature as Cassel's inequality.

The following Grüss type integral inequality for real or complex-valued functions also holds.

Proposition 5. *Let* $f, g, h \in L^2_{\rho}(\Omega, \mathbb{K})$ *with* $\int_{\Omega} \rho(s) |h(s)|^2 d\mu(s) = 1$ *and* $a, A, b, B \in \mathbb{K}$ *such that* $\mathrm{Re}(A\overline{a}), \mathrm{Re}(B\overline{b}) > 0$ *and*

$$\begin{aligned} \mathrm{Re}\left[(Ah(s) - f(s)) \left(\overline{f(s)} - \overline{a}\overline{h(s)} \right) \right] &\geq 0, \\ \mathrm{Re}\left[(Bh(s) - g(s)) \left(\overline{g(s)} - \overline{b}\overline{h(s)} \right) \right] &\geq 0, \end{aligned}$$

for $\mu-$*a.e.* $s \in \Omega$. *Then we have the inequalities*

$$\begin{aligned} &\left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} \rho(s) f(s) \overline{h(s)} d\mu(s) \int_{\Omega} \rho(s) h(s) \overline{g(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} \cdot \frac{|A-a| |B-b|}{\sqrt{\mathrm{Re}(A\overline{a}) \mathrm{Re}(B\overline{b})}} \left| \int_{\Omega} \rho(s) f(s) \overline{h(s)} d\mu(s) \int_{\Omega} \rho(s) h(s) \overline{g(s)} d\mu(s) \right| \end{aligned} \tag{7.14}$$

The constant $\frac{1}{4}$ *is best possible.*

The proof follows by Theorem 4.

Remark 8. *All the other inequalities in Sections 3 – 6 may be used in a similar way to obtain the corresponding integral inequalities. We omit the details.*

References

[1] X.H. CAO, Bessel sequences in a Hilbert space. *Gongcheng Shuxue Xuebao* **17** (2000), no. 2, 92–98.

[2] A. De ROSSI, A strengthened Cauchy-Schwarz inequality for biorthogonal wavelets. *Math. Inequal. Appl.* **2** (1999), no. 2, 263–282.

[3] J. B. DIAZ and F. T. METCALF, A complementary triangle inequality in Hilbert and Banach spaces. *Proc. Amer. Math. Soc.* **17** (1966), 88–97.

[4] S.S. DRAGOMIR, A generalization of Grüss inequality in inner product spaces and applications. *J. Math. Anal. Appl.* **237** (1999), no. 1, 74–82.

[5] S.S. DRAGOMIR, A note on Bessel's inequality, *Austral. Math. Soc. Gaz.* **28** (2001), no. 5, 246–248.

[6] S.S. DRAGOMIR, Some Grüss type inequalities in inner product spaces, *J. Inequal. Pure & Appl. Math.*, **4**(2003), No. 2, Article 42, [`On line: http://jipam.vu.edu.au/v4n2/032_03.html`]

[7] S.S. DRAGOMIR, A counterpart of Schwarz's inequality in inner product spaces, *RGMIA Res. Rep. Coll.,* **6**(2003), *Supplement*, Article 18, `[On line http://rgmia.vu.edu.au/v6(E).html]`

[8] S.S. DRAGOMIR, A generalisation of the Cassels and Grueb-Reinboldt inequalities in inner product spaces, Preprint, *Mathematics ArXiv*, math.CA/0307130, `[On line http://front.math.ucdavis.edu/]`

[9] S.S. DRAGOMIR, Some companions of the Grüss inequality in inner product spaces, *RGMIA Res. Rep. Coll.* **6**(2003), *Supplement*, Article 8, `[On line http://rgmia.vu.edu.au/v6(E).html]`

[10] S.S. DRAGOMIR, On Bessel and Grüss inequalities for orthornormal families in inner product spaces, *RGMIA Res. Rep. Coll.* **6**(2003), *Supplement*, Article 12, `[On line http://rgmia.vu.edu.au/v6(E).html]`

[11] S.S. DRAGOMIR, A counterpart of Bessel's inequality in inner product spaces and some Grüss type related results, *RGMIA Res. Rep. Coll.* **6**(2003), *Supplement*, Article 10, `[On line http://rgmia.vu.edu.au/v6(E).html]`

[12] S.S. DRAGOMIR, Some new results related to Bessel and Grüss inequalities for orthornormal families in inner product spaces, *RGMIA Res. Rep. Coll.* **6**(2003), *Supplement*, Article 13, `[On line http://rgmia.vu.edu.au/v6(E).html]`

[13] S. S. DRAGOMIR and B. MOND, On the superadditivity and monotonicity of Schwarz's inequality in inner product spaces. *Makedon. Akad. Nauk. Umet. Oddel. Mat.-Tehn. Nauk. Prilozi* **15** (1994), no. 2, 5–22 (1996).

[14] S.S. DRAGOMIR and J. SÁNDOR, On Bessel's and Gram's inequalities in pre-Hilbertian spaces. *Period. Math. Hungar.* **29** (1994), no. 3, 197–205.

[15] H. GUNAWAN, On n-inner products, n-norms, and the Cauchy-Schwarz inequality. *Sci. Math. Jpn.* **55** (2002), no. 1, 53–60.

[16] H. GUNAWAN, A generalization of Bessel's inequality and Parseval's identity. *Period. Math. Hungar.* **44** (2002), no. 2, 177–181.

[17] S.M. KHALEELULLA, On Diaz-Metcalf's complementary triangle inequality. *Kyungpook Math. J.* **15** (1975), 9–11.

[18] P.M. MILIČIĆ, On a complementary inequality of the triangle inequality (French) *Mat. Vesnik* **41** (1989), no. 2, 83–88.

[19] D. K. RAO, A triangle inequality for angles in a Hilbert space. *Rev. Colombiana Mat.* **10** (1976), no. 3, 95–97.

[20] P. F. RENAUD, A matrix formulation of Grüss inequality, *Linear Algebra Appl.* **335** (2001), 95–100.

[21] N. UJEVIĆ, A generalisation of Grüss inequality in prehilbertian spaces, *Math. Inequal. & Appl.,* (to appear).

School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, MCMC 8001, Victoria, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html