ON SOME GRONWALL TYPE INEQUALITIES WITH ITERATED INTEGRALS

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Abstract: The main objective of the present paper is to establish some new Gronwall type inequalities involving iterated integrals.

1. Introduction

Let $u : [\alpha, \alpha + h] \to R$ be a continuous real-valued function satisfying the

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inequality

$$0 \le u(t) \le \int_\alpha^t [a + bu(s)]\, ds \quad \text{for} \quad t \in [\alpha, \alpha + h],$$

where a, b are nonnegative constants. Then $u(t) \le ahe^{bh}$ for $t \in [\alpha, \alpha + h]$. This result was proved by T. H. Gronwall [8] in the year 1919, and is the prototype for the study of several integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Among the several publications on this subject, the paper of Bellman [3] is very well known. It is clear that Bellman's result contains that of Gronwall. This is the reason why inequalities of this type were called "Gronwall-Bellman inequalities" or "Inequalities of Gronwall type". The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types (see Gronwall [8] and Guiliano [9]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Some applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [13], Bihari [4], and Langenhop [10]. During the past few years several authors (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

Bainov and Simeonov proved the following interesting integral inequality involving iterated integrals, which appear in [1, p. 101]:

Let Let $u(t), a(t)$, and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that

$$\begin{aligned} u(t) \le{}& a(t) + b(t)\Bigg[\int_\alpha^t k_1(t, t_1)u(t_1)\, dt_1 + \cdots \\ &+ \int_\alpha^t \bigg(\int_\alpha^{t_1} \cdots \bigg(\int_\alpha^{t_{n-1}} k_n(t, t_1, \dots, t_n)u(t_n)\, dt_n\bigg) \dots\bigg) dt_1\Bigg] \end{aligned} \tag{1.1}$$

for $t \in J$, where $k_i(t, t_1, \dots, t_i)$ are nonnegative continuous functions in $J_{i+1}, i = 1, 2, \dots, n$, which are nondecreasing in $t \in J$ for all fixed $(t_1, \dots, t_i) \in J_i, i = 1, 2, \dots, n$. Then, for $t \in J$

$$u(t) \le a(t) + b(t) \int_\alpha^t \widehat{R}[a](t, s) \exp\bigg(\int_s^t \widehat{R}[b](t, \tau)\, d\tau\bigg)\, ds,$$

where, for $(t,s) \in J_2$,

$$\widehat{R}[w](t,s)) = k_1(t,s)w(s) + \int_\alpha^s k_2(t,s,t_2)w(t_2)dt_2 + \\ + \sum_{i=3}^n \int_\alpha^s \left(\int_\alpha^{t_2} \cdots \left(\int_\alpha^{t_{i-1}} k_i(t,s,t_2,\ldots,t_i)w(t_i)\,dt_i \right) \cdots \right) dt_2,$$

for each continuous function $w(t)$ *in* J.

In this paper we consider simple inequalities involving iterated integrals in the inequality (1.1) for functions when the function u in the right-hand side of the inequality (1.1) is replaced by the function u^p for some p, We also provide some integral inequalities involving iterated integrals.

2. The results

In this section, we state and prove some new nonlinear integral inequalities involving iterated integrals. Throughout the paper, all the functions which appear in the inequalities are assumed to be real-valued.

Before considering our first integral inequality, we need the following lemmas, which appears in [1, p. 2, p. 38].

Lemma 2.1. *Let* $b(t)$ *and* $f(t)$ *be continuous function for* $t \geq \alpha$, *let* $v(t)$ *be a differentiable function for* $t \geq \alpha$, *and suppose*

$$v'(t) \leq b(t)v(t) + f(t), \quad t \geq \alpha$$

and $v(\alpha) \leq v_0$. *Then, for* $t \geq \alpha$,

$$v(t) \leq v_0 \exp\left(\int_\alpha^t b(s)\,ds \right) + \int_\alpha^t f(s) \exp\left(\int_s^t b(\tau)\,d\tau \right) ds.$$

Lemma 2.2. *Let* $v(t)$ *be a positive differential function satisfying the inequality*

$$v'(t) \leq b(t)v(t) + k(t)v^p(t), \quad t \in J = [\alpha, \beta],$$

where the functions b *and* k *are continuous in* J, *and* $p \geq 0, p \neq 1$, *is a constant. Then*

$$v(t) \leq \exp\left(\int_\alpha^t b(s)\,ds \right) \left[v^q(\alpha) + q \int_\alpha^t k(s) \exp\left(-q \int_\alpha^s b(\tau)\,d\tau \right) ds \right]^{1/q}$$

for $t \in [\alpha, \beta_1)$, *where* β_1 *is chosen so that the expression between [...] is positive in the subinterval* $[\alpha, \beta_1)$.

In the next theorems, we consider some simple inequalities involving iterated integrals. Let $\alpha < \beta$, and set $J_i = \{(t_1, t_2, \ldots, t_i) \in R^i : \alpha \le t_i \le \cdots \le t_1 \le \beta\}$, $i = 1, \cdots, n$.

Theorem 2.3. *Let* $u(t), a(t)$ *and* $b(t)$ *be nonnegative continuous functions in* $J = [\alpha, \beta]$ *and let* $p > 1$ *be a constant. Suppose that* $\frac{a(t)}{b(t)}$ *is nondecreasing in* J *and*

$$\begin{aligned} u(t) \le a(t) + b(t)\bigg[\int_\alpha^t k_1(t, t_1) u^p(t_1)\, dt_1 + \cdots \\ + \int_\alpha^t \bigg(\int_\alpha^{t_1} \cdots \bigg(\int_\alpha^{t_{n-1}} k_n(t, t_1, \cdots, t_n) u^p(t_n)\, dt_n\bigg) \cdots \bigg) dt_1\bigg] \end{aligned} \tag{2.1}$$

for any $t \in J$, *where* $k_i(t, t_1, \ldots, t_i)$ *are nonnegative continuous functions in* J_{i+1} *for* $i = 1, 2, \cdots, n$, *which are nondecreasing in* $t \in J$ *for all fixed* $(t_1, \cdots, t_i) \in J_i$, $i = 1, 2, \cdots, n$. *Then, for any* $t \in [\alpha, \beta_p)$

$$u(t) \le a(t)\left[1 - (p-1)\int_\alpha^t \left(\frac{a(s)}{b(s)}\right)^{p-1} R[b^p](t, s)\, ds\right]^{\frac{1}{1-p}} \tag{2.2}$$

where, for $(t, s) \in J_2$,

$$\beta_p = \sup\{t \in J : (p-1)\int_\alpha^t \left(\frac{a(s)}{b(s)}\right)^{p-1} R[b^p](t, s)\, ds < 1\},$$

and

$$\begin{aligned} R[w](t, s) = k_1(t, s) w(s) + \int_\alpha^s k_2(t, s, t_2) w(t_2) dt_2 \\ + \sum_{i=3}^n \int_\alpha^s \bigg(\int_\alpha^{t_2} \cdots \bigg(\int_\alpha^{t_{i-1}} k_i(t, s, t_2, \cdots, t_i) w(t_i)\, dt_i\bigg) \cdots \bigg) dt_2, \end{aligned}$$

for each continuous function $w(t)$ *in* J.

Proof. For a fixed $T \in (\alpha, \beta]$ and $\alpha \le t \le T$ we have

$$u(t) \le a(t) + b(t) v(t), \tag{2.3}$$

where

$$v(t) = \int_\alpha^t k_1(T,t_1)u^p(t_1)\,dt_1 + \cdots$$
$$+ \int_\alpha^t \biggl(\int_\alpha^{t_1} \cdots \biggl(\int_\alpha^{t_{n-1}} k_n(T,t_1,\cdots,t_n)u^p(t_n)\,dt_n\biggr)\cdots\biggr)dt_1.$$

Since $\frac{\partial k_i}{\partial t}(T,t_1,\dots,t_i) = 0$ for $i = 1,\dots,n$ and $t \in [\alpha,T]$, we have

$$v'(t) = R[u^p](T,t) \le (R[b^p](T,t))\biggl(\frac{a(t)}{b(t)} + v(t)\biggr)^p,$$

that is,

$$v'(t) \le Q(T,t)[a(T)/b(T) + v(t)], \tag{2.4}$$

where $Q(T,t) = (R[b^p](T,t))[a(t)/b(t)+v(t)]^{p-1}$. Lemma 2.1 and (2.4) imply

$$v(t) + \frac{a(T)}{b(T)} \le \frac{a(T)}{b(T)} \exp\biggl(\int_\alpha^t Q(T,s)\,ds\biggr), \quad \alpha \le t \le T.$$

Hence, for $t = T$,

$$v(t) + \frac{a(t)}{b(t)} \le \frac{a(t)}{b(t)} \exp\biggl(\int_\alpha^t Q(T,s)\,ds\biggr). \tag{2.5}$$

From (2.10), we successively obtain

$$\biggl[v(t) + \frac{a(t)}{b(t)}\biggr]^{p-1} \le \biggl[\frac{a(t)}{b(t)}\biggr]^{p-1} \exp\biggl(\int_\alpha^t (p-1)Q(T,s)\,ds\biggr),$$
$$Q(T,t) \le (R[b^p](T,t))\biggl[\frac{a(t)}{b(t)}\biggr]^{p-1} \exp\biggl(\int_\alpha^t (p-1)Q(T,s)\,ds\biggr),$$
$$Z(T,t) \le (p-1)(R[b^p](T,t))\biggl[\frac{a(t)}{b(t)}\biggr]^{p-1} \exp\biggl(\int_\alpha^t (p-1)Q(T,s)\,ds\biggr),$$

where $Z(T,t) = (p-1)Q(T,t)$. Consequently, we have

$$Z(T,s)\exp\biggl(-\int_\alpha^s Z(T,s)\,ds\biggr) \le (p-1)R[b^p](T,s)\biggl[\frac{a(s)}{b(s)}\biggr]^{p-1}$$

or

$$\frac{d}{ds}\left[-\exp\left(-\int_\alpha^s Z(T,\tau)\,d\tau\right)\right] \le (p-1)R[b^p](T,s)\left[\frac{a(s)}{b(s)}\right]^{p-1}.$$

Integrating this from α to t yields

$$1-\exp\left(-\int_\alpha^t Z(T,s)\,ds\right) \le (p-1)\int_\alpha^t \left(\frac{a(s)}{b(s)}\right)^{p-1} R[b^p](T,s)\,ds,$$

from which we conclude that

$$\exp\left(\int_\alpha^t Q(T,s)\,ds\right) \le \left[1-(p-1)\int_\alpha^t \left(\frac{a(s)}{b(s)}\right)^{p-1} R[b^p](T,s)\,ds\right]^{\frac{1}{1-p}}.$$

This, together with (2.3) and (2.5), implies

$$u(t) \le a(t)\left[1-(p-1)\int_\alpha^t \left(\frac{a(s)}{b(s)}\right)^{p-1} R[b^p](T,s)\,ds\right]^{\frac{1}{1-p}}.$$

In particular, for $T=t$ we find (2.2). This completes the proof.

Theorem 2.4. *Let $u(t)$ and $b(t)$ be nonnegative continuous functions in $J=[\alpha,\beta]$, and suppose that*

$$u(t) \le b(t)\Big[a(t)+\int_\alpha^t k_1(t,t_1)u^p(t_1)\,dt_1+\cdots$$
$$+\int_\alpha^t\left(\int_\alpha^{t_1}\cdots\left(\int_\alpha^{t_{n-1}} k_n(t,t_1,\dots,t_n)u^p(t_n)\,dt_n\right)\dots\right)dt_1\Big]$$

for $t\in J$, where $p\ge 0, p\ne 1$ be a constant, $a(t)>0$ is nondecreasing continuous function in $t\in J$, and $k_i(t,t_1,\dots,t_i)$ are nonnegative continuous functions in $J_{i+1}, i=1,2,\dots,n$, which are nondecreasing in $t\in J$ for all fixed $(t_1,\dots,t_i)\in J_i, i=1,2,\dots,n$. Then

$$u(t) \le b(t)\left[a^q(t)+q\int_\alpha^t R[b^p](t,s)\,ds\right]^{1/q} \tag{2.6}$$

for $t\in[\alpha,\beta_1)$, where $q=1-p$ and β_1 is chosen so that the expression between [...] is positive in the subinterval $[\alpha,\beta_1)$.

Proof. For a fixed $T \in (\alpha, \beta]$ and $\alpha \leq t \leq T$ we have

$$\begin{aligned} u(t) &\leq b(t)v(t) \\ &\equiv b(t)\Big[a(T) + \int_\alpha^t k_1(T, t_1)u^p(t_1)\,dt_1 + \cdots \\ &\quad + \int_\alpha^t \Big(\int_\alpha^{t_1} \cdots \Big(\int_\alpha^{t_{n-1}} k_n(T, t_1, \dots, t_n)u^p(t_n)\,dt_n\Big)\dots\Big)dt_1\Big]. \end{aligned}$$

Since $v(\alpha) = a(T)$, $v(t)$ is nondecreasing and continuous in J, and $\frac{\partial k_i}{\partial t}(T, t_1, \dots, t_i) \equiv 0$ for $i = 1, \dots, n$ and $t \in [\alpha, T]$, we have

$$\begin{aligned} v'(t) &= R[u^p](T, t) \leq R[b^p v^p](T, t) \\ &\leq (R[b^p](t))v^p(T, t). \end{aligned} \tag{2.7}$$

Lemma 2.2 and (2.7) imply

$$v(t) \leq \left[a^q(T) + q\int_\alpha^t R[b^p](T, s)\,ds\right]^{1/q}$$

from which, we obtain

$$u(t) \leq b(t)\left[a^q(T) + q\int_\alpha^t R[b^p](T, s)\,ds\right]^{1/q}$$

for $\alpha \leq t \leq T$. In particular, for $T = t$ we find (2.6). This completes the proof. □

Theorem 2.5. *Let $u(t)$, $a(t)$ and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that*

$$\begin{aligned} u(t) \leq\, & a(t) + b(t)\Big[\int_\alpha^t k_1(t, t_1)u^p(t_1)\,dt_1 + \cdots \\ &+ \int_\alpha^t \Big(\int_\alpha^{t_1} \cdots \Big(\int_\alpha^{t_{n-1}} k_n(t, t_1, \dots, t_n)u^p(t_n)\,dt_n\Big)\dots\Big)dt_1\Big] \end{aligned}$$

for $t \in J$, where $0 < p \leq 1$ be a constant, $\frac{a(t)}{b(t)} \geq 1$ is nondecreasing in J and $k_i(t, t_1, \dots, t_i)$ are nonnegative continuous functions in $J_{i+1}, i = 1, 2, \dots, n$, *which are nondecreasing in* $t \in J$ *for all fixed* $(t_1, \ldots, t_i) \in J_i, i = 1, 2, \ldots, n$. *Then*

$$u(t) \le a(t) \exp\left(\int_\alpha^t R[b^p](t, \tau)\, d\tau\right) \tag{2.8}$$

for $t \in [\alpha, \beta]$.

Proof. For a fixed $T \in (\alpha, \beta]$ and $\alpha \le t \le T$ we have

$$\begin{aligned} u(t) &\le a(t) + b(t)w(t) \\ &\equiv a(t) + b(t)\left[\int_\alpha^t k_1(T, t_1)u^p(t_1)\, dt_1 + \cdots \right. \\ &\quad \left. + \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(T, t_1, \ldots, t_n)u^p(t_n)\, dt_n\right) \ldots \right) dt_1\right]. \end{aligned}$$

Since $w(\alpha) = 0$, $w(t)$ is nondecreasing and continuous in J, and $\frac{\partial k_i}{\partial t}(T, t_1, \ldots, t_i) \equiv 0$ for $i = 1, \ldots, n$ and $t \in [\alpha, T]$, we have

$$\begin{aligned} w'(t) = R[u^p](T, t) &\le R[b^p](T, t)\left(\frac{a(t)}{b(t)} + w(t)\right)^p \\ &\le R[b^p](T, t)\left(\frac{a(t)}{b(t)} + w(t)\right). \end{aligned} \tag{2.9}$$

Lemma 2.1 and (2.9) imply

$$w(t) \le \int_\alpha^t R[b^p](T, s)\left(\frac{a(s)}{b(s)}\right) \exp\left(\int_s^t R[b^p](T, \tau)\, d\tau\right) ds$$

from which, we obtain

$$u(t) \le a(t) + b(t) \int_\alpha^t R[b^p](T, s)\left(\frac{a(s)}{b(s)}\right) \exp\left(\int_s^t R[b^p](T, \tau)\, d\tau\right) ds. \tag{2.10}$$

Indeed, (2.10) implies that

$$\begin{aligned} u(t) &\le a(t)\left[1 + \int_\alpha^t R[b^p](T, s) \exp\left(\int_s^t R[b^p](T, \tau)\, d\tau\right) ds\right] \\ &= a(t) \exp\left(\int_\alpha^t R[b^p](T, \tau)\, d\tau\right) \end{aligned}$$

for $\alpha \le t \le T$. In particular, for $T = t$ we find (2.8). This completes the proof. $\square$

Theorem 2.6. *Let $u, f_1, \dots, f_n$ be nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that*

(2.11)
$$\begin{aligned} u(t) \le & a + \int_\alpha^t f_1(t_1) u^p(t_1)\, dt_1 + \cdots \\ & + \int_\alpha^t f_1(t_1) \left(\int_\alpha^{t_1} f_2(t_2) \cdots \left(\int_\alpha^{t_{n-1}} f_n(t_n) u^p(t_n)\, dt_n \right) \cdots \right) dt_1 \end{aligned}$$

for $t \in J$, where $a \ge 1$ and $0 < p \le 1$ are a constant. Then

$$u(t) \le a R_1(t), \quad t \in J, \tag{2.12}$$

where

$$R_n(t) = \exp\left(\int_\alpha^t f_n(s)\, ds \right), \quad t \in J,$$

and

$$R_n(t) = 1 + \int_\alpha^t f_i(t) R_{i+1}(s) \exp\left(\int_\alpha^s f_i(\tau)\, d\tau \right) ds$$

for $t \in J, i = n-1, \dots, 1$.

Proof. We set

$$u_1(t) = a + L_1[u^p](t), \quad u_{j+1}(t) = u_j + L_{j+1}[u^p](t)$$

for $t \in J, j = 1, \dots, n-1$, where

$$\begin{aligned} L_k[u^p](t) = & \int_\alpha^t f_k(t_k) u^p(t_k)\, dt_k + \cdots \\ & + \int_\alpha^t f_k(t_k) \left(\int_\alpha^t f_{k+1}(t_{k+1}) \cdots \left(\int_\alpha^t f_n(t_n)\, dt_n \right) \cdots \right) dt_k \end{aligned}$$

for $t \in J, k = 1, \dots, n$. Now (2.11) implies

$$u(t) \le u_1(t). \tag{2.13}$$

Taking into account that

$$\begin{aligned} & u_k(t) \le u_{k+1}(t), \\ & (L_k[u^p])' = f_k(u^p(t) + L_{k+1}[u^p]), \quad k = 1, \dots, n-1, \end{aligned}$$

and
$$(L_n[u^p])' = f_n(t)u^p(t).$$
We successively find
$$\begin{aligned}
u_1'(t) &= (L_1[u^p](t))' = f_1[u^p(t) + L_2[u^p]] \\
&\le f_1[u_1(t) + L_2[u^p]] = f_1 u_2, \\
u_k'(t) &\le (f_1 + \cdots + f_{k-1})u_k(t) + f_k u_k(t), \quad k = 2, \ldots, n-1, \\
u_n'(t) &\le (f_1 + \cdots + f_n)u_n(t).
\end{aligned} \tag{2.14}$$
Since $u_k(\alpha) = a, k = 1, \ldots, n$, (2.14) gives by successive application of Lemma 2.1,
$$u_k(t) \le aR_k(t)\exp\left(\int_\alpha^t \sum_{j=1}^{k-1} f_j(s)\,ds\right), \quad k = n, n-1, \ldots, 1.$$
For $k = 1$ this and (2.13) imply (2.12). □

Remark 2.1. In the case when $a \ge 0, p = 1$, the inequality given in (2.11) reduces to the inequality established earlier by Ráb in [16](see, also [1, Theorem 11.6, p.102]). □

Corollary 2.7. *Let u, f, g are nonnegative continuous functions in $J = [\alpha, \beta]$, $u_0 \ge 1$ and suppose that*
$$u(t) \le u_0 + \int_\alpha^t f(s)\left[u^p(s) + \int_\alpha^s g(\tau)u^p(\tau)\,d\tau\right]ds$$
for $t \in J$, where $0 < p \le 1$ is a constant. Then
$$u(t) \le u_0\left[1 + \int_\alpha^t f(s)\exp\left(\int_\alpha^s (f(\tau) + g(\tau))\,d\tau\right)ds\right]$$
for $t \in J$.

Theorem 2.8. *Let $u, f_i, i = 1, \ldots, n$ be nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that*
$$\begin{aligned}
u(t) \le{}& a(t) + \int_\alpha^t f_1(t_1)u^p(t_1)\,dt_1 + \cdots \\
&+ \int_\alpha^t f_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2)\cdots\left(\int_\alpha^{t_{n-1}} f_n(t_n)u^p(t_n)\,dt_n\right)\cdots\right)dt_1
\end{aligned} \tag{2.15}$$

for $t \in J$, where $a(t) \geq 1$ is continuous function in J and $0 < p \leq 1$ is a constant. Then

$$u(t) \leq a(t) + \int_\alpha^t f_1(s)[a(s) + v_2(s)]\, ds, \tag{2.16}$$

where

$$u_n(t) = \int_\alpha^t (f_1(s) + \cdots + f_n(s))a(s) \exp\biggl(\int_s^t (f_1(\tau) + \cdots + f_n(\tau))\, d\tau\biggr)\, ds,$$

and

$$u_k(t) = \int_\alpha^t \biggl[\biggl(\sum_{j=1}^k f_j(s)\biggr)a(s) + f_k(s)v_{k+1}(s)\biggr] \exp\biggl(\int_s^t \sum_{j=1}^{k-1} f_j(\tau)\, d\tau\biggr)\, ds$$

for $t \in J, k = n-1, \dots, 2$.

Proof. Let $L_k[u^p](t)$ be defined as in Theorem 2.6, and put

$$v_1(t) = L_1[u^p](t), \quad v_{k+1}(t) = v_k + L_{k+1}[u^p](t)$$

for $t \in J, k = 1, \dots, n-1$. Then (2.15) implies

$$u(t) \leq a(t) + v_1(t), \tag{2.17}$$

and we successively find

$$\begin{aligned}
v_1'(t) &= (L_1[u^p](t))' = f_1[u^p(t) + L_2[u^p]] \\
&\leq f_1[a(t) + v_1(t) + L_2[u^p]] = f_1[a(t) + v_2(t)], \\
v_k'(t) &\leq (f_1 + \cdots + f_{k-1})v_k(t) + (f_1 + \cdots + f_k)a + f_k v_{k+1}(t), \\
&\quad k = 2, \dots, n-1, \\
v_n'(t) &\leq (f_1 + \cdots + f_n)v_n(t) + (f_1 + \cdots + f_n)a(t).
\end{aligned} \tag{2.18}$$

Since $v_k(\alpha) = 0, k = 1, \dots, n$, solving the system (2.18) ‘backward’, and applying Lemma 2.1, we arrive at

$$v_1(t) \leq \int_\alpha^t f_1(s)[a(s) + v_2(s)]\, ds, \tag{2.19}$$

where

$$v_k(t) = \int_\alpha^t \left[\left(\sum_{j=1}^{k} f_j(s) \right) a(s) + f_k(s) v_{k+1}(s) \right] \exp\left(\int_s^t \sum_{j=1}^{k-1} f_j(\tau)\, d\tau \right) ds,$$

for $t \in J, k = n-1, \dots, 2$, and

$$u_n(t) = \int_\alpha^t (f_1(s) + \cdots + f_n(s)) a(s) \exp\left(\int_s^t (f_1(\tau) + \cdots + f_n(\tau))\, d\tau \right) ds.$$

The inequalities (2.17) and (2.19) imply (2.16). □

Remark 2.2. In the case when $a(t) \geq 0, p = 1$, the inequality given in (2.15) reduces to the inequality established earlier by Young in [18](see, also [1, Theorem 11.7, p.103]). □

Corollary 2.9. *Let u, f, g, h are nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that*

$$u(t) \leq u_0 + \int_\alpha^t (f(s) u^p(s) + h(s))\, ds + \int_\alpha^t f(s) \left(\int_\alpha^s g(\tau) u^p(\tau)\, d\tau \right) ds$$

for $t \in J$, where $u_0 + \int_\alpha^t h(s)\, ds \geq 1$ is continuous function in J and $0 < p \leq 1$ is a constant. Then

$$u(t) \leq u_0 + \int_\alpha^t h(s)\, ds + \int_\alpha^t f(s) \left[u_0 + \int_\alpha^s h(\tau)\, d\tau \right] \exp\left(\int_\alpha^s (f(\tau) + g(\tau)\, \tau \right) ds.$$

Indeed, this follows from Theorem 2.8 with $f_1 = f, f_2 = g$, and $a(t) = u_0 + \int_\alpha^t h(s)\, ds$.

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