INEQUALITIES AND MONOTONICITY OF THE RATIO FOR THE GEOMETRIC MEANS OF A POSITIVE ARITHMETIC SEQUENCE WITH UNIT DIFFERENCE

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Abstract. For any nonnegative integer k and natural numbers n and m, we have the following inequalities on the ratio for the geometric means of a positive arithmetic sequence with unit difference:

$$\frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{1/n}}{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{1/(n+m)}} \le \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}},$$

where $\alpha \in [0,1]$ is a constant. The equality above is valid for $n=1$ and $m=1$. Moreover, some monotonicity results for the sequences involving $\sqrt[n]{\prod_{i=k+1}^{n+k}(i+\alpha)}$ are obtained, and the related inequalities are generalized.

1. Introduction

It is known that, for $n \in \mathbb{N}$, the following inequalities were given in [6]:

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1. \tag{1}$$

In [1], the left inequality in (1) was refined by

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{2}$$

for all positive real numbers r. Both bounds are best possible.

There is a rich literature on refinements, extensions, and generalizations of the inequalities in (2), for examples, [2, 7, 8, 10, 14] and references therein.

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A48.

Key words and phrases. Arithmetic sequence, geometric mean, inequality, monotonicity, ratio.

The author was supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, China.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Using analytic method and Stirling's formula, in [9, 4, 12, 13], for $n, m \in \mathbb{N}$ and k being a nonnegative integer, the author and others proved the following

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \leq \sqrt{\frac{n+k}{n+m+k}}\,, \tag{3}$$

the equality in (3) holds for $n = 1$ and $m = 1$. These extend and refine inequalities in (1).

Meanwhile, an inequality involving the ratio of gamma function was obtained in [4, 12]: For positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{4}$$

In this paper, using the ideas and method in [3, 5, 11] and the mathematical induction, we further generalize the inequalities in (3) and obtain the following inequalities of the ratio for the geometric means of a positive arithmetic sequence with unit difference and monotonicity results.

Theorem 1. *Let k be a nonnegative integer, n and m positive integers, and $\alpha \in [0,1]$ a constant. Then*

$$\frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{1/n}}{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{1/(n+m)}} \leq \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}}. \tag{5}$$

If $n = 1$ and $m = 1$, then equality in the right hand side inequality of (5) hold.

Theorem 2. *The sequence*

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i+\alpha)}}{n+k+1+\alpha} \tag{6}$$

is strictly decreasing with $n \in \mathbb{N}$, and strictly increasing with nonnegative integer k and $\alpha \in [0,1]$.

The sequence

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i+\alpha)}}{\sqrt{n+k+\alpha}} \tag{7}$$

is strictly increasing with $n \in \mathbb{N}$, nonnegative integer k, and $\alpha \in [0,1]$.

The sequence

$$\frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{1/n}}{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{1/(n+m)}} \tag{8}$$

is strictly increasing with nonnegative integer k and $\alpha \in [0,1]$ for fixed numbers $n, m \in \mathbb{N}$.

2. Proofs of Theorem 1 and Theorem 2

The left hand side inequality of (5) can be rearranged as

$$\begin{gathered}
\frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{\frac{1}{n}}}{n+k+1+\alpha} \ge \frac{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{\frac{1}{n+m}}}{n+m+k+1+\alpha},\\
\left[\prod_{i=k+1}^{n+k}\frac{i+\alpha}{n+k+1+\alpha}\right]^{\frac{1}{n}} \ge \left[\prod_{i=k+1}^{n+m+k}\frac{i+\alpha}{n+m+k+1+\alpha}\right]^{\frac{1}{n+m}},\\
\frac{1}{n}\sum_{i=k+1}^{n+k}\ln\frac{i+\alpha}{n+k+1+\alpha} \ge \frac{1}{n+m}\sum_{i=k+1}^{n+m+k}\ln\frac{i+\alpha}{n+m+k+1+\alpha},
\end{gathered}$$

which is equivalent to

$$\frac{1}{n}\sum_{i=k+1}^{n+k}\ln\frac{i+\alpha}{n+k+1+\alpha} \ge \frac{1}{n+1}\sum_{i=k+1}^{n+k+1}\ln\frac{i+\alpha}{n+k+2+\alpha}. \tag{9}$$

The same argument gives us that the right hand side inequality of (5) is equivalent to

$$\frac{1}{n}\sum_{i=k+1}^{n+k}\ln\frac{i+\alpha}{\sqrt{n+k+\alpha}} \le \frac{1}{n+1}\sum_{i=k+1}^{n+k+1}\ln\frac{i+\alpha}{\sqrt{n+k+1+\alpha}}. \tag{10}$$

Proof of the left hand side inequality in (5). Since $\ln x$ is increasing and concave on $(0,1]$, from definition of the concave function and monotonicity of the function $\ln x$, it follows that

$$\begin{aligned}
&\frac{i-k}{n+1}\ln\frac{i+1+\alpha}{n+k+2+\alpha} + \frac{n+k-i+1}{n+1}\ln\frac{i+\alpha}{n+k+2+\alpha}\\
&\le \ln\left(\frac{i-k}{n+1}\cdot\frac{i+1+\alpha}{n+k+2+\alpha} + \frac{n+k-i+1}{n+1}\cdot\frac{i+\alpha}{n+k+2+\alpha}\right)\\
&\le \ln\frac{i+\alpha}{n+k+1+\alpha}.
\end{aligned} \tag{11}$$

Summing up on both sides of (11) leads to

$$\begin{aligned}
&\sum_{i=k+1}^{n+k}\left[\frac{i-k}{n+1}\ln\frac{i+1+\alpha}{n+k+2+\alpha}+\frac{n+k-i+1}{n+1}\ln\frac{i+\alpha}{n+k+2+\alpha}\right]\\
&=\frac{n}{n+1}\sum_{i=k+1}^{n+k}\ln\frac{i+\alpha}{n+k+2+\alpha}+\frac{n}{n+1}\ln\frac{n+k+1+\alpha}{n+k+2+\alpha}\\
&\le\sum_{i=k+1}^{n+k}\ln\frac{i+\alpha}{n+k+1+\alpha},
\end{aligned}$$

which is equivalent to

$$\frac{n}{n+1}\sum_{i=k+1}^{n+k+1}\ln\frac{i+\alpha}{n+k+2+\alpha}\le\sum_{i=k+1}^{n+k}\ln\frac{i+\alpha}{n+k+1+\alpha}.$$

The proof of the left hand side of inequality (5) is complete. □

Proof of the right hand side inequality in (5). By standard argument, the inequality (10) can be rearranged as

$$\frac{1}{n}\sum_{i=k+1}^{n+k}\ln(i+\alpha)\le\frac{n+1}{2}\ln(n+k+\alpha)-\frac{n-1}{2}\ln(n+k+1+\alpha). \tag{12}$$

If $n=1$, inequality (12) holds with equality.

If $n=2$, it is easy to verify by a standard argument that inequality (12) is valid.

Assume that inequality (12) holds for some positive integer $n\in\mathbb{N}$. Straightforward computation yields

$$\begin{aligned}
&\frac{1}{n+1}\sum_{k+1}^{n+k+1}\ln(i+\alpha)\\
&=\frac{n}{n+1}\left[\frac{1}{n}\sum_{k+1}^{n+k}\ln(i+\alpha)+\frac{1}{n}\ln(n+k+1+\alpha)\right]\\
&\le\frac{n}{n+1}\left[\frac{n+1}{2}\ln(n+k+\alpha)-\frac{n-1}{2}\ln(n+k+1+\alpha)\right]\\
&\quad+\frac{\ln(n+k+1+\alpha)}{n+1}\\
&=\frac{n}{2}\ln(n+k+\alpha)-\frac{n-2}{2}\ln(n+k+1+\alpha).
\end{aligned}$$

Therefore, by induction on n, to prove inequality (12), it is sufficient to verify the following

$$\begin{aligned} n\ln(n+k+\alpha)-(n-2)\ln(n+k+1+\alpha) \\ \le (n+1)\ln(n+k+1+\alpha)-(n-1)\ln(n+k+2+\alpha), \end{aligned}$$

which is equivalent to

$$n[\ln(n+k+\alpha)-\ln(n+k+1+\alpha)] \le (n-1)[\ln(n+k+1+\alpha)-\ln(n+k+2+\alpha)],$$

which can be further rewritten as

$$\frac{n}{n-1} \ge \frac{\ln(n+k+\alpha)-\ln(n+k+1+\alpha)}{\ln(n+k+1+\alpha)-\ln(n+k+2+\alpha)}. \tag{13}$$

Using Cauchy's mean-value theorem for derivative, it follows that, there exists a number $\xi \in (0,1)$ satisfying

$$\frac{\ln(n+k+\alpha)-\ln(n+k+1+\alpha)}{\ln(n+k+1+\alpha)-\ln(n+k+2+\alpha)} = \frac{n+k+1+\alpha+\xi}{n+k+\alpha+\xi}.$$

Since $k+1+\alpha+\xi > 1$ and $n \ge 2$ and the function $\frac{x}{x-1}$ is decreasing for $x > 1$, we have

$$\frac{n+k+1+\alpha+\xi}{n+k+\alpha+\xi} \le \frac{n}{n-1},$$

which implies the inequality (13) holds. The right hand side of inequality (5) follows. □

Proof of Theorem 2. The monotonicities with n of the sequences (6) and (7) follow from Theorem 1 easily.

Direct calculating yields

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i+\alpha)}}{n+k+1+\alpha} = \frac{n+k+2+\alpha}{n+k+1+\alpha}\sqrt[n]{\frac{k+1+\alpha}{n+k+1+\alpha}} \cdot \frac{\sqrt[n]{\prod_{i=k+2}^{n+k+1}(i+\alpha)}}{n+k+2+\alpha}.$$

Let

$$\psi(t) = \ln(n+2+\alpha+t) + \frac{1}{n}\ln(1+\alpha+t) - \left(1+\frac{1}{n}\right)\ln(n+1+\alpha+t)$$

for $t \ge 0$. Direct computing and simplifying yields

$$\psi'(t) = \frac{n+1}{(1+\alpha+t)(n+1+\alpha+t)(n+2+\alpha+t)} > 0,$$

and then $\psi(t)$ is increasing. Therefore, $\exp(\psi(t))$ is increasing, and

$$\lim_{t\to\infty}\exp(\psi(t))=\lim_{t\to\infty}\left(\frac{n+t+2+\alpha}{n+t+1+\alpha}\sqrt[n]{\frac{t+1+\alpha}{n+t+1+\alpha}}\right)=1.$$

Thus

$$\frac{n+k+2+\alpha}{n+k+1+\alpha}\sqrt[n]{\frac{k+1+\alpha}{n+k+1+\alpha}}<1.$$

The sequence (6) increases strictly with k.

The fact that

$$\frac{1}{n}\sum_{i=k+1}^{n+k}\frac{1}{i+\alpha}-\frac{1}{n+k+1+\alpha}>0$$

implies the following sequence

$$\frac{1}{n}\sum_{i=k+1}^{n+k}\ln(i+\alpha)-\ln(n+k+1+\alpha)$$

is strictly increasing with α, and then the sequence (6) is also strictly increasing with α.

Easy computing yields

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i+\alpha)}}{\sqrt{n+k+\alpha}}=\sqrt{\frac{n+k+1+\alpha}{n+k+\alpha}}\sqrt[n]{\frac{k+1+\alpha}{n+k+1+\alpha}}\cdot\frac{\sqrt[n]{\prod_{i=k+2}^{n+k+1}(i+\alpha)}}{n+k+1+\alpha}.$$

Let

$$\phi(t)=\frac{1}{n}\ln(1+\alpha+t)+\left(\frac{1}{2}-\frac{1}{n}\right)\ln(n+1+\alpha+t)-\frac{1}{2}\ln(n+\alpha+t)$$

for $t\geq 0$. Computing directly and simplifying yields

$$\phi'(t)=\frac{2n+t+\alpha-1}{2(1+\alpha+t)(n+\alpha+t)(n+1+\alpha+t)}>0,$$

and then $\phi(t)$ is increasing. Therefore, $\exp(\phi(t))$ is increasing, and

$$\lim_{t\to\infty}\exp(\phi(t))=\lim_{t\to\infty}\left(\sqrt{\frac{n+t+1+\alpha}{n+t+\alpha}}\sqrt[n]{\frac{t+1+\alpha}{n+t+1+\alpha}}\right)=1.$$

Thus

$$\sqrt{\frac{n+k+1+\alpha}{n+k+\alpha}}\sqrt[n]{\frac{k+1+\alpha}{n+k+1+\alpha}}<1.$$

The sequence (7) increases strictly with k.

The fact that

$$\frac{1}{n}\sum_{i=k+1}^{n+k}\frac{1}{i+\alpha}-\frac{1}{2}\cdot\frac{1}{n+k+\alpha}>0$$

implies the following sequence

$$\frac{1}{n}\sum_{i=k+1}^{n+k}\ln(i+\alpha)-\frac{1}{2}\ln(n+k+\alpha)$$

is strictly increasing, and then the sequence (7) is strictly increasing with α.

Straightforward calculation gives us

$$\begin{aligned}
&\frac{\left[\prod_{i=k+2}^{n+k+1}(i+\alpha)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+2}^{n+m+k+1}(i+\alpha)\right]^{\frac{1}{n+m}}}\\
&\qquad=\left[\frac{(n+k+1+\alpha)^{n+m}}{(n+m+k+1+\alpha)^{n}(k+1+\alpha)^{m}}\right]^{\frac{1}{n(n+m)}}\frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{\frac{1}{n+m}}}.
\end{aligned}$$

Let

$$\tau(t)=(m+t)\ln(k+1+\alpha+t)-t\ln(m+k+1+\alpha+t)-m\ln(k+1+\alpha)$$

for $t\geq 0$. Then

$$\begin{aligned}
\tau(0)&=0,\\
\tau'(t)&=\frac{m+t}{k+\alpha+1+t}-\frac{t}{k+m+\alpha+1+t}-\ln\frac{k+m+\alpha+1+t}{k+\alpha+1+t}\\
&\geq\frac{m+t}{k+\alpha+1+t}-\frac{t}{k+m+\alpha+1+t}-\frac{m}{k+\alpha+1+t}\\
&\geq 0,
\end{aligned}$$

and $\tau(t)$ is increasing and nonnegative, which implies

$$\frac{(n+k+1+\alpha)^{n+m}}{(n+m+k+1+\alpha)^{n}(k+1+\alpha)^{m}}>1.$$

Therefore, the sequence (8) is strictly increasing with k.

The inequality

$$\frac{1}{n}\sum_{i=k+1}^{n+k}\frac{1}{i+\alpha}>\frac{1}{n+m}\sum_{i=k+1}^{n+m+k}\frac{1}{i+\alpha},\tag{14}$$

which is equivalent to

$$\sum_{i=k+1}^{n+k}\frac{1}{i+\alpha}>\frac{n}{n+k+1+\alpha}$$

being valid clearly, implies that the sequence (8) is strictly increasing with α. $\square$

3. Open problem

It is clear that it is natural to pose the following open problem.

Open Problem. *For all nonnegative integers k and natural numbers n and m, we have*

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{1/n}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{1/(n+m)}} \le \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \quad (15)$$

where a and b are positive constants.

We will discuss this open problem in a subsequent paper.

Acknowledgements. This paper was done while the author's visit to the RGMIA between November 1, 2001 and January 31, 2002, as a Visiting Professor with grants from the Victoria University of Technology and Jiaozuo Institute of Technology.

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