AN APPROXIMATION FOR THE FINITE-FOURIER TRANSFORM OF TWO INDEPENDENT VARIABLES

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Abstract. In this paper we develop some approximations of the two-dimensional Finite-Fourier transform in terms of the complex exponential mean. A cubature formula is developed as a numerical application and explored via a numerical experiment.

1. Introduction

The Fourier transform has long been a principle analytical tool in such diverse fields as linear systems, optics, random process modeling, probability theory, quantum physics, and boundary-value problems [3]. In particular, it has been very successfully applied to the restoration of astronomical data [2]. The Fourier transform, a pervasive and versatile tool, is used in many fields of science as a mathematical or physical tool to alter a problem into one that can be more easily solved. Some scientists understand Fourier theory as a physical phenomenon, not simply as a mathematical tool. In some branches of science, the Fourier transform of one function may yield another physical function [1]. Utilizing some integral identities and inequalities developed in [4, 5, 6], we point out some approximations of the two-dimensional Finite-Fourier transform in terms of the complex exponential mean \( E(z, w) \) and estimate the error of approximation for different classes of continuous mappings defined on finite intervals.

In this paper \( f : [a, b] \times [c, d] \to \mathbb{R} \) will be a continuous mapping defined on the finite interval \([a, b] \times [c, d]\) and \( \mathcal{F}(f) \) its Finite-Fourier transform. That is

\[
\mathcal{F}(f)(u, v; a, b, c, d) = \int_a^b \int_c^d f(x, y) e^{-2\pi i(ux+vy)} dy dx, \tag{1}
\]

\((u, v) \in [a, b] \times [c, d]\). For a function of one variable we use the notation

\[
\mathcal{F}(g)(u; a, b) = \int_a^b g(x) e^{-2\pi iux} dx.
\]

2. Some Integral Inequalities

In this section we employ an identity obtained in [4] and develop inequalities for the estimation of the two dimensional Fourier transform. The following inequality holds.
Theorem 1. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b] \times [c, d]$ and assume that $f''_{x,y} := \frac{\partial^2 f}{\partial x \partial y}$ exists on $(a, b) \times (c, d)$, then we have the inequality

$$\left| \mathcal{F}(f)(u, v; a, b, c, d) - J_1 - J_2 + J_3 \right|$$

$$\leq \begin{cases} \frac{(b-a)^2}{9} (d-c)^2 \| f''_{x,y} \|_\infty, & \text{if } f''_{x,y} \in L_\infty ([a, b] \times [c, d]); \\ \left[ \frac{2 [(b-a)(d-c)]^{\frac{q+1}{q}}}{(q+1)(q+2)} \right]^\frac{1}{q} \| f''_{x,y} \|_p, & \text{if } f''_{x,y} \in L_p ([a, b] \times [c, d]), \\ (b-a)(d-c) \| f''_{x,y} \|_1, & \text{if } f''_{x,y} \in L_1 ([a, b] \times [c, d]) \end{cases}$$

for all $(u, v) \in [a, b] \times [c, d]$, where

- $J_1 := J_1(u, v; a, b, c, d) = E(u) \int_a^b \mathcal{F}(f(s, \cdot))(v; c, d) \, ds,$
- $J_2 := J_2(u, v; a, b, c, d) = E(v) \int_c^d \mathcal{F}(f(\cdot, t))(u; a, b) \, dt,$
- $J_3 := J_3(u, v; a, b, c, d) = E(u) E(v) \int_a^b \int_c^d f(s, t) \, dt \, ds$

with

$E(u) := E(-2\pi i ub, -2\pi iua), \text{ and } E(v) := E(-2\pi ivd, -2\pi ivc), \text{ given that}$

$E$ is the exponential mean of complex numbers, that is

$$E(z, w) := \begin{cases} \frac{e^z - e^w}{z - w}, & \text{if } z \neq w \text{ for } z, w \in \mathbb{C}. \\ e^w, & \text{if } z = w \end{cases}$$

Furthermore we define the usual Lebesgue norms

$$\| f''_{x,y} \|_\infty = \sup_{(s, t) \in [a, b] \times [c, d]} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right| < \infty, \text{ and}$$

$$\| f''_{x,y} \|_p = \left( \int_a^b \int_c^d \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right|^p dt \, ds \right)^{\frac{1}{p}} < \infty.$$
Proof. Using the identity obtained by Barnett and Dragomir in [4],

\[
\begin{align*}
  f(x, y) & = \frac{\int_a^b f(s, y) \, ds}{b-a} + \frac{\int_c^d f(x, t) \, dt}{d-c} \\
  & - \frac{\int_a^b \int_c^d f(s, t) \, dt \, ds}{(b-a)(d-c)} \\
  & + \frac{\int_a^b \int_c^d P(x, s) Q(y, t) f''_{x,y}(s, t) \, dt \, ds}{(b-a)(d-c)}
\end{align*}
\]  

(3)

provided that \( f \) is continuous on \([a, b] \times [c, d]\) and

\[
P(x, s) = \begin{cases} 
  s-a, & a \leq s \leq x \\
  s-b, & x < s \leq b
\end{cases}
\]

and

\[
Q(y, t) = \begin{cases} 
  t-c, & c \leq t \leq y \\
  t-d, & y < t \leq d.
\end{cases}
\]

If we replace \( f(x, y) \) in (1) by its representation from (3), we get

\[
\begin{align*}
  \mathcal{F}(f)(u, v; a, b, c, d) & = \int_a^b \int_c^d \left( e^{-2\pi i(u x + v y)} \int_a^b f(s, y) \, ds \right) \, dy \, dx \\
  & + \int_a^b \int_c^d \left( e^{-2\pi i(u x + v y)} \int_c^d f(x, t) \, dt \right) \, dy \, dx \\
  & - \int_a^b \int_c^d \left( \int_a^b \int_c^d f(s, t) \, dt \, ds \right) \, dy \, dx \\
  & + R(f, u, v; a, b, c, d),
\end{align*}
\]  

(4)

where

\[
R(f, u, v; a, b, c, d)
= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left( e^{-2\pi i(u x + v y)} \right) \\
\times \left[ \int_a^b \int_c^d P(x, s) Q(y, t) f''_{x,y}(s, t) \, dt \, ds \right] \, dy \, dx.
\]  

(5)

Let

\[
\mathcal{J}_1 = \int_a^b \int_c^d \left( \frac{e^{-2\pi i u x}}{b-a} \right) \int_a^b f(s, y) \, ds \, dy, \quad \text{then}
\]

\[
\mathcal{J}_1 = \int_a^b \frac{e^{-2\pi i u x}}{b-a} \left( \int_c^d e^{-2\pi i v y} \left( \int_a^b f(s, y) \, ds \right) \, dy \right)
= e^{-2\pi i ub} - e^{-2\pi i ua} \int_a^b \left( \int_c^d e^{-2\pi i v y} f(s, y) \, dy \right) \, ds
= E(u) \int_a^b \mathcal{F}(f(s, \cdot))(v; c, d) \, ds.
\]
In a similar fashion we obtain
\[ J_2 = \int_a^b \int_c^d \left( \frac{e^{-2\pi i(ux+vy)}}{d-c} \int_c^d f(x,t) \, dt \right) \, dy dx \]

\[ = E(v) \int_c^d \mathcal{F}(f(t, \cdot)) (u; a, b) \, dt \]

and
\[ J_3 = \int_a^b \int_c^d \left( \frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \cdot \int_a^b \int_c^d f(s,t) \, dt ds \right) \, dy dx \]

\[ = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t) \, dt ds \times \int_a^b \int_c^d e^{-2\pi iux} \cdot e^{-2\pi ivy} \, dy dx \]

\[ = E(u) E(v) \int_a^b \int_c^d f(s,t) \, dt ds. \]  

(6)

Using the properties of modulus on (4), we have
\[ |\mathcal{F}(f)(u,v; a, b, c, d) - J_1 - J_2 + J_3| \]

\[ = \left| \int_a^b \int_c^d \int_a^b \int_c^d \left( \frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \cdot P(x,s) Q(y,t) \times f''_{x,y}(s,t) \, dt ds \right) \, dy dx \right| \]

\[ \leq \int_a^b \int_c^d \int_a^b \int_c^d \left| \frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \right| \left| P(x,s) \right| \left| Q(y,t) \right| \times \left| f''_{x,y}(s,t) \right| \, dt ds dy dx \]  

(7)

\[ = \int_a^b \int_c^d \int_a^b \int_c^d \frac{|P(x,s)| |Q(y,t)|}{(b-a)(d-c)} \times \left| f''_{x,y}(s,t) \right| \, dt ds dy dx. \]  

(8)

Now, we observe that
\[ \int_a^b \int_c^d \int_a^b \int_c^d \left| P(x,s) \right| \left| Q(y,t) \right| \times \left| f''_{x,y}(s,t) \right| \, dt ds dy dx \]

\[ \leq \left\| f''_{x,y} \right\|_{\infty} \left[ \int_a^b \left( \int_a^b \left| P(x,s) \right| \, ds \right) \, dx \right] \int_c^d \left( \int_c^d \left| Q(y,t) \right| \, dt \right) \, dy \]

(9)

\[ = \left\| f''_{x,y} \right\|_{\infty} \left[ \int_a^b \left\{ \frac{(s-a)^2}{2} \right\}^x \, dx \right] \times \int_c^d \left\{ \frac{(t-c)^2}{2} \right\}^y \, dy \]

\[ = \left\| f''_{x,y} \right\|_{\infty} \left[ \left( \int_a^b \frac{(x-a)^2}{2} \, dx \right) + \left( \int_a^b \frac{(b-x)^2}{2} \, dx \right) \right] \times \left( \int_c^d \frac{(y-c)^2}{2} \, dy + \int_c^d \frac{(d-y)^2}{2} \, dy \right) \]

\[ = \left\| f''_{x,y} \right\|_{\infty} \left[ \frac{3}{2} (b-a)^3 \cdot \frac{3}{2} (d-c)^3 \right]. \]
Substituting in (8) with (9), we obtain the first inequality in (2).

Applying Hölder’s integral inequality for double integrals, we get

\[
\int_a^b \int_c^d \int_a^b \int_c^d |P(x,s)Q(y,t)| |f''_{x,y}(s,t)| \, dt \, ds \, dy \, dx \\
\leq \left( \int_a^b \int_c^d \int_a^b \int_c^d |P(x,s)Q(y,t)|^q \, dt \, ds \, dy \, dx \right)^{\frac{1}{q}} \\
\times \left( \int_a^b \int_c^d \int_a^b \int_c^d |f''_{x,y}(s,t)|^p \, dt \, ds \, dy \, dx \right)^{\frac{1}{p}} \\
= \|f''_{x,y}\|^p \frac{(b-a)(d-c)^{\frac{1}{p}}}{\left( \int_a^b \left( \int_a^b |P(x,s)|^q \, ds \right) \, dx \right)^{\frac{1}{q}}} \\
\times \left( \int_c^d \left( \int_c^d |Q(y,t)|^q \, dt \right) \, dy \right)^{\frac{1}{q}} \\
= \|f''_{x,y}\|^p \left( \int_a^b \left( \int_a^b \left( \frac{(x-a)^{q+1}}{q+1} + \frac{(b-x)^{q+1}}{q+1} \right) \, dx \right)^{\frac{1}{q}} \\
\times \left( \int_c^d \left( \int_c^d \left( \frac{(y-c)^{q+1}}{q+1} + \frac{(d-y)^{q+1}}{q+1} \right) \, dy \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \right) \\
= \|f''_{x,y}\|^p \left[ \frac{2^q (b-a)^{1+\frac{q}{2}} (d-c)^{1+\frac{q}{2}}}{((q+1)(q+2))^{\frac{q}{2}}} \right]. \\
\tag{10}
\]

Utilizing (8) with (11), we get the second inequality of (2).

Finally, we obtain that

\[
\int_a^b \int_c^d \int_a^b \int_c^d |P(x,s)Q(y,t)| \times |f''_{x,y}(s,t)| \, dt \, ds \, dy \, dx \\
\leq \sup_{(x,s) \in [a,b]^{2}} |P(x,s)| \sup_{(y,t) \in [c,d]^{2}} |Q(y,t)| \times \int_a^b \int_c^d \int_a^b \int_c^d |f''_{x,y}| \, dt \, ds \, dy \, dx \\
= (b-a)(d-c) \int_a^b \int_c^d \int_a^b \int_c^d |f''_{x,y}| \, dt \, ds \, dy \, dx \\
= \|f''_{x,y}\|_1 \left( (b-a)^2 (d-c)^2 \right). \\
\tag{12}
\]

Substituting in (8) with (12), gives the final inequality in (2), where we have used the fact that

\[
\max \{X, Y\} = \frac{X+Y}{2} + \left| \frac{Y-X}{2} \right|. 
\]

Thus the theorem is completely proved. \(\blacksquare\)
3. A Numerical Cubature Formula

To illustrate the use of a cubature formula, we form a composite rule from the inequality (2).

Let us consider the arbitrary divisions \( I_n : a = x_0 < x_1 < \cdots < x_n = b \) on \([a, b]\) and \( J_m : c = y_0 < y_1 < \cdots < y_m = d \) on \([c, d]\), define the sum

\[
\mathcal{F}(f, I_n, J_m, u, v) = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \mathcal{I}_1(SD) + \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \mathcal{I}_2(SD) - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \mathcal{I}_3(SD) \tag{13}
\]

where

\[
(SD) := (u, v; x_k, x_{k+1}, y_l, y_{l+1});
\]

\[
h_k := x_{k+1} - x_k \ (k = 0, 1, 2, \cdots, n - 1) \quad \text{and} \quad v_l := y_{l+1} - y_l \ (l = 0, 1, \cdots, m - 1)
\]

Under the above assumptions the following theorem can be obtained.

**Theorem 2.** Let \( f : [a, b] \times [c, d] \to \mathbb{R} \) be continuous mapping on \([a, b] \times [c, d]\), then we have the cubature formula

\[
\mathcal{F}(f)(u, v; a, b, c, d) = \mathcal{F}(f, I_n, J_m, u, v) + R(f, I_n, J_m, u, v),
\]

where \( \mathcal{F}(f, I_n, J_m, \cdot, \cdot) \) approximates the Fourier Transform \( \mathcal{F}(f) \) at every point \((u, v) \in [a, b] \times [c, d] \), and the remainder term \( R(f, I_n, J_m, \cdot, \cdot) \) satisfies the bounds

\[
|R(f, I_n, J_m, u, v)| \leq \begin{cases} 
\frac{1}{9} \left( \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} h_k^2 v_l^2 \right) \| f''_{xy} \|_\infty \\
\frac{2}{(q + 1)(q + 2)} \left( \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} h_k v_l \right) \| f''_{xy} \|_1 \\
\kappa(h) \tau(\nu) \| f''_{xy} \|_p 
\end{cases} \tag{15}
\]

where

\[
\kappa(h) := \max \{ h_k \mid k = 0, \cdots, n - 1 \} \quad \text{and} \quad \tau(\nu) := \max \{ \nu_l \mid l = 0, \cdots, m - 1 \}.
\]

**Proof.** Applying Theorem 1 over every subinterval \([x_k, x_{k+1}]\) and \([y_l, y_{l+1}]\), we can state that

\[
\left| \mathcal{F}(f) (SD) - \mathcal{I}_1(SD) - \mathcal{I}_2(SD) + \mathcal{I}_3(SD) \right| 
\leq \begin{cases} 
\frac{1}{9} h_k^2 v_l^2 \sup_{(s, t) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right| \\
\frac{2}{(q + 1)(q + 2)} \left( \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} h_k v_l \right) \| f''_{xy} \|_1 \\
h_k v_l \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right|^p dtds 
\end{cases}
\]

\[
\leq \begin{cases} 
\frac{1}{9} h_k^2 v_l^2 \sup_{(s, t) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right| \\
\frac{2}{(q + 1)(q + 2)} \left( \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} h_k v_l \right) \| f''_{xy} \|_1 \\
h_k v_l \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right|^p dtds 
\end{cases}
\]
where

\[
\mathcal{D}^2 := \left( \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right|^p \, dt \, ds \right)^{\frac{1}{p}},
\]

Summing over \( k \) from 0 to \( n - 1 \) and \( l \) from 0 to \( m - 1 \), and using the triangle inequality, we obtain

\[
|R(f, I_n, J_m, u, v)| = |\mathcal{F}(f)(u, v; a, b, c, d) - \mathcal{F}(f, I_n, J_m, u, v)|
\]

\[
\leq \left[ \frac{1}{9} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \sup_{(s, t) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right| n^2 v^2 \right]^{\frac{2}{3}} \mathcal{D}^2
\]

\[
\leq \left[ \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} h_k v_l \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right|^p \, dt \, ds \right]^{\frac{1}{p}}
\]

where

\[
\sup_{(s, t) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right| \leq \sup_{(s, t) \in [a, b] \times [c, d]} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right| = \|f''_{x,y}\|_{\infty}
\]

thus the first inequality in (15) is obtained. Using Hölder’s discrete inequality, we have

\[
\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} [h_k v_l]^{q+1} \left( \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right|^p \, dt \, ds \right)^{\frac{1}{p}}
\]

\[
\leq \left[ \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left( [h_k v_l]^{q+1} \right)^{\frac{1}{q}} \right]^{\frac{q}{q}} \times \left[ \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left( \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s, t)}{\partial x \partial y} \right|^p \, dt \, ds \right)^{\frac{1}{p}} \right]^{\frac{1}{p}}
\]

\[
= \left( \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} (h_k v_l)^{q+1} \right)^{\frac{1}{q}} \|f''_{x,y}\|_p
\]
which proves the second inequality in (15). For the last inequality, we observe that

\[
\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} h_k v_l \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right| dt ds \leq \kappa(h) \tau(v) \sum_{l=0}^{m-1} h_k v_l \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right| dt ds \\
= \kappa(h) \tau(v) \int_a^b \int_c^d \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right| dt ds
\]

and the theorem is completely proved.

In practical applications, it is convenient to consider the equidistant partitioning of the region \([a, b] \times [c, d]\). Thus let

\[
I_n : x_k = a + k \cdot \frac{b-a}{n}, \quad k = 0, 1, \ldots, n \quad \text{and} \quad J_m : y_l = c + l \cdot \frac{d-c}{m}, \quad l = 0, 1, \ldots, m,
\]

and we defined the sum

\[
\mathcal{F}_{n,m}(f, I_n, J_m, u, v) = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} I_1(ES) + \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} J_2(ES) - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} I_3(ES)
\]

(16)

where \((ES) := (u, v; a + k \cdot \frac{b-a}{n}, a + (k+1) \cdot \frac{b-a}{n}, c + l \cdot \frac{d-c}{m}, c + (l+1) \cdot \frac{d-c}{m})\).

The following corollary of Theorem 2 holds:

**Corollary 1.** Let \(f\) be as defined in Theorem 2. Then we have

\[
\mathcal{F}(f)(u, v; a, b, c, d) = \mathcal{F}_{n,m}(f, I_n, J_m, u, v) + R_{n,m}(f, I_n, J_m, u, v),
\]

(17)

where \(\mathcal{F}_{n,m}(f, I_n, J_m, \ldots)\) approximates the Fourier Transform \(\mathcal{F}(f)\) at every point \((u, v) \in [a, b] \times [c, d]\), and the remainder term \(R_{n,m}(f, I_n, J_m, \ldots)\) satisfies the bounds

\[
|R_{n,m}(f, I_n, J_m, u, v)| \leq \left\{ \begin{array}{ll}
\frac{(b-a)^2 (d-c)^2}{9nm} \|f''_{x,y}\|_{\infty}; \\
\left[ \frac{2[(b-a)(d-c)]^{1+q}}{(q+1)(q+2)} \right]^{\frac{2}{q}} \frac{nm}{\|f''_{x,y}\|_p}; \\
\frac{(b-a)(d-c)}{nm} \|f''_{x,y}\|_1.
\end{array} \right.
\]

(18)
4. Numerical Experiment

To illustrate the use of the cubature formula, we will employ (13) to approximate the finite Fourier transform of

\[ f(x, y) = e^{3x - 2y}(x - y), \quad 0 \leq x, y \leq 1. \] (19)

Since \( \mathcal{F}(f) \) can be computed analytically we can gauge the performance of the cubature rule as well as compare it to the theoretical error bound (18). The results are shown in Table 1 where \( n^2 \) is the number of uniform partitions of the domain \([0, 1] \times [0, 1]\). It is clearly evident that the cubature rule performs extremely well and achieves single precision accuracy when \( n = 16 \). Halving the interval size will increase the accuracy by approximately one and a half orders, and a simple analysis shows that the rate of convergence is at least \( O((nm)^{-2}) \). The contrasts with the theoretical error which is \( O(1/(nm)) \). Extending the Peano kernel, that is using a higher order identity to that of (3), may provide a higher order theoretical error result.

In Figure 1, we show a three dimensional plot of the finite Fourier transformed obtained using (13).

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<th>Num. Error</th>
<th>Ratio</th>
<th>Th. Error</th>
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<td>3.11</td>
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<td>0.82E+00</td>
</tr>
<tr>
<td>8</td>
<td>0.16E-05</td>
<td>30.63</td>
<td>0.20E+00</td>
</tr>
<tr>
<td>16</td>
<td>0.23E-07</td>
<td>67.49</td>
<td>0.51E-01</td>
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<tr>
<td>32</td>
<td>0.34E-09</td>
<td>68.02</td>
<td>0.13E-01</td>
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<tr>
<td>64</td>
<td>0.77E-11</td>
<td>44.09</td>
<td>0.32E-02</td>
</tr>
</tbody>
</table>

Table 1: Numerical error (column 2) and theoretical error (column 4) in approximating the finite Fourier transform of (19) using equation (13).

\[ f(x, y) = e^{3x - 2y}(x - y), \quad 0 \leq x, y \leq 1 \] evaluated using the rule (13).
5. Conclusion

The current work has modelled a means for estimating the partition required in order to be guaranteed a certain accuracy for the two-dimensional Finite-Fourier transform in term of the complex exponential mean.

References


