A GENERALISATION OF AN OSTROWSKI INEQUALITY IN INNER PRODUCT SPACES

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Abstract. A generalisation of inner product spaces of an inequality due to Ostrowski and applications for sequences and integrals are given.

1. Introduction

In 1951, A.M. Ostrowski [2, p. 289] obtained the following result (see also [1, p. 92]).

Theorem 1. Suppose that \( \mathbf{a} = (a_1, \ldots, a_n) \), \( \mathbf{b} = (b_1, \ldots, b_n) \) and \( \mathbf{x} = (x_1, \ldots, x_n) \) are real \( n \)-tuples such that \( \mathbf{a} \neq 0 \) and

\[
\sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i x_i = 1.
\]

Then

\[
\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} \frac{a_i^2}{\sum_{i=1}^{n} b_i^2} - \left( \sum_{i=1}^{n} a_i b_i \right)^2, \tag{1.2}
\]

with equality if and only if

\[
x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2}, \quad k = 1, \ldots, n. \tag{1.3}
\]

Another similar result due to Ostrowski which is far less known and obtained in the same work [2, p. 130] (see also [1, p. 94]), is the following one.

Theorem 2. Let \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{x} \) be \( n \)-tuples of real numbers with \( \mathbf{a} \neq 0 \) and

\[
\sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 = 1.
\]

Then

\[
\sum_{i=1}^{n} \frac{a_i^2}{b_i^2} \geq \left( \sum_{i=1}^{n} b_i x_i \right)^2. \tag{1.5}
\]

If \( \mathbf{a} \) and \( \mathbf{b} \) are not proportional, then the equality holds in (1.5) iff

\[
x_k = q \cdot \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\left( \sum_{k=1}^{n} a_k^2 \right)^2 \left( \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 \right)^2}, \quad k \in \{1, \ldots, n\}, \tag{1.6}
\]

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The case of equality which was neither mentioned in [1] nor in [2] is considered in Remark 1.

In the present paper, by the use of an elementary argument based on Schwarz’s inequality, a natural generalisation in inner-product spaces of (1.5) is given. The case of equality is analyzed. Applications for sequences and integrals are also provided.

2. The Results

The following theorem holds.

**Theorem 3.** Let \((H, \langle \cdot, \cdot \rangle)\) be a real or complex inner product space and \(a, b \in H\) two linearly independent vectors. If \(x \in H\) is such that

\[(i) \quad \langle x, a \rangle = 0 \text{ and } \|x\| = 1,\]

then

\[
\frac{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}{\|a\|^2} \geq |\langle x, b \rangle|^2.
\]

The equality holds in (2.1) iff

\[
\begin{align*}
\langle x, a \rangle &= \nu \left( b - \frac{\langle a, b \rangle}{\|a\|^2} \cdot a \right), \\
\end{align*}
\]

where \(\nu \in K \ (C, \mathbb{R})\) is such that

\[
|\nu| = \frac{\|a\|}{\left[ \|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right]^{\frac{1}{2}}}. 
\]

**Proof.** We use Schwarz’s inequality in the inner product space \(H\), i.e.,

\[
\|u\|^2 \|v\|^2 \geq |\langle u, v \rangle|^2, \quad u, v \in H
\]

with equality iff there is a scalar \(\alpha \in K\) such that

\[
\begin{align*}
u &= \alpha v. \\
\end{align*}
\]

If we apply (2.4) for \(u = z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c, v = d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c\), where \(c \neq 0\) and \(c, d, z \in H\), and taking into account that

\[
\begin{align*}
\left\| z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c \right\|^2 &= \frac{\|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2}{\|c\|^2}, \\
\left\| d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\|^2 &= \frac{\|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2}{\|c\|^2}
\end{align*}
\]

and

\[
\begin{align*}
\langle z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \rangle &= \frac{\langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle}{\|c\|^2},
\end{align*}
\]

we deduce the inequality

\[
\begin{align*}
\left[ \|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2 \right] \left[ \|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2 \right] \geq \left[ \langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle \right]^2.
\end{align*}
\]
with equality iff there is a $\beta \in K$ such that

$$z = \frac{\langle z, c \rangle}{\|c\|^2} \cdot c + \beta \left( d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right).$$

If in (2.6) we choose $z = x$, $c = a$ and $d = b$, where $a$ and $x$ satisfy (i), then we deduce

$$\|a\|^2 \left[ \|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right] \geq \left[ \langle x, b \rangle \|a\|^2 \right]^2$$

which is clearly equivalent to (2.1).

The equality holds in (2.1) iff

$$x = \nu \left( b - \frac{\langle a, b \rangle}{\|a\|^2} \cdot a \right),$$

where $\nu \in K$ satisfies the condition

$$1 = \|x\| = |\nu| \left\| b - \frac{\langle a, b \rangle}{\|a\|^2} \cdot a \right\| = |\nu| \left[ \frac{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}{\|a\|^2} \right]^{\frac{1}{2}},$$

and the theorem is thus proved. \qed

The following particular cases hold.

1. If $a, b, x \in \ell^2 (K)$, $K = \mathbb{C}, \mathbb{R}$, where

$$\ell^2 (K) := \left\{ x = (x_i)_{i \in \mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with $a, b$ linearly independent and

$$\sum_{i=1}^{\infty} x_i a_i = 0, \quad \sum_{i=1}^{\infty} |x_i|^2 = 1,$$

then

$$\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - |\sum_{i=1}^{\infty} a_i b_i|^2 \geq \left| \sum_{i=1}^{\infty} x_i b_i \right|^2.$$

The equality holds in (2.9) iff

$$x_i = \nu \left[ b_i - \frac{\sum_{k=1}^{\infty} a_k b_k}{\sum_{k=1}^{\infty} |a_k|^2} \cdot a_i \right], \quad i \in \{1, 2, \ldots \}$$

with $\nu \in K$ is such that

$$|\nu| = \left( \frac{\sum_{k=1}^{\infty} |a_k|^2}{\sum_{k=1}^{\infty} |a_k|^2 \sum_{k=1}^{\infty} |b_k|^2 - |\sum_{k=1}^{\infty} a_k b_k|^2} \right)^{\frac{1}{2}}.$$
2. If \( f, g, h \in L^2(\Omega, m) \), where \( \Omega \) is an \( m \)-measurable space and
\[
L^2(\Omega, m) := \left\{ f : \Omega \rightarrow \mathbb{K}, \int_\Omega |f(x)|^2 \, dm(x) < \infty \right\},
\]
with \( f, g \) being linearly independent and
\[
\int_\Omega h(x) \frac{f(x)}{\overline{f(x)}} \, dm(x) = 0, \quad \int_\Omega |h(x)|^2 \, dm(x) = 1,
\]
then
\[
\frac{\int_\Omega |f(x)|^2 \, dm(x) \int_\Omega |g(x)|^2 \, dm(x) - \left| \int_\Omega f(x) \overline{g(x)} \, dm(x) \right|^2}{\int_\Omega |f(x)|^2 \, dm(x)} \geq \left| \int_\Omega h(x) \overline{g(x)} \, dm(x) \right|^2.
\]
The equality holds in (2.13) iff
\[
h(x) = \nu \left[ g(x) - \frac{\int_\Omega g(x) \overline{f(x)} \, dm(x)}{\int_\Omega |f(x)|^2 \, dm(x)} \, f(x) \right]
\]
for a.e. \( x \in \Omega \)
and \( \nu \in \mathbb{K} \) with
\[
|\nu| = \left( \frac{\left( \int_\Omega |f(x)|^2 \, dm(x) \right)}{\left( \int_\Omega |f(x)|^2 \, dm(x) \int_\Omega |g(x)|^2 \, dm(x) - \left| \int_\Omega f(x) \overline{g(x)} \, dm(x) \right|^2 \right)^{1/2}} \right)^{1/2}.
\]

References

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