INTEGRAL EXPRESSION AND INEQUALITIES OF MATHIEU TYPE SERIES

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Abstract. Let $r > 0$ be a positive real number and $a = (a_1, a_2, \ldots, a_k, \ldots)$ a positive sequence such that the series $g(x) = \sum_{k=1}^{\infty} e^{-a_k x}$ converges for $x > 0$, then

$$\sum_{k=1}^{\infty} \frac{a_k}{(a_k^2 + r^2)^2} = \frac{1}{2} \int_{0}^{\infty} xg(x) \sin(rx) \, dx.$$  

If $a = (a_1, a_2, \ldots, a_k, \ldots)$ is a positive arithmetic sequence with difference $d > 0$, then several inequalities of Mathieu type series $\sum_{k=1}^{\infty} \frac{a_k}{(a_k^2 + r^2)^2}$ are obtained for $r > 0$ under some conditions on $a$.

1. Introduction

In 1890, Mathieu defined $S(r)$ in [12] as

$$S(r) = \sum_{k=1}^{\infty} \frac{2k}{(k^2 + r^2)^2}, \quad r > 0,$$  

and conjectured that $S(r) < \frac{1}{r^2}$. We call formula (1) Mathieu’s series. In [2, 11], Berg and Makai proved

$$\frac{1}{r^2 + \frac{1}{2}} < S(r) < \frac{1}{r^2}.$$  


$$\frac{1}{r^2 + \frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2 + \frac{1}{6}},$$  

where $\zeta$ denotes the zeta function and the number $\zeta(3)$ is the best possible. The integral form of Mathieu’s series (1) was given in [6, 7] by

$$S(r) = \frac{1}{r} \int_{0}^{\infty} \frac{x}{e^x - 1} \sin(rx) \, dx.$$  

Recently, the following results were obtained in [14, 15]:

(1) Let $\Phi_1$ and $\Phi_2$ be two integrable functions such that $\frac{x}{e^x - 1} - \Phi_1(x)$ and $\Phi_2(x) - \frac{x}{e^x - 1}$ are increasing. Then, for any positive number $r$, we have

$$\frac{1}{r} \int_{0}^{\infty} \Phi_2(x) \sin(rx) \, dx \leq S(r) \leq \frac{1}{r} \int_{0}^{\infty} \Phi_1(x) \sin(rx) \, dx.$$  

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where $J_S$ for above was partially solved, for example, among other things, an integral expression established in [example, example, example].

There has been a much rich literature on the study of Mathieu's series, for example, [14, 15], the open problem stated above was considered and an integral expression was obtained: Let $\alpha = 0$ and $\alpha > 0$, then, for any positive real number $a$, we have

$$\frac{2n}{(a^2 + r^2)^{n/2}} < \frac{1}{r} \int_0^{\pi/r} \frac{x}{e^x - 1} \sin(rx) \, dx < \frac{1 + \exp(-\pi/r)}{r^2 + \frac{1}{4}}. \quad (8)$$

In [9, 14, 15], the following open problem was proposed by B.-N. Guo and F. Qi respectively: Let

$$S(r, t, \alpha) = \sum_{n=1}^{\infty} \frac{2n^{n/2}}{(n^2 + r^2)^{n/2}} \quad (10)$$

for $t > 0$, $r > 0$ and $\alpha > 0$. Can one obtain an integral expression of $S(r, t, \alpha)$? Give some sharp inequalities for the series $S(r, t, \alpha)$.

In [17], the open problem stated above was considered and an integral expression of $S(r, t, 2)$ was obtained: Let $a > 0$ and $p \in \mathbb{N}$, then

$$S(a, p, 2) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + a^2)^{p+1}} = \frac{2}{(2a)^{p+1}} \int_0^{\infty} \frac{t^p \cos\left(\frac{\pi x}{2} - at\right)}{e^x - 1} \, dt - 2 \sum_{k=1}^{p} \frac{(k - 1)(2a)^{k-2p-1} (- (p + 1))}{k!(p - k + 1)} \int_0^{\infty} \frac{t^k \cos\left(\frac{\pi x}{2} (2p - k + 1) - at\right)}{e^x - 1} \, dt. \quad (11)$$

Using the quadrature formulas, some new inequalities of Mathieu series (1) were established in [8]. By the help of Laplace transform, the open problem mentioned above was partially solved, for example, among other things, an integral expression for $S\left(r, \frac{1}{2}, 2\right)$ was given as follows:

$$S\left(r, \frac{1}{2}, 2\right) = \frac{2}{r} \int_0^{\infty} \frac{J_0(\sqrt{2}t)}{e^t - 1} \, dt, \quad (12)$$

where $J_0$ is Bessel function of order zero.

In this paper, we are about to investigate the following Mathieu type series

$$S(r, a) = \sum_{k=1}^{\infty} \frac{a_k}{(a_k^2 + r^2)^{1/2}}, \quad (13)$$
where \( a = (a_1, a_2, \ldots, a_k, \ldots) \) is a positive sequence satisfying \( \lim_{k \to \infty} a_k = \infty \), and obtain an integral expression and some inequalities of \( S(r, a) \) under some suitable conditons. At last, two open problems are proposed.

2. **Integral expression of Mathieu type series** (13)

Let \( a = (a_1, a_2, \ldots, a_k, \ldots) \) be a positive sequence satisfying \( \lim_{k \to \infty} a_k = \infty \), let

\[
b_k(r, a) = \frac{a_k}{(a_k^2 + r^2)^2}
\]

and

\[
S(r, a) = \sum_{k=1}^{\infty} b_k(r, a).
\]

**Theorem 1.** Let \( r > 0 \) and \( a = (a_1, a_2, \ldots, a_k, \ldots) \) be a positive sequence such that the series

\[
g(x) \triangleq \sum_{k=1}^{\infty} e^{-a_k x}
\]

converges for \( x > 0 \). Then we have

\[
S(r, a) = \frac{1}{2r} \int_0^{\infty} x g(x) \sin(rx) \, dx.
\]

**Proof.** By direct computation, we have

\[
b_k(r, a) = \frac{i}{4r} \left[ \frac{1}{(a_k + ir)^2} - \frac{1}{(a_k - ir)^2} \right],
\]

where \( i^2 = -1 \).

From the definition of gamma function, it is easy to see that

\[
\frac{\Gamma(t)}{u^t} = \int_{0}^{\infty} x^{t-1} e^{-ux} \, dx.
\]

Set \( u = a_k \pm ir \) in formula (19), then

\[
\Gamma(t) = \int_{0}^{\infty} x^{t-1} e^{-(a_k \pm ir)x} \, dx,
\]

\[
\Gamma(t) \sum_{k=1}^{\infty} \left[ \frac{1}{(a_k + ir)^t} - \frac{1}{(a_k - ir)^t} \right] = -2i \int_{0}^{\infty} x^{t-1} e^{-(a_k \pm ir)x} \sin(rx) \, dx,
\]

\[
\Gamma(t) \sum_{k=1}^{\infty} \left[ \frac{1}{(a_k + ir)^t} - \frac{1}{(a_k - ir)^t} \right] = -2i \int_{0}^{\infty} g(x) x^{t-1} \sin(rx) \, dx.
\]

Since \( \Gamma(2) = 1 \), we have

\[
S(r; f) = \frac{1}{2r} \int_{0}^{\infty} g(x) x \sin(rx) \, dx.
\]

The proof is complete. \( \square \)

**Remark 1.** If let \( a_k = k \) in (17), then we can easily obtain the formula (4) in [7]. Note that the proof of Theorem 1 uses the technique which was used by O. E. Emersleben in [7].
Corollary 1. If \( a = (a_1, a_2, \ldots, a_k, \ldots) \) is a positive arithmetic sequence with difference \( d > 0 \), then for any positive real number \( r > 0 \), we have

\[
S(r, a) = \frac{1}{2r} \int_0^{\infty} \frac{xe^{(d-a_1)x}}{e^{dx} - 1} \sin(rx) \, dx. \tag{24}
\]

Proof. Since \( a = (a_1, a_2, \ldots, a_k, \ldots) \) is an arithmetic sequence with difference \( d > 0 \), then \( \{e^{-xa_k}\} \) is a geometric sequence with constant ratio \( e^{-dx} < 1 \), thus

\[
g(x) = \sum_{k=1}^{\infty} e^{-xa_k} = \frac{e^{(d-a_1)x}}{e^{dx} - 1}.
\]

Then formula (24) follows from (17). \( \square \)

3. Inequalities of Mathieu type series (24)

Now we give a general estimate of Mathieu type series (24) as follows.

Theorem 2. Let \( a = (a_1, a_2, \ldots, a_k, \ldots) \) be a positive arithmetic sequence with difference \( d > 0 \). Let \( \Phi_1 \) and \( \Phi_2 \) be two integrable functions such that \( \frac{x e^{(d-a_1)x}}{e^{dx} - 1} - \Phi_1(x) \) and \( \Phi_2(x) - \frac{x e^{(d-a_1)x}}{e^{dx} - 1} \) are increasing. Then for any positive number \( r \),

\[
\frac{1}{2r} \int_0^{\infty} \Phi_2(x) \sin(rx) \, dx \leq S(r, a) \leq \frac{1}{2r} \int_0^{\infty} \Phi_1(x) \sin(rx) \, dx. \tag{25}
\]

Proof. The function \( \psi(x) = \sin(rx) \) has a period \( \frac{2\pi}{r} \), and \( \psi(x) = -\psi(x + \frac{\pi}{r}) \).

Since \( f(x) = \frac{x e^{(d-a_1)x}}{e^{dx} - 1} - \Phi_1(x) \) is increasing, for any \( \alpha > 0 \), we have \( f(x + \alpha) \geq f(x) \). Therefore, from Lemma 1 or Corollary 1 in [14, 15], we obtain

\[
\int_{2k\pi/r}^{2(k+1)\pi/r} \left[ x e^{(d-a_1)x} - \Phi_1(x) \right] \sin(rx) \, dx \leq 0, \tag{26}
\]

\[
\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{x e^{(d-a_1)x}}{e^{dx} - 1} \sin(rx) \, dx \leq \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x) \sin(rx) \, dx. \tag{27}
\]

Then, from formula (13), we have

\[
S(r, a) = \frac{1}{2r} \sum_{k=0}^{\infty} \int_{2k\pi/r}^{2(k+1)\pi/r} \frac{x e^{(d-a_1)x}}{e^{dx} - 1} \sin(rx) \, dx
\leq \frac{1}{2r} \sum_{k=0}^{\infty} \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x) \sin(rx) \, dx \tag{28}
\]

Then, from formula (13), we have

\[
S(r, a) = \frac{1}{2r} \sum_{k=0}^{\infty} \int_{2k\pi/r}^{2(k+1)\pi/r} \frac{x e^{(d-a_1)x}}{e^{dx} - 1} \sin(rx) \, dx
\leq \frac{1}{2r} \sum_{k=0}^{\infty} \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x) \sin(rx) \, dx
\leq \frac{1}{2r} \int_0^{\infty} \Phi_1(x) \sin(rx) \, dx.
\]

The right hand side of inequality (25) follows.

Similar arguments yield the left hand side of inequality (25). \( \square \)

For \( x > 0 \), we have

\[
\frac{1}{e^x} < \frac{x}{e^x - 1} < \frac{1}{e^{x/2}}. \tag{29}
\]
Theorem 3. Let \( a = (a_1, a_2, \ldots, a_k, \ldots) \) be a positive arithmetic sequence with difference \( d > 0 \) and \( a_1 > \frac{d}{2} \). For positive number \( r > 0 \), we have

\[
\frac{1}{d} \left\{ \frac{(1 + e^{-\pi a_1/r})}{2(a_1^2 + r^2)} (1 - e^{-2\pi a_1/r}) - \frac{2}{(2a_1 - d)^2 + 4r^2} \left[ 1 - e^{-\pi(2a_1-d)/r} \right] \right\} \leq S(r, a)
\]

\[
\leq \frac{1}{d} \left\{ \frac{2}{(2a_1 - d)^2 + 4r^2} \left[ 1 - e^{-\pi(2a_1-d)/r} \right] - \frac{(e^{-2\pi a_1/r} + e^{-\pi a_1/r})}{2(a_1^2 + r^2)} \left( 1 - e^{-2\pi a_1/r} \right) \right\}
\]

(30)

Proof. For \( r > 0 \), using (24), by direct calculation, we have

\[
S(r, a) = \frac{1}{2r} \sum_{k=0}^{\infty} \int_{2k\pi/r}^{(2k+1)\pi/r} \int_{(2k+1)\pi/r}^{(2k+2)\pi/r} e^{r(d-a_1)x}\sin(rx) \frac{dx}{e^{dx} - 1} \, dx.
\]

(31)

The inequality (29) gives us

\[
\frac{r (1 + e^{-\pi a_1/r})}{d(a_1^2 + r^2)} \left( 1 - e^{-2\pi a_1/r} \right) = \sum_{k=0}^{\infty} \int_{2k\pi/r}^{(2k+1)\pi/r} \frac{\sin(rx)}{e^{a_1x}} \, dx
\]

\[
\leq \sum_{k=0}^{\infty} \int_{2k\pi/r}^{(2k+1)\pi/r} e^{r(d-a_1)x}\sin(rx) \frac{dx}{e^{dx} - 1} \, dx
\]

\[
\leq \sum_{k=0}^{\infty} \int_{2k\pi/r}^{(2k+1)\pi/r} \frac{\sin(rx)}{e^{a_1x}} \, dx = \frac{4r \left[ 1 - e^{-\pi(2a_1-d)/(2r)} \right]}{d(a_1^2 - d^2 + 4r^2) \left[ 1 - e^{-\pi(2a_1-d)/r} \right]}
\]

(32)

and

\[
-\frac{4r \left[ e^{-\pi(2a_1-d)/r} + e^{-\pi(2a_1-d)/(2r)} \right]}{d(a_1^2 - d^2 + 4r^2) \left[ 1 - e^{-\pi(2a_1-d)/r} \right]} = \sum_{k=0}^{\infty} \int_{2k\pi/r}^{(2k+1)\pi/r} \frac{\sin(rx)}{e^{a_1x}} \, dx
\]

\[
\leq \sum_{k=0}^{\infty} \int_{2k\pi/r}^{(2k+1)\pi/r} e^{r(d-a_1)x}\sin(rx) \frac{dx}{e^{dx} - 1} \, dx
\]

\[
\leq \sum_{k=0}^{\infty} \int_{2k\pi/r}^{(2k+1)\pi/r} \frac{\sin(rx)}{e^{a_1x}} \, dx = -\frac{r \left( e^{-2\pi a_1/r} + e^{-\pi a_1/r} \right)}{d(a_1^2 + r^2) \left( 1 - e^{-2\pi a_1/r} \right)}.
\]

(33)

Substituting (32) and (33) into (31) yields (30). The proof is complete. \( \square \)

Theorem 4. Let \( a = (a_1, a_2, \ldots, a_k, \ldots) \) be a positive arithmetic sequence with difference \( d > 0 \) and \( a_1 > \frac{d}{2} \). For any positive number \( r > 0 \), we have

\[
S(r, a) \leq \frac{1}{2r} \int_0^{\pi/r} x e^{r(d-a_1)x}\sin(rx) \frac{dx}{e^{dx} - 1} < \frac{2}{d((d - 2a_1)^2 + 4r^2)} \left[ 1 + e^{\pi(d-2a_1)/(2r)} \right].
\]

(34)
Proof. Straightforward computation yields

\[
\int_0^\infty \frac{xe^{(d-a_1)x} \sin(rx)}{e^{ax} - 1} \, dx - \int_0^{\pi/r} \frac{xe^{(d-a_1)x} \sin(rx)}{e^{ax} - 1} \, dx
\]

\[
= \sum_{k=1}^{\infty} \int_{k\pi/r}^{(k+1)\pi/r} \frac{xe^{(d-a_1)x} \sin(rx)}{e^{ax} - 1} \, dx
\]

\[
= \sum_{i=1}^{\infty} \left( \int_{2i\pi/r}^{(2i+1)\pi/r} \frac{xe^{(d-a_1)x} \sin(rx)}{e^{ax} - 1} \, dx
\]

\[
= \frac{1}{r^2} \sum_{i=1}^{\infty} \left[ \int_0^\pi \frac{(2i\pi + x) \exp((d-a_1)(2i\pi + x))}{\exp\left(\frac{(d-a_1)(2i\pi + x)}{r}\right) - 1} \sin(2i\pi + x) \, dx
\]

\[
- \frac{[(2i-1)\pi + x] \exp((d-a_1)(2i-1)\pi + x)}{\exp\left(\frac{(d-a_1)(2i-1)\pi + x)}{r}\right) - 1} \right] \sin x \, dx. \tag{35}
\]

For \( x > 0 \), we have

\[
\frac{d}{dx} \left( \frac{xe^{(d-a_1)x}}{e^{ax} - 1} \right) = -\frac{x \left[ a_1 + \frac{1}{x} \left( \frac{dx}{e^{ax} - 1} \right) \right] e^{(d-a_1)x}}{e^{ax} - 1}, \tag{36}
\]

\[
\frac{d}{dx} \left( \frac{dx}{e^{ax} - 1} \right) = 1 - 2e^{dx} + e^{2dx} - d^2x^2e^{dx}, \tag{37}
\]

\[
\frac{d}{dx} \left[ 1 - 2e^{dx} + e^{2dx} - d^2x^2e^{dx} \right] = 2d^2xe^{dx} \left[ e^{dx} - 1 - \frac{dx}{2} \right] > 0, \tag{38}
\]

\[
\lim_{x \to 0} \left[ \frac{1}{x} \left( \frac{dx}{e^{ax} - 1} - 1 \right) \right] = -\frac{d}{2}, \tag{39}
\]

then \( \frac{d}{dx} \left( \frac{dx}{e^{ax} - 1} \right) > 0 \) and \( \frac{1}{x} \left( \frac{dx}{e^{ax} - 1} - 1 \right) > -\frac{d}{2} \). From \( a_1 > \frac{d}{2} \), it follows that

\[
\frac{d}{dx} \left( \frac{xe^{(d-a_1)x}}{e^{ax} - 1} \right) < 0, \quad \text{the function } \frac{xe^{(d-a_1)x}}{e^{ax} - 1} \text{ decreases, and then for } x > 0 \text{ and } i \in \mathbb{N},
\]

\[
\frac{(2i\pi + x) \exp((d-a_1)(2i\pi + x))}{\exp\left(\frac{(d-a_1)(2i\pi + x)}{r}\right) - 1} < \frac{[(2i-1)\pi + x] \exp((d-a_1)(2i-1)\pi + x)}{\exp\left(\frac{(d-a_1)(2i-1)\pi + x)}{r}\right) - 1}, \tag{40}
\]

thus, from inequality (29), we have

\[
\int_0^\infty \frac{xe^{(d-a_1)x}}{e^{ax} - 1} \sin(rx) \, dx < \int_0^{\pi/r} \frac{xe^{(d-a_1)x}}{e^{ax} - 1} \sin(rx) \, dx
\]

\[
< \int_0^{\pi/r} \frac{\sin(rx)}{x e^{(a_1 - \frac{d}{2})x}} \, dx = \frac{4r}{d} \left[ 1 + e^{\pi(d-2a_1)/(2r)} \right] \left[ (d-2a_1)^2 + 4r^2 \right]. \tag{41}
\]

Inequality (34) follows from combination of (41) with (24). \qed

Remark 2. The proof of Theorem 4 can be shortened by observing that

\[
\frac{xe^{(d-a_1)x}}{e^{dx} - 1} = \frac{x \exp\left(\frac{d}{2} - a_1\right) x}{2 \sinh\left(\frac{dx}{2}\right)} \tag{42}
\]

and \( \sinh x \) and \( \exp\left(\frac{a_1 - d}{2}\right) x \) are both increasing with \( x > 0 \) for \( a_1 > \frac{d}{2} \).
This observation was given by Professor Lothar Berg at FB Mathematik der Universität, Universitätsp. 1, D-18055 Rostock, Germany, through an e-mail to the author on May 19, 2003.

By exploiting a technique presented by E. Makai in [11] and used by the author in [14], we obtain the following inequalities of Mathieu type series (13).

**Theorem 5.** Suppose $r$ is a positive number, then for a positive sequence $a = (a_1, a_2, \ldots, a_k, \ldots)$ and a positive real number $\alpha > 0$ satisfying $a_{k+1}^{\alpha/2} - a_k^{\alpha/2} = 1$, we have

$$\frac{1}{2r^2 + 1} < \sum_{k=1}^{\infty} \frac{a_k^{\alpha/2}}{(a_k^\alpha + r^2)^2} < \frac{1}{2r^2}. \quad (43)$$

**Proof.** By standard argument, we obtain

$$\frac{1}{(a_k^{\alpha/2} - \frac{1}{2})^2 + r^2 + \frac{1}{4}} - \frac{1}{(a_k^{\alpha/2} + \frac{1}{2})^2 + r^2 - \frac{1}{4}} = \frac{2a_k^{\alpha/2}}{(a_k^\alpha + r^2 - a_k^{\alpha/2})(a_k^\alpha + r^2 + a_k^{\alpha/2})}$$

$$> \frac{2a_k^{\alpha/2}}{(a_k^\alpha + r^2)^2}$$

$$> \frac{2a_k^{\alpha/2}}{(a_k^\alpha + r^2)^2 + r^2 + \frac{1}{4}}$$

$$= \frac{1}{(a_k^{\alpha/2} - \frac{1}{2})^2 + r^2 + \frac{1}{4}} - \frac{1}{(a_k^{\alpha/2} + \frac{1}{2})^2 + r^2 + \frac{1}{4}},$$

summing for $k = 1, 2, \ldots$ yields inequalities in (43). \qed

## 4. Two Open Problems

Now we will propose two open problems for interesting readers to discuss.

**Open Problem 1.** Let $r > 0$, $t > 0$, $\alpha > 0$, $\beta > 0$ and $a = (a_1, a_2, \ldots, a_k, \ldots)$ be a positive sequence, define

$$S(r, t, \alpha, \beta, a) = \sum_{k=1}^{\infty} \frac{a_k^\beta}{(a_k^\alpha + r^2)t}. \quad (44)$$

1. Under what conditions does the sequence $S(r, t, \alpha, \beta, a)$ converge?
2. Can one obtain an integral expression for the series $S(r, t, \alpha, \beta, a)$?
3. Can one establish a sharp double inequality for the series $S(r, t, \alpha, \beta, a)$?

**Open Problem 2.** For $r > 0$, we have

$$\left[ \int_0^\infty \frac{x \sin(rx)}{e^x - 1} \, dx \right]^2 > 2r^2 \int_0^\infty \frac{x^2 f(x)}{e^{rx}} \, dx,$$

where $f(x) = \sum_{k=1}^{\infty} ke^{-k^2 x}$. \quad (45)
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