NEW DOUBLE INEQUALITIES FOR MATHIEU TYPE SERIES

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Abstract. In this paper using the trapezoidal quadrature rule, we established new double inequality, for Mathieu's series of following type:

\[ S(a, p, \alpha) = \sum_{n=1}^{\infty} \frac{2n^{\alpha/2}}{(n^\alpha + a^2)^{p+1}}, \text{ where } a > 0, \ p > 0, \ \alpha > 0. \]

As a corollary of this inequality the solution of the problem posed by Tomovski and Trenčevski in [7], for \( \alpha = 2 \) and \( p > 0 \) is completed.

1. Introduction

In [6] Feng Qi introduced the series of following type

\[ S(a, p, \alpha) = \sum_{n=1}^{\infty} \frac{2n^{\alpha/2}}{(n^\alpha + a^2)^{p+1}}, \text{ where } a > 0, \ \alpha > 0 \text{ and } p > 0. \]

For \( \alpha = 2 \) this series first was defined in [3] by Diananda.

Concerning the series \( S(a, p, 2) \) in [3], P.H. Diananda proved the following Theorem:

**Theorem A.** If \( a, p > 0 \) then

\[ S(a, p, 2) < \frac{1}{pa^{2p}}. \]

For \( \alpha = 2, \ p = 1 \) the series \( S(a, p, \alpha) \) was introduced in [5] by Mathieu. Thus the series:

\[ S(a, 1, 2) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + a^2)^2} \]

is called as Mathieu’s series.

Mathieu (see [5]) obtained inequality for series \( S(a, 1, 2) \) which is a corollary of Theorem A for \( p = 1 \).

**Theorem B.** If \( a > 0 \), then

\[ S(a, 1, 2) < \frac{1}{a^2} \]

This theorem was refined by several authors (see [2],[3],[8]) but the best result was obtained recently by H. Alzer and J.L. Brenner. Namely they proved the following theorem.

**Theorem C.** [1] For all real numbers \( a \neq 0 \), we have:

\[ \frac{1}{a^2 + 1/(2\zeta(3))} < S(a, 1, 2) < \frac{1}{a^2 + 1/6} \]

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The constants $1/(2\zeta(3))$ and $1/6$ are best possible, where $\zeta$ denotes the zeta function. On the other hand, Feng Qi in [6] established double inequality for series $S(a, 1, \alpha)$.

**Theorem D.** [6] Suppose $a$ is a positive number, then for any positive real number $\alpha$, we have:

$$\frac{1}{a^2 + \frac{1}{2}} < S(a, 1, \alpha) < \frac{1}{a^2}$$

In [7] we established double inequality for series $S(a, p, 2)$, when $a > 0$ and $p \in \mathbb{N}$.

**Theorem E.** [7] If $a > 0$, $p \in \mathbb{N}$, then

$$|S(a, p, 2)| \leq \frac{2(2a)^{-p}}{p!a^{p+1}} \left( \sum_{k=0}^{\infty} (-1)^k \frac{(k\pi)^p}{\exp(k\pi/a) - 1} + \sum_{k=0}^{\infty} (-1)^k \frac{((k + \frac{1}{2})\pi)^p}{\exp((k + \frac{1}{2})\pi/a) - 1} \right) + \sum_{k=1}^{p} \frac{2(2a)^{-p+k-1}}{k!a^{k+1}} \left( \frac{(-p + 1)}{p - k} \right) \frac{1 - k}{p - k + 1} \times \sum_{j=0}^{\infty} (-1)^j \frac{(j\pi)^k}{\exp(j\pi/a) - 1}$$

$$+ \sum_{j=0}^{\infty} ((j + \frac{1}{2})\pi)^k \frac{1}{\exp((j + \frac{1}{2})\pi/a) - 1}$$

2. **The integral expression for series $S(a, p, \alpha)$**

In this section, we shall establish an integral expression of $S(a, p, \alpha)$, where $a > 0$, $p > 0$, $\alpha > 0$. This is an open problem, posed by Feng Qi in [6].

**Theorem 1.** For $a > 0$, $p > 0$, $\alpha > 0$ the following integral expression of $S(a, p, \alpha)$ holds:

$$S(a, p, \alpha) = \frac{2}{\Gamma(p + 1)} \int_0^{\infty} x^pe^{-a^2x} g(x) dx,$$

where

$$g(x) = \sum_{n=1}^{\infty} n^{\alpha/2} e^{-n^\alpha x}$$

**Proof.** Using the well-known formula:

$$\frac{1}{t^{p+1}} = \frac{2}{\Gamma(p + 1)} \int_0^{\infty} x^p e^{-xt} dx,$$

we obtain

$$\frac{2\alpha^{\alpha/2}}{(n^\alpha + a^2)^{p+1}} = \frac{2}{\Gamma(p + 1)} \int_0^{\infty} x^p n^{\alpha/2} e^{-(n^\alpha + a^2)x} dx.$$

Applying the Cauchy integration test, we obtain that $\sum_{n=1}^{\infty} n^{\alpha/2} e^{-n^\alpha x}$ is convergent for all $x > 0$ and $\alpha > 0$, i.e. $g(x) = \sum_{n=1}^{\infty} n^{\alpha/2} e^{-n^\alpha x}$.

Thus
Theorem 2. \( S(a, p, \alpha) = \frac{2}{\Gamma(p+1)} \int_0^\infty x^p e^{-a^2 x} \left( \sum_{n=1}^{\infty} n^\alpha/2 e^{-n^\alpha x} \right) \)

\[ = \frac{2}{\Gamma(p+1)} \int_0^\infty x^p e^{-a^2 x} g(x) dx. \]

\[ \square \]

3. Main results

Our main results are as follows

Theorem 2. If \( a > 0, p > 0, \alpha > 0 \), then

\[ \frac{2\Gamma(p+\frac{1}{2} + \frac{1}{2})\Gamma(p - \frac{1}{2} + \frac{1}{2})}{\alpha \Gamma(p+1)a^{2p-\frac{1}{2}+1}} - \frac{2\Gamma(p + \frac{1}{2})}{\sqrt{2}\pi \Gamma(p+1)a^{2p+1}} < S(a, p, \alpha) < \]

\[ \frac{2\Gamma(p + \frac{1}{2} + 1)\Gamma(p - \frac{1}{2} + \frac{1}{2})}{\alpha \Gamma(p+1)a^{2p-2/\alpha+1}} + \frac{2\Gamma(p + \frac{1}{2})}{\sqrt{2}\pi \Gamma(p+1)a^{2p+1}}. \]

Proof. Let \( f(x) = x^{\alpha/2} e^{-a^2 x}, x > 0, t > 0 \)

Since \( \int_0^\infty f(x) dx = \frac{\Gamma(\frac{1}{2} + \frac{1}{2})}{\alpha t^{1/\alpha + 1/2}} \) and \( \max f(x) = \frac{1}{\sqrt{2\pi t}} \), applying the trapezoidal quadrature rule (see[4]):

\[ \int_0^\infty f(x) dx - \max f(x) < \sum_{n=1}^{\infty} f(n) < \int_0^\infty f(x) dx + \max f(x) - \int_0^1 f(x) dx, \]

we obtain

\[ \frac{\Gamma(\frac{1}{2} + \frac{1}{2})}{\alpha t^{1/\alpha + 1/2}} - \frac{1}{\sqrt{2\pi t}} < g(t) < \frac{\Gamma(\frac{1}{2} + \frac{1}{2}, 1)}{\alpha t^{1/\alpha + 1/2}} + \frac{1}{\sqrt{2\pi t}}, \]

where \( g \) is the function defined in Theorem 1 and \( \Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt. \)

Hence

\[ \frac{2\Gamma(p + \frac{1}{2})}{\alpha \Gamma(p+1)} \int_0^\infty t^{p-\frac{1}{2} - \frac{1}{2} - a^2 t} dt - \frac{2}{\sqrt{2}\pi \Gamma(p+1)} \int_0^\infty t^{p-\frac{1}{2} - \frac{1}{2} - a^2 t} dt < \]

\[ S(a, p, \alpha) < \frac{2\Gamma(p + \frac{1}{2} + 1)\Gamma(p - \frac{1}{2} + \frac{1}{2})}{\alpha \Gamma(p+1)a^{2p-2/\alpha+1}} + \frac{2\Gamma(p + \frac{1}{2})}{\sqrt{2}\pi \Gamma(p+1)a^{2p+1}}. \]

\[ \square \]
As a corollary of this theorem we obtain new double inequalities for series \( S(a, p, 2) \ a > 0, \ p > 0 \), which complete the result of partially solved problem by Tomovski and Trenčevski (see [7]), when \( p \in \mathbb{N} \).

**Corollary.** If \( a, p > 0 \) then the following inequalities hold:

\[
\frac{2}{\Gamma(p+1)} \left( \frac{\Gamma(p)}{2a^{2p}} - \frac{\Gamma(p + \frac{1}{2})}{\sqrt{2\pi a^{2p+1}}} \right) < S(a, p, 2) < \frac{2}{\Gamma(p+1)} \left( \frac{\Gamma(p)}{2\pi a^{2p}} + \frac{\Gamma(p + \frac{1}{2})}{\sqrt{2\pi a^{2p+1}}} \right)
\]

**Proof.** By putting in Theorem 1, \( \alpha = 2 \) the proof is completed. \( \square \)

**References**


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