THE BEST LOWER AND UPPER BOUNDS OF HARMONIC SEQUENCE

CHAO-PING CHEN AND FENG QI

Abstract. For any natural number $n \in \mathbb{N}$,

$$\frac{1}{2n + \frac{1}{1-\gamma} - 2} \leq \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \tag{1}$$

where $\gamma = 0.57721566490153286\cdots$ denotes Euler's constant. The constants $\frac{1}{1-\gamma} - 2$ and $\frac{1}{3}$ are the best possible.

1. Introduction

Let n be a natural number, then we have

$$\frac{1}{2n} - \frac{1}{8n^2} < \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n}, \tag{2}$$

where $\gamma = 0.57721566\cdots$ is Euler's constant.

The inequality (2) is called in literature Franel's inequality [4, Ex. 18]. Because of the well known importance of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$, there exists a very rich literature on inequalities of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$. For example, [1], [3, pp. 68–78] and references therein.

L. Tóth and S. Mare in [5, p. 264] proposed the following problems:

(1) Prove that for every positive integer n we have

$$\frac{1}{2n + \frac{2}{5}} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \tag{3}$$

where γ is Euler's constant.

(2) Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but that $\frac{1}{3}$ cannot be replaced by a slightly larger number.

In 1997 and 1999, K. Wu and B.-Ch. Yang in [8] and Sh.-R. Wei and B.-Ch. Yang in [7] verified inequality (3).

In this short note, we shall give the best lower and upper bounds of the sequence $\sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma$ and refine inequality (3), obtain the following

---

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Inequality, harmonic sequence, Binet formula, psi function, trigamma function.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, Doctor Fund of Jiaozuo Institute of Technology, CHINA.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Theorem 1. *For any natural number $n \in \mathbb{N}$, we have*

$$\frac{1}{2n + \frac{1}{1-\gamma} - 2} \le \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \tag{4}$$

where $\gamma = 0.57721566490153286\cdots$ denotes Euler's constant. The constants $\frac{1}{1-\gamma} - 2$ and $\frac{1}{3}$ are the best possible.

2. Lemma

In order to prove inequality (3), the following lemma is necessary.

Lemma 1. *For $x > 0$, we have*

$$\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \ln x < \frac{1}{2x} \tag{5}$$

and

$$\frac{1}{2x^2} - \frac{1}{6x^3} < \frac{1}{x} - \psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}, \tag{6}$$

where $\psi = \frac{\Gamma'}{\Gamma}$ is the logarithmic derivative of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \,\mathrm{d}t. \tag{7}$$

Proof. It is a well known fact ([1] and [6, p. 103]) that for $x > 0$ and a nonnegative integer m,

$$\psi(x+1) = \psi(x) + \frac{1}{x} \tag{8}$$

and

$$\frac{m!}{x^{m+1}} = \int_0^\infty t^m e^{-xt} \,\mathrm{d}t. \tag{9}$$

The first Binet's formula ([1] and [6, p. 106]) states that for $x > 0$

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} - \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1 - e^{-t}}\right) \frac{e^{-xt}}{t} \,\mathrm{d}t. \tag{10}$$

Differentiating (10), integrating by part and using formulas (9) and (8), it is deduced that

$$\psi(x+1) - \ln x = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt} \,\mathrm{d}t. \tag{11}$$

Using formulas (9) and (11) and the series expansion of e^x at $x = 0$ yields

$$\begin{aligned}
&\psi(x+1) - \ln x - \frac{1}{2x} + \frac{1}{12x^2} \\
&= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{1}{12} t\right) e^{-xt} \,\mathrm{d}t \\
&= \int_0^\infty \frac{12(e^t - 1) - 12t - 6t(e^t - 1) + t^2(e^t - 1)}{12t(e^t - 1)} e^{-xt} \,\mathrm{d}t \\
&= \int_0^\infty \left[\frac{1}{12t(e^t - 1)} \sum_{n=5}^\infty \frac{(n-3)(n-4)}{n!} t^n\right] e^{-xt} \,\mathrm{d}t \\
&> 0
\end{aligned} \tag{12}$$

and

$$\begin{aligned}\psi(x+1)-\ln x-\frac{1}{2x}&=\int_0^\infty\left(\frac{1}{t}-\frac{1}{e^t-1}-\frac{1}{2}\right)e^{-xt}\,\mathrm{d}t\\&=-\int_0^\infty\left[\frac{1}{2t(e^t-1)}\sum_{n=3}^\infty\frac{n-2}{n!}t^n\right]e^{-xt}\,\mathrm{d}t\\&<0.\end{aligned}\tag{13}$$

Hence, inequality (5) follows.

Differentiation of (11) immediately produces

$$\frac{1}{x}-\psi'(x+1)=\int_0^\infty\left(1-\frac{t}{e^t-1}\right)e^{-xt}\,\mathrm{d}t.\tag{14}$$

Exploiting formulas (9) and (14) and the series expansion of e^x at $x=0$ yields

$$\begin{aligned}&\frac{1}{x}-\psi'(x+1)-\frac{1}{2x^2}+\frac{1}{6x^3}\\&=\int_0^\infty\left(1-\frac{t}{e^t-1}-\frac{1}{2}t+\frac{1}{12}t^2\right)e^{-xt}\,\mathrm{d}t\\&=\int_0^\infty\left[\frac{1}{12(e^t-1)}\sum_{n=5}^\infty\frac{(n-3)(n-4)}{n!}t^n\right]e^{-xt}\,\mathrm{d}t\\&>0\end{aligned}\tag{15}$$

and

$$\begin{aligned}&\frac{1}{x}-\psi'(x+1)-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}\\&=\int_0^\infty\left(1-\frac{t}{e^t-1}-\frac{1}{2}t+\frac{1}{12}t^2-\frac{1}{720}t^4\right)e^{-xt}\,\mathrm{d}t\\&=\int_0^\infty\left[\frac{1}{720(e^t-1)}\sum_{n=7}^\infty\left(\frac{720}{n!}-\frac{360}{(n-1)!}+\frac{60}{(n-2)!}-\frac{1}{(n-4)!}\right)t^n\right]e^{-xt}\,\mathrm{d}t.\end{aligned}\tag{16}$$

Noticing that for $n\ge 7$,

$$\begin{aligned}&\frac{720}{n!}-\frac{360}{(n-1)!}+\frac{60}{(n-2)!}-\frac{1}{(n-4)!}\\&=-\frac{120+218(n-7)+119(n-7)^2+22(n-7)^3+(n-7)^4}{n!}<0,\end{aligned}\tag{17}$$

we obtain

$$\frac{1}{x}-\psi'(x+1)-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}<0.\tag{18}$$

Therefore, inequality (6) holds. The proof is complete. □

3. Proof of Theorem 1

In [1], [2, p. 593] and [6, p. 104] it is given that $\psi(n)=\sum_{k=1}^{n-1}\frac{1}{k}-\gamma$. Thus, inequality (4) can be rearranged as

$$\frac{1}{3}<\frac{1}{\psi(n+1)-\ln n}-2n\le\frac{1}{1-\gamma}-2.\tag{19}$$

Define for $x>0$

$$\phi(x)=\frac{1}{\psi(x+1)-\ln x}-2x.\tag{20}$$

Differentiating ϕ and utilizing (5) and (6) reveals that for $x > \frac{12}{5}$,

$$\begin{aligned}
&(\psi(x+1) - \ln x)^2 \phi'(x) \\
&= \frac{1}{x} - \psi'(x+1) - 2(\psi(x+1) - \ln x)^2 \\
&< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left(\frac{1}{2x} - \frac{1}{12x^2}\right)^2 \\
&= \frac{12-5x}{360x^5} < 0,
\end{aligned} \tag{21}$$

and $\phi(x)$ decreases with $x > \frac{12}{5}$.

Straightforward calculation produces

$$\phi(1) = \frac{1}{1-\gamma} - 2 = 0.36527211862544155\cdots, \tag{22}$$

$$\phi(2) = \frac{1}{\frac{3}{2} - \gamma - \ln 2} - 4 = 0.35469600731465752\cdots, \tag{23}$$

$$\phi(3) = \frac{1}{\frac{11}{6} - \gamma - \ln 3} - 6 = 0.34898948531361115\cdots. \tag{24}$$

Therefore, the sequence

$$\phi(n) = \frac{1}{\psi(n+1) - \ln n} - 2n, \quad n \in \mathbb{N} \tag{25}$$

is decreasing strictly, and for $n \in \mathbb{N}$

$$\lim_{n\to\infty} \phi(n) < \phi(n) \le \phi(1) = \frac{1}{1-\gamma} - 2. \tag{26}$$

Making use of approximating expansion of ψ in [1], [2, p. 594], or [6, p. 108] gives

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \quad (x \to \infty), \tag{27}$$

and then

$$\lim_{n\to\infty} \phi(n) = \lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \frac{\frac{1}{3} + O(x^{-2})}{1 + O(x^{-1})} = \frac{1}{3}. \tag{28}$$

The proof is complete.

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tebles*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, Washington, 1965, 1972.

[2] Group of compilation, *Shùxué Shǒucè* (*Handbook of Mathematics*), The People's Education Press, Beijing, China, 1979. (Chinese)

[3] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 2nd edition, Hunan Education Press, Changsha City, Hunan Province, China, 1993. (Chinese)

[4] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. I and II, Springer-Verlag, Berlin, Heidelberg, 1972.

[5] L. Tóth and S. Mare, *E 3432*, Amer. Math. Monthly **98** (1991), no. 3, 264.

[6] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn* (*Introduction to Special Function*), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)

[7] Sh.-R. Wei and B.-Ch. Yang, *An improvement of Franel's inequality*, J. Central Univ. Nationalities **8** (1999), no. 1, 66–68. (Chinese)

[8] K. Wu and B.-Ch. Yang, *Some refinements of Franel's inequality*, J. South China Normal Univ. Natur. Sci. Ed. (1997), no. 3, 5–8. (Chinese)

(Ch.-P. Chen) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

(F. Qi) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA
E-mail address: qifeng@jzit.edu.cn, fengqi618@member.ams.org
URL: http://rgmia.vu.edu.au/qi.html