THE BEST LOWER AND UPPER BOUNDS OF HARMONIC SEQUENCE

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Abstract. For any natural number \( n \in \mathbb{N} \),
\[
\frac{1}{2n + \frac{1}{n-\gamma} - 2} \leq \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}.
\]
(1)

where \( \gamma = 0.57721566490153286 \ldots \) denotes Euler’s constant. The constants \( \frac{1}{n-\gamma} - 2 \) and \( \frac{1}{3} \) are the best possible.

1. Introduction

Let \( n \) be a natural number, then we have
\[
\frac{1}{2n} - \frac{1}{8n^2} < \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n},
\]
(2)

where \( \gamma = 0.57721566 \ldots \) is Euler’s constant.

The inequality (2) is called in literature Franel’s inequality [4, Ex. 18]. Because of the well known importance of the harmonic sequence \( \sum_{i=1}^{n} \frac{1}{i} \), there exists a very rich literature on inequalities of the harmonic sequence \( \sum_{i=1}^{n} \frac{1}{i} \). For example, [1], [3, pp. 68–78] and references therein.

L. Tóth and S. Mare in [5, p. 264] proposed the following problems:

(1) Prove that for every positive integer \( n \) we have
\[
\frac{1}{2n} < \frac{1}{8n^2} + \cdots + \frac{1}{n} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}},
\]
(3)

where \( \gamma \) is Euler’s constant.

(2) Show that \( \frac{2}{5} \) can be replaced by a slightly smaller number, but that \( \frac{1}{3} \) cannot be replaced by a slightly larger number.


In this short note, we shall give the best lower and upper bounds of the sequence
\[
\sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma
\]
and refine inequality (3), obtain the following

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Theorem 1. For any natural number \( n \in \mathbb{N} \), we have
\[
\frac{1}{2n + \frac{1}{1 - \gamma} - 2} \leq \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}},
\]
where \( \gamma = 0.57721566490153286 \ldots \) denotes Euler’s constant. The constants \( \frac{1}{1 - \gamma} - 2 \) and \( \frac{1}{3} \) are the best possible.

2. Lemma

In order to prove inequality (3), the following lemma is necessary.

Lemma 1. For \( x > 0 \), we have
\[
\frac{1}{2x} - \frac{1}{12x^2} < \psi(x + 1) - \ln x < \frac{1}{2x},
\]
and
\[
\frac{1}{2x^2} - \frac{1}{6x^3} < \frac{1}{x} - \psi'(x + 1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5},
\]
where \( \psi = \frac{x'}{x} \) is the logarithmic derivative of the gamma function
\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt.
\]

Proof. It is a well known fact ([1] and [6, p. 103]) that for \( x > 0 \) and a nonnegative integer \( m \),
\[
\psi(x + 1) = \psi(x) + \frac{1}{x}
\]
and
\[
\frac{m!}{x^{m+1}} = \int_{0}^{\infty} t^{m} e^{-xt} \, dt.
\]

The first Binet’s formula ([1] and [6, p. 106]) states that for \( x > 0 \)
\[
\ln \Gamma(x) = (x - \frac{1}{2}) \ln x - x + \ln \sqrt{2\pi} - \int_{0}^{\infty} \left( \frac{1}{2} + \frac{1}{2} - \frac{1}{1 - e^{-t}} \right) e^{-xt} \, dt.
\]

Differentiating (10), integrating by part and using formulas (9) and (8), it is deduced that
\[
\psi(x + 1) - \ln x = \int_{0}^{\infty} \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} \, dt.
\]

Using formulas (9) and (11) and the series expansion of \( e^x \) at \( x = 0 \) yields
\[
\psi(x + 1) - \ln x - \frac{1}{2x} + \frac{1}{12x^2} = \int_{0}^{\infty} \left( \frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{1}{12t} \right) e^{-xt} \, dt = \int_{0}^{\infty} \frac{12(e^t - 1) - 12t - 6t(e^t - 1) + t^2(e^t - 1)}{12t(e^t - 1)} e^{-xt} \, dt = \int_{0}^{\infty} \sum_{n=3}^{\infty} \frac{(n-3)(n-4)}{n!} t^n \, dt e^{-xt} \, dt
\]
\[
> 0
\]
and

\[ \psi(x + 1) - \ln x - \frac{1}{2x} = \int_0^\infty \left( 1 - \frac{1}{e^t - 1} - \frac{1}{2} \right) e^{-xt} \, dt \]
\[ = -\int_0^\infty \left[ \frac{1}{2t(e^t - 1)} \sum_{n=3}^\infty \frac{n-2}{n!} t^n \right] e^{-xt} \, dt \]
\[ < 0. \quad (13) \]

Hence, inequality (5) follows.

Differentiation of (11) immediately produces

\[ \frac{1}{x} - \psi'(x + 1) = \int_0^\infty \left( 1 - \frac{t}{e^t - 1} \right) e^{-xt} \, dt. \quad (14) \]

Exploiting formulas (9) and (14) and the series expansion of \( e^x \) at \( x = 0 \) yields

\[ \frac{1}{x} - \psi'(x + 1) - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} \]
\[ = \int_0^\infty \left( 1 - \frac{t}{e^t - 1} - \frac{1}{2} t + \frac{1}{12} t^2 - \frac{1}{720} t^4 \right) e^{-xt} \, dt \]
\[ = \int_0^\infty \left[ \frac{1}{12(e^t - 1)} \sum_{n=5}^\infty \frac{(n-3)(n-4)}{n!} t^n \right] e^{-xt} \, dt \]
\[ > 0. \quad (15) \]

and

\[ \frac{1}{x} - \psi'(x + 1) - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} \]
\[ = \int_0^\infty \left( 1 - \frac{t}{e^t - 1} - \frac{1}{2} t + \frac{1}{12} t^2 - \frac{1}{720} t^4 \right) e^{-xt} \, dt \]
\[ = \int_0^\infty \left[ \frac{1}{720(e^t - 1)} \sum_{n=7}^\infty \frac{720}{n!} \left( \frac{360}{(n-1)!} + \frac{60}{(n-2)!} - \frac{1}{(n-4)!} \right) t^n \right] e^{-xt} \, dt. \quad (16) \]

Noticing that for \( n \geq 7 \),

\[ \frac{720}{n!} - \frac{360}{(n-1)!} + \frac{60}{(n-2)!} - \frac{1}{(n-4)!} \]
\[ = \frac{120 + 218(n-7) + 119(n-7)^2 + 22(n-7)^3 + (n-7)^4}{n!} < 0, \quad (17) \]

we obtain

\[ \frac{1}{x} - \psi'(x + 1) - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < 0. \quad (18) \]

Therefore, inequality (6) holds. The proof is complete. \( \square \)

3. Proof of Theorem 1

In [1], [2, p. 593] and [6, p. 104] it is given that \( \psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma \). Thus, inequality (4) can be rearranged as

\[ \frac{1}{3} < \frac{1}{\psi(n + 1) - \ln n} - 2n \leq \frac{1}{1 - \gamma} - 2. \quad (19) \]

Define for \( x > 0 \)

\[ \phi(x) = \frac{1}{\psi(x + 1) - \ln x} - 2x. \quad (20) \]
Differentiating $\phi$ and utilizing (5) and (6) reveals that for $x > \frac{12}{5}$,

\[
(\psi(x+1) - \ln x)^2 \phi'(x) = \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left(\frac{1}{2x} - \frac{1}{12x^2}\right)^2
\]

\[
< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left(\frac{1}{2x} - \frac{1}{12x^2}\right)^2
\]

\[
= \frac{1}{12 - 5x} - \frac{360x^5}{6x^3} < 0,
\]

and $\phi(x)$ decreases with $x > \frac{12}{5}$.

Straightforward calculation produces

\[
\phi(1) = \frac{1}{1 - \gamma} - 2 = 0.36527211862544155 \cdots,
\]

\[
\phi(2) = \frac{1}{2 - \gamma - \ln 2} - 4 = 0.35469600731465752 \cdots,
\]

\[
\phi(3) = \frac{1}{3 - \gamma - \ln 3} - 6 = 0.34898948531361115 \cdots.
\]

Therefore, the sequence

\[
\phi(n) = \frac{1}{\psi(n+1) - \ln n} - 2n, \quad n \in \mathbb{N}
\]

is decreasing strictly, and for $n \in \mathbb{N}$

\[
\lim_{n \to \infty} \phi(n) < \phi(n) \leq \phi(1) = \frac{1}{1 - \gamma} - 2.
\]

Making use of approximating expansion of $\psi$ in [1], [2, p. 594], or [6, p. 108] gives

\[
\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \quad (x \to \infty),
\]

and then

\[
\lim_{n \to \infty} \phi(n) = \lim_{x \to \infty} \phi(x) = \lim_{x \to \infty} \frac{1}{1 + O(x^{-1})} = \frac{1}{3}.
\]

The proof is complete.

References


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