ON APPLICATIONS OF THE INTEGRAL OF PRODUCTS OF FUNCTIONS AND ITS BOUNDS

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Abstract. The Steffensen inequality and bounds for the Čebyšev functional are utilised to obtain bounds for some classical special functions. The technique relies on determining bounds on integrals of products of functions. The above techniques are used to obtain novel and useful bounds for the Bessel function of the first kind, the Beta function and the Zeta function.

1. Introduction and Review of some Recent Results

There are a number of results that provide bounds for integrals of products of functions. The main techniques that shall be employed in the current article involve the Steffensen inequality and a variety of bounds related to the Čebyšev functional. There have been some developments in both of these in the recent past with which the current author has been involved. These have been put to fruitful use in a variety of areas of applied mathematics including quadrature rules, in the approximation of integral transforms, as well as in applied probability problems (see [16], [9] and [4]).

It is the intention that in the current article the techiques will be utilised to obtain useful bounds for special functions. The methodologies will be demonstrated through obtaining bounds for the Bessel function of the first kind, the Beta function and the Zeta function.

It is instructive to introduce some techniques for approximating and bounding integrals of the product of functions. We first introduce inequalities due to Steffensen and then review bounds for the Čebyšev functional.

The following theorem is due to Steffensen [23] (see also [4]).

Theorem 1. *Let $h : [a, b] \to \mathbb{R}$ be a nonincreasing mapping on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ be an integrable mapping on $[a, b]$ with*

$$-\infty < \phi \le g(t) \le \Phi < \infty \quad \text{for all } x \in [a, b],$$

then

$$\phi \int_a^{b-\lambda} h(x)\,dx + \Phi \int_{b-\lambda}^{b} h(x)\,dx \le \int_a^b h(x)\,g(x)\,dx \le \Phi \int_a^{a+\lambda} h(x)\,dx + \phi \int_{a+\lambda}^{b} h(x)\,dx, \tag{1.1}$$

Date: June 20, 2003.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20; Secondary 26D10.

Key words and phrases. Čebyšev functional, Grüss inequality, Bessel, Beta and Zeta function bounds.

where

$$\lambda = \int_a^b G(x)\,dx, \quad G(x) = \frac{g(x) - \phi}{\Phi - \phi} \quad \Phi \neq \phi. \tag{1.2}$$

Remark 1. *We note that the result (1.1) may be rearranged to give Steffensen's better known result that*

$$\int_{b-\lambda}^b h(x)\,dx \leq \int_a^b h(x)\,G(x)\,dx \leq \int_a^{a+\lambda} h(x)\,dx, \tag{1.3}$$

where λ is as given by (1.2) and $0 \leq G(x) \leq 1$.

Equation (1.3) has a very pleasant interpretation, as observed by Steffensen, that if we divide by λ then

$$\frac{1}{\lambda}\int_{b-\lambda}^b h(x)\,dx \leq \frac{\int_a^b G(x)\,h(x)\,dx}{\int_a^b G(x)\,dx} \leq \frac{1}{\lambda}\int_a^{a+\lambda} h(x)\,dx. \tag{1.4}$$

Thus, the weighted integral mean of $h(x)$ is bounded by the integral means over the end intervals of length λ, the total weight.

Now, for two measurable functions $f, g : [a,b] \to \mathbb{R}$, define the functional, which is known in the literature as Čebyšev's functional, by

$$T(f,g) := \mathcal{M}(fg) - \mathcal{M}(f)\,\mathcal{M}(g), \tag{1.5}$$

where the integral mean is given by

$$\mathcal{M}(f) := \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.6}$$

The integrals in (1.5) are assumed to exist.

Further, the weighted Čebyšev functional is defined by

$$T(f,g;p) := \mathcal{M}(fg;p) - \mathcal{M}(f;p)\,\mathcal{M}(g;p), \tag{1.7}$$

where the weighted integral mean is given by

$$\mathcal{M}(f;p) = \frac{\int_a^b p(x)\,f(x)\,dx}{\int_a^b p(x)\,dx}, \tag{1.8}$$

with $0 < \int_a^b p(x)\,dx < \infty$.

We note that,

$$T(f,g;1) \equiv T(f,g)$$

and

$$\mathcal{M}(f;1) \equiv \mathcal{M}(f).$$

We further note that bounds for (1.5) and (1.7) may be looked upon as approximating the integral mean of the product of functions in terms of the product of integral means which are more easily calculated explicitly. Bounds are perhaps best procured from identities. It is worthwhile noting that a number of identities relating to the Čebyšev functional already exist. (The reader is referred to [21] Chapters IX and X.) Korkine's identity is well known, see [21, p. 296] and is given by

$$T(f,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(x) - f(y))\,(g(x) - g(y))\,dxdy. \tag{1.9}$$

It is identity (1.9) that is often used to prove an inequality due to Grüss for functions bounded above and below, [21].

The Grüss inequality is given by

$$|T(f,g)| \leq \frac{1}{4}\left(\Phi_f - \phi_f\right)\left(\Phi_g - \phi_g\right), \tag{1.10}$$

where $\phi_f \leq f(x) \leq \Phi_f$ for $x \in [a,b]$.

If we let $S(f)$ be an operator defined by

$$S(f)(x) := f(x) - \mathcal{M}(f), \tag{1.11}$$

which shifts a function by its integral mean, then the following identity holds. Namely,

$$T(f,g) = T(S(f),g) = T(f,S(g)) = T(S(f),S(g)), \tag{1.12}$$

and so

$$T(f,g) = \mathcal{M}(S(f)g) = \mathcal{M}(fS(g)) = \mathcal{M}(S(f)S(g)) \tag{1.13}$$

since $\mathcal{M}(S(f)) = \mathcal{M}(S(g)) = 0$.

For the last term in (1.13) or (1.14) only one of the functions needs to be shifted by its integral mean. If the other were to be shifted by *any* other quantity, the identities would still hold. A weighted version of (1.13) related to

$$T(f,g) = \mathcal{M}((f(x) - \gamma)S(g)) \tag{1.14}$$

for γ arbitrary was given by Sonin [24] (see [21, p. 246]).

The weighted identity corresponding to (1.14) is of course given by

$$T(f,g;p) = \mathcal{M}((f(\cdot) - \gamma)S(g;p)(\cdot);p), \tag{1.15}$$

where

$$S(g;p)(x) = f(x) - \mathcal{M}(g;p). \tag{1.16}$$

The interested reader is also referred to Dragomir [15] and Fink [17] for extensive treatments of the Grüss and related inequalities.

Identity (1.9) may also be used to prove the Čebyšev inequality which states that for $f(\cdot)$ and $g(\cdot)$ synchronous, namely $(f(x) - f(y))(g(x) - g(y)) \geq 0$, a.e. $x, y \in [a,b]$, then

$$T(f,g) \geq 0. \tag{1.17}$$

As mentioned earlier, there are many identities involving the Čebyšev functional (1.5) or more generally (1.7). Recently, Cerone [4] obtained, for $f, g : [a,b] \to \mathbb{R}$ where f is of bounded variation and g continuous on $[a,b]$, the identity

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t), \tag{1.18}$$

where

$$\psi(t) = (t-a)G(t,b) - (b-t)G(a,t) \tag{1.19}$$

with

$$G(c,d) = \int_c^d g(x)\,dx. \tag{1.20}$$

The following theorem was proved in [4].

Theorem 2. *Let $f,g:[a,b]\to\mathbb{R}$, where f is of bounded variation and g is continuous on $[a,b]$. Then*

$$(b-a)^2\left|T(f,g)\right| \le \begin{cases} \sup\limits_{t\in[a,b]}\left|\psi(t)\right|\bigvee\limits_a^b(f), & \\ L\int_a^b\left|\psi(t)\right|dt, & \text{for } f\ L-\text{Lipschitzian,} \\ \int_a^b\left|\psi(t)\right|df(t), & \text{for } f \text{ monotonic nondecreasing,}\end{cases} \tag{1.21}$$

where $\bigvee_a^b(f)$ is the total variation of f on $[a,b]$.

An equivalent identity and theorem were also obtained for the weighted Čebyšev functional (1.7).

The bounds for the Čebyšev functional were utilised to procure approximations to moments and moment generating functions in [4].

In [11], bounds were obtained for the approximations of moments although the work in [4] places less stringent assumptions on the behaviour of the probability density function.

In a subsequent paper to [4], Cerone and Dragomir [9] obtained a refinement of the classical Čebyšev inequality (1.17) as embodied in the following theorem.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ a continuous function on $[a,b]$ so that $\varphi(t)\ge 0$ for each $t\in(a,b)$. Then one has the inequality:*

$$T(f,g)\ge\frac{1}{(b-a)^2}\left|\int_a^b\left[(t-a)\left|G(t,b)\right|-(b-t)\left|G(a,t)\right|\right]df(t)\right|\ge 0, \tag{1.22}$$

where

$$\varphi(t)=\frac{G(t,b)}{b-t}-\frac{G(a,t)}{t-a} \tag{1.23}$$

and $G(c,d)$ is as defined in (1.20).

Bounds were also found for $\left|T(f,g)\right|$ in terms of the Lebesgue norms $\|\phi\|_p$, $p\ge 1$ effectively utilising (1.21) and noting that $\psi(t)=(t-a)(b-t)\varphi(t)$.

It should be mentioned here that the author in [6] demonstrated relationships between the Čebyšev functional $T(f,g;a,b)$, the generalised trapezoidal functional $GT(f;a,x,b)$ and the Ostrowski functional $\Theta(f;a,x,b)$ defined by

$$T(f,g;a,b):=M(fg;a,b)-M(f;a,b)\,M(g;a,b)$$

$$GT(f;a,x,b):=\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-M(f;a,b)$$

and

$$\Theta(f;a,x,b):=f(x)-M(f;a,b)$$

where the integral mean is of course defined by

$$M(f;a,b):=\frac{1}{b-a}\int_a^b f(x)\,dx=\mathcal{M}(f)=\mathcal{M}(f;1),\quad\text{from (1.6), (1.8).} \tag{1.24}$$

This was made possible through the fact that both $GT\left(f;a,x,b\right)$ and $\Theta\left(f;a,x,b\right)$ satisfy identities like (1.18) involving appropriate Peano kernels. Namely,

$$GT\left(f;a,x,b\right)=\int_{a}^{b}q\left(x,t\right)df\left(t\right),\quad q\left(x,t\right)=\frac{t-x}{b-a};\ x,t\in\left[a,b\right]$$

and

$$\Theta\left(f;a,x,b\right)=\int_{a}^{b}p\left(x,t\right)fd\left(t\right),\quad \left(b-a\right)p\left(x,t\right)=\begin{cases}t-a, & t\in\left[a,x\right]\\ \\ t-b, & t\in\left(x,b\right]\end{cases}$$

respectively.

The reader is referred to [13], [16] and the references therein for applications of these to numerical quadrature.

For other Grüss type inequalities, see the books [21] and [22], and the papers [12] – [17], where further references are given.

Recently, Cerone and Dragomir [8] – [10] have pointed out generalisations of the above results for integrals defined on two different intervals $[a,b]$ and $[c,d]$ and more generally in a measurable space setting (see also, Cerone [5]).

In the current paper we shall mainly utilize the Steffensen result as depicted in Theorem 1 and the following results bounding the Čebyšev functional to determine bounds on a variety of special functions.

From (1.15) and (1.16) we note that

$$P\cdot\left|T\left(f,g;p\right)\right|=\left|\int_{a}^{b}p\left(x\right)\left(f\left(x\right)-\gamma\right)\left(g\left(x\right)-\mathcal{M}\left(g;p\right)\right)dx\right| \tag{1.25}$$

to give

$$P\cdot\left|T\left(f,g;p\right)\right|\leq\begin{cases}\inf_{\gamma\in\mathbb{R}}\left\|f\left(\cdot\right)-\gamma\right\|\int_{a}^{b}p\left(x\right)\left|g\left(x\right)-\mathcal{M}\left(g;p\right)\right|dx,\\ \\ \left(\int_{a}^{b}p\left(x\right)\left(f\left(x\right)-\mathcal{M}\left(f;p\right)\right)^{2}dx\right)^{\frac{1}{2}}\\ \qquad\qquad\times\left(\int_{a}^{b}p\left(x\right)\left(g\left(x\right)-\mathcal{M}\left(g;p\right)\right)^{2}dx\right)^{\frac{1}{2}},\end{cases} \tag{1.26}$$

where

$$\int_{a}^{b}p\left(x\right)\left(h\left(x\right)-\mathcal{M}\left(h;p\right)\right)^{2}dx=\int_{a}^{b}p\left(x\right)h^{2}\left(x\right)dx-P\cdot\mathcal{M}^{2}\left(h;p\right) \tag{1.27}$$

and it may be easily shown by direct calculation that,

$$\inf_{\gamma\in\mathbb{R}}\left[\int_{a}^{b}p\left(x\right)\left(f\left(x\right)-\gamma\right)^{2}dx\right]=\int_{a}^{b}p\left(x\right)\left(f\left(x\right)-\mathcal{M}\left(f;p\right)\right)^{2}dx. \tag{1.28}$$

Some of the above results are used to find bounds for the Bessel function (Section 2), the Beta function (Section 3) and the Zeta function (Section 4).

2. Bounding the Bessel Function

In this section we investigate techniques for determining bounds on the Bessel function of the first kind.

In Abramowitz and Stegun [1] equation (9.1.21) defines the Bessel of the first kind by

$$J_{\nu}(z) = \gamma_{\nu}(z) \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \cos(zt)\, dt, \quad \mathrm{Re}(\nu) > -\frac{1}{2}, \tag{2.1}$$

where

$$\gamma_{\nu}(z) = \frac{2\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}. \tag{2.2}$$

For the current work the interest is in both z and ν real.

Theorem 4. *For z real then*

$$\begin{aligned} &\frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) - B\left(\frac{1}{2},\nu+\frac{1}{2};(1-\lambda)^2\right) \\ &\leq \frac{J_{\nu}(z)}{\gamma_{\nu}(z)} \\ &\leq B\left(\frac{1}{2},\nu+\frac{1}{2};\lambda^2\right) - \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right), \quad \nu > \frac{1}{2} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &B\left(\frac{1}{2},\nu+\frac{1}{2};\lambda^2\right) - \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) \\ &\leq \frac{J_{\nu}(z)}{\gamma_{\nu}(z)} \\ &\leq \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) - B\left(\frac{1}{2},\nu+\frac{1}{2};(1-\lambda)^2\right), \quad -\frac{1}{2} < \nu < \frac{1}{2}, \end{aligned} \tag{2.4}$$

where

$$B(\alpha,\beta;x) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1}\, du, \qquad \textit{the incomplete Beta function,} \tag{2.5}$$

$$B(\alpha,\beta) = B(\alpha,\beta;1) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \textit{the Beta function,} \tag{2.6}$$

and

$$2\lambda - 1 = \frac{\sin z}{z}. \tag{2.7}$$

Proof. Consider the case $\nu > \frac{1}{2}$ then $h(t) = \left(1-t^2\right)^{\nu-\frac{1}{2}}$ is nonincreasing for $t \in [0,1]$. Further, taking $g(t) = \cos zt$ we have that $-1 \leq g(t) \leq 1$ for $t \in [0,1]$ and, from (1.2)

$$\lambda = \frac{1}{2}\int_0^1 (\cos zt + 1) = \frac{1}{2}\left(1 + \frac{\sin z}{z}\right).$$

Thus, from Theorem 1, we have

$$\begin{aligned}
&-\int_0^{1-\lambda}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt+\int_{1-\lambda}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt\\
&<\frac{J_\nu(z)}{\gamma_\nu(z)}\\
&<\int_0^{\lambda}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt-\int_{\lambda}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt,
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_0^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt-2\int_0^{1-\lambda}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt\\
&<\frac{J_\nu(z)}{\gamma_\nu(z)}\\
&<2\int_0^{\lambda}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt-\int_0^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt.
\end{aligned}\tag{2.8}$$

If we let

$$G(\alpha)=\int_0^{\alpha}\left(1-t^2\right)^{\nu-\frac{1}{2}}dt\tag{2.9}$$

then (2.8) becomes

$$G(1)-2G(1-\lambda)<\frac{J_\nu(z)}{\gamma_\nu(z)}<2G(\lambda)-G(1).\tag{2.10}$$

A simple change of variable $u=t^2$ in (2.9) gives

$$G(\alpha)=\frac{1}{2}\int_0^{\alpha^2}u^{-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}du$$

and so

$$G(\alpha)=\frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2},\alpha^2\right),\tag{2.11}$$

where $B(\alpha,\beta;x)$ is the incomplete beta function as given by (2.5).

Thus substituting (2.11) into (2.10) produces (2.3).

For $-\frac{1}{2}<\nu<\frac{1}{2}$ then $h(t)$ is nondecreasing for $t\in[0,1]$ and thus the inequalities in (2.2) are reversed, or equivalently, the bounds are swapped to produce (2.4). $\square$

Remark 2. *If we take $\nu=\frac{1}{2}$ in either (2.3) or (2.4) then equality is obtained. Namely,*

$$\frac{J_{\frac{1}{2}}(z)}{\gamma_{\frac{1}{2}}(z)}=\frac{\sin z}{z}.$$

Remark 3. *We note from (2.1) that we may obtain a classical bound (see* [1, p. 362]*) for $J_\nu(z)$, namely*

$$|J_\nu(z)|\le\frac{2\left(\frac{|z|}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1\left(1-t^2\right)^{\nu-\frac{1}{2}}dt,$$

where from (2.9) and (2.11)

$$\int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} dt = \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\nu+1\right)} \tag{2.12}$$

to give

$$\left|J_\nu\left(z\right)\right| \le \left|\frac{z}{2}\right|^\nu \frac{1}{\Gamma\left(\nu+1\right)}. \tag{2.13}$$

The following theorem gives a bound on the deviation of the Bessel function from an approximant. This is accomplished via bounds on the Čebyšev functional for which there are numerous results.

Theorem 5. *The following result holds for the Bessel function of the first kind* $J_\nu\left(z\right)$. *Namely,*

$$\begin{aligned} &\left|J_\nu\left(z\right) - \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+1\right)}\cdot\frac{\sin z}{z}\right| \\ &\quad\le \left(\frac{|z|}{2}\right)^\nu\left[\frac{2}{\sqrt{\pi}}\cdot\frac{\Gamma\left(2\nu\right)}{\Gamma^2\left(\nu+\frac{1}{2}\right)\Gamma\left(2\nu+\frac{1}{2}\right)} - \frac{1}{\Gamma^2\left(\nu+1\right)}\right] \\ &\qquad\times\left[\left(\frac{\cos z}{4}\right)^2+\frac{1}{2}-\left(\frac{\sin z}{z}-\frac{\cos z}{4}\right)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{2.14}$$

Proof. From (2.1) and (2.2) consider,

$$Q_\nu\left(z\right) = \frac{J_\nu\left(z\right)}{\gamma_\nu\left(z\right)} = \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}}\cos\left(zt\right)dt. \tag{2.15}$$

Let $f\left(t\right) = \left(1-t^2\right)^{\nu-\frac{1}{2}}$ and $g\left(t\right) = \cos zt$.

Now,

$$\mathcal{M}\left(g\right) = \int_0^1 \cos\left(zt\right)dt = \frac{\sin z}{z} \tag{2.16}$$

and from (2.12)

$$\mathcal{M}\left(f\right) = \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} dt = \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\nu+1\right)}. \tag{2.17}$$

Thus, from (1.26)

$$\begin{aligned} \left|Q_\nu\left(z\right) - \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\nu+1\right)}\cdot\frac{\sin z}{z}\right| &\le \left(\int_0^1 f^2\left(t\right)dt - \mathcal{M}^2\left(f\right)\right)^{\frac{1}{2}} \\ &\quad\times\left(\int_0^1 g^2\left(t\right)dt - \mathcal{M}^2\left(g\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{2.18}$$

We have, from (2.17),

$$\int_0^1 f^2\left(t\right)dt = \int_0^1 \left(1-t^2\right)^{2\nu-1} dt = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(2\nu\right)}{\Gamma\left(2\nu+\frac{1}{2}\right)} \tag{2.19}$$

and

$$\int_0^1 g^2(t)\,dt = \int_0^1 \cos^2(zt)\,dt = \frac{1}{2}\left(1 + \frac{\sin z}{z} \cdot \cos z\right). \tag{2.20}$$

Substitution of (2.19) and (2.20) gives

$$\begin{aligned} &\left| Q_\nu(z) - \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu+1)} \cdot \frac{\sin z}{z} \right| \\ &\qquad \le \left[\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(2\nu)}{\Gamma\left(2\nu + \frac{1}{2}\right)} - \frac{\pi}{4} \cdot \left(\frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu+1)} \right)^2 \right]^{\frac{1}{2}} \\ &\qquad\qquad \times \left[\left(\frac{\cos z}{4} \right)^2 + \frac{1}{2} - \left(\frac{\sin z}{z} - \frac{\cos z}{4} \right)^2 \right]^{\frac{1}{2}}, \end{aligned} \tag{2.21}$$

and so (2.14) is obtained on multiplication of (2.21) by $|\gamma_\nu(z)|$. □

3. Bounding the Beta Function

The incomplete beta function is defined by

$$B(x, y; z) = \int_0^z t^{x-1}(1-t)^{y-1}\,dt, \quad 0 < z \le 1. \tag{3.1}$$

We shall restrict our attention to $x > 1$ and $y > 1$.

In this region we observe that

$$0 \le t^{x-1} \le z^{x-1} \quad \text{and} \quad (1-z)^{y-1} \le (1-t)^{y-1} \le 1 \tag{3.2}$$

with t^{x-1}, an increasing function and $(1-t)^{y-1}$, a decreasing function, for $t \in [0, z]$.

The following theorem follows from utilizing Steffensen's result as depicted in Theorem 1.

Theorem 6. *For $x > 1$ and $y > 1$ with $0 \le z \le 1$ we have the incomplete Beta function defined by (3.1) satisfying the following bounds*

$$\max\{L_1(z), L_2(z)\} \le B(x, y; z) \le \min\{U_1(z), U_2(z)\}, \tag{3.3}$$

where

$$L_1(z) = \frac{z^{x-1}}{y}\left[\left(1 - z + \frac{z}{x}\right)^y - (1-z)^y\right], \quad U_1(z) = \frac{z^{x-1}}{y}\left[1 - \left(1 - \frac{z}{x}\right)^y\right] \tag{3.4}$$

and

$$\begin{aligned} L_2(z) &= \frac{\lambda_2^x(z)}{x} + (1-z)^{y-1}\,\frac{z^x - \lambda_2^x(z)}{x}, \\ U_2(z) &= (1-z)^{y-1}\,\frac{(x - \lambda_2(z))^x}{x} + \frac{z^x - (z - \lambda_2(z))^x}{x} \end{aligned} \tag{3.5}$$

with

$$\lambda_2(z) = \frac{1 - (1-z)\left[1 - z(1-y)\right]}{y\left[1 - (1-z)^{y-1}\right]}. \tag{3.6}$$

Proof. If we take $h(t) = (1-t)^{y-1}$ and $g(t) = t^{x-1}$, then for $y > 1$ and $x > 1$, $h(t)$ is a decreasing function of t and $0 \le g(t) \le z^{x-1}$. Thus, from (1.1)

$$z^{x-1} \int_{z-\lambda_1}^{z} (1-t)^{y-1} \, dt \le \int_0^z t^{x-1} (1-t)^{y-1} \, dt \le z^{x-1} \int_0^{\lambda_1} (1-t)^{y-1} \, dt, \tag{3.7}$$

where

$$\lambda_1 = \lambda_1(z) = \int_0^z \frac{t^{x-1}}{z^{x-1}} dt = \frac{z}{x}.$$

Now,

$$\int_0^{\lambda_1} (1-t)^{y-1} \, dt = \frac{1-(1-\lambda_1)^y}{y}$$

and

$$\int_{z-\lambda_1}^{z} (1-t)^{y-1} \, dt = \frac{(1-z+\lambda_1)^y - (1-z)^y}{y},$$

so that, from (3.7),

$$\frac{z^{x-1}}{y} \left[\left(1 - z + \frac{z}{x}\right)^y - (1-z)^y \right] \le B(x,y;z) \le \frac{z^{x-1}}{y} \left[1 - \left(1 - \frac{z}{x}\right)^y \right]. \tag{3.8}$$

If $h(t)$ is an increasing function then the inequalities in (1.1) are reversed. Thus, if $h(t) = t^{x-1}$ and $g(t) = (1-t)^{y-1}$, then for $x > 1$ and $y > 1$, $h(t)$ is an increasing function of t and $(1-z)^{y-1} \le g(t) \le 1$. From (1.1) we have

$$\begin{aligned} \int_0^{\lambda_2} t^{x-1} dt + (1-z)^{y-1} \int_{\lambda_2}^{z} t^{x-1} dt &\le \int_0^z t^{x-1} (1-t)^{y-1} \, dt \\ &\le (1-z)^{y-1} \int_0^{z-\lambda_2} t^{x-1} dx + \int_{z-\lambda_2}^{z} t^{x-1} dx, \end{aligned} \tag{3.9}$$

where

$$\lambda_2 = \lambda_2(z) = \int_0^z \frac{(1-t)^{y-1} - (1-z)^{y-1}}{1-(1-z)^{y-1}} dt = \frac{1-(1-z)\left[1-z(1-y)\right]}{y\left[1-(1-z)^{y-1}\right]}$$

as given by (3.6).

Hence, from (3.9)

$$\begin{aligned} &\frac{\lambda_2^x(z)}{x} + (1-z)^{y-1} \frac{z^x - \lambda_2^x(z)}{x} \\ &\le B(x,y;z) \\ &\le (1-z)^{y-1} \frac{(x-\lambda_2(z))^x}{x} + \frac{z^x - (z-\lambda_2(z))^x}{x}. \end{aligned} \tag{3.10}$$

Combining the results (3.8) and (3.10) produces the result (3.4) with obvious use of notation. □

Corollary 1. *For $x > 1$ and $y > 1$ we have the Beta function*

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt,$$

which is symmetric in x and y satisfies the following bounds,

$$\max\left\{\frac{1}{xy^{x}},\frac{1}{yx^{y}}\right\} \leq B(x,y;z) \leq \min\left\{\frac{1}{y}\left[1-\left(1-\frac{1}{x}\right)^{y}\right],\frac{1}{x}\left[1-\left(1-\frac{1}{y}\right)^{x}\right]\right\}. \tag{3.11}$$

Proof. Put $z=1$ in (3.6) to give $\lambda_2(1)=\frac{1}{y}$ followed by the obvious correspondences from (3.3) – (3.5). □

The following theorem relates to the Beta function.

Theorem 7. *For $x>1$ and $y>1$ the following bounds hold for the Beta function, namely,*

$$0\leq\frac{1}{xy}-B(x,y)\leq\frac{1}{2}\min\{A(x),A(y)\}, \tag{3.12}$$

where

$$A(x)=\frac{1}{x}\left[\frac{1}{x^{\frac{1}{x-1}}}\left(1-\frac{2}{x}\right)+1\right]. \tag{3.13}$$

Proof. We have from (1.14),

$$\begin{aligned}0\leq|T(f,g)|&=|\mathcal{M}(fg)-\mathcal{M}(f)\mathcal{M}(g)|\\&\leq\mathcal{M}(|f(\cdot)-\gamma|\,|g(\cdot)-\mathcal{M}(g)|).\end{aligned}$$

That is,

$$|T(f,g)|\leq\inf_{\gamma}\|f(\cdot)-\gamma\|_{\infty}\mathcal{M}|g(\cdot)-\mathcal{M}(g)| \tag{3.14}$$

If we take $f(t)=t^{x-1}$, $g(t)=(1-t)^{y-1}$ then $\mathcal{M}(f)=\frac{1}{x}$ and $\mathcal{M}(g)=\frac{1}{y}$, so that we have from (3.14)

$$\begin{aligned}0&\leq\frac{1}{xy}-B(x,y)\\&\leq\inf_{\gamma}\sup_{t\in[0,1]}\left|t^{x-1}-\gamma\right|\int_0^1\left|(1-t)^{y-1}-\frac{1}{y}\right|dy\\&=\inf_{\gamma}\max\{\gamma,1-\gamma\}\int_0^1\left|(1-t)^{y-1}-\frac{1}{y}\right|dy.\end{aligned} \tag{3.15}$$

Now,

$$\inf_{\gamma}\max\{\gamma,1-\gamma\}=\inf_{\gamma}\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]=\frac{1}{2}$$

and

$$\begin{aligned}\int_0^1\left|(1-t)^{y-1}-\frac{1}{y}\right|dy&=\int_0^1\left|u^{y-1}-\frac{1}{y}\right|du\\&=\int_0^{u_*}\left(\frac{1}{y}-u^{y-1}\right)du+\int_{u_*}^1\left(u^{y-1}-\frac{1}{y}\right)du\\&=\frac{1}{y}\left[u_*-u_*^{y}-(u_*^{y}-1)\right]\\&=\frac{1}{y}\left[u_*\left(1-2u_*^{y}\right)+1\right],\end{aligned}$$

where $u_*^{y-1} = \frac{1}{y}$.

Thus

$$0 \le \frac{1}{xy} - B(x,y) \le \frac{1}{2y}\left[\frac{1}{y^{\frac{1}{y-1}}}\left(1-\frac{2}{y}\right)+1\right] = \frac{A(y)}{2},$$

where $A(y)$ is as given by (3.13).

We may interchange the role of x and y because of the symmetry and so (3.12) results. □

Remark 4. *Computer experimentation indicates that $A(x)$ is a strictly decreasing function so that* $\min\{A(x), A(y)\} = A(\max\{x,y\})$.

The following pleasing result is valid.

Theorem 8. *For $x > 1$ and $y > 1$ we have*

$$\begin{aligned} 0 \le \frac{1}{xy} - B(x,y) &\le \frac{x-1}{x\sqrt{2x-1}} \cdot \frac{y-1}{y\sqrt{2y-1}} \\ &\le 0.090169437\ldots, \end{aligned} \tag{3.16}$$

where the upper bound is obtained at $x = y = \frac{3+\sqrt{5}}{2} = 2.618033988\ldots$.

Proof. We have from (1.26) – (1.28)

$$(b-a)\left|T(f,g)\right| \le \left(\int_a^b f^2(t)\,dt - \mathcal{M}^2(f)\right)^{\frac{1}{2}} \times \left(\int_a^b g^2(t)\,dt - \mathcal{M}^2(g)\right)^{\frac{1}{2}}.$$

That is, taking $f(t) = t^{x-1}$, $g(t) = (1-t)^{y-1}$ then

$$0 \le \frac{1}{xy} - B(x,y) \le \left(\int_0^1 t^{2x-2}dt - \frac{1}{x^2}\right)^{\frac{1}{2}} \times \left(\int_0^1 (1-t)^{2y-2}\,dt - \frac{1}{y^2}\right)^{\frac{1}{2}}. \tag{3.17}$$

Now,

$$\int_0^1 t^{2x-2}dt = \frac{1}{2x-1} \quad \text{and} \quad \int_0^1 (1-t)^{2y-2}\,dt = \frac{1}{2y-1}$$

and so from (3.17) we have the first inequality in (3.16).

Now, consider

$$C(x) = \frac{x-1}{x\sqrt{2x-1}}. \tag{3.18}$$

The maximum occurs when $x = x^* = \frac{3+\sqrt{5}}{2}$ to give $C(x^*) = 0.3002831\ldots$. Hence, because of the symmetry we have the upper bound as stated in (3.16). □

Remark 5. *In a recent paper Alzer* [2] *shows that*

$$0 \le \frac{1}{xy} - B(x,y) \le b_A = \max_{x \ge 1}\left(\frac{1}{x^2} - \frac{\Gamma^2(x)}{\Gamma(2x)}\right) = 0.08731\ldots, \tag{3.19}$$

where 0 and b_A are shown to be the best constants. This uniform bound is only smaller for a small area around $\left(\frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$ while the first upper bound in (3.16) provides a better bound over a much larger region of the $x-y$ plane.

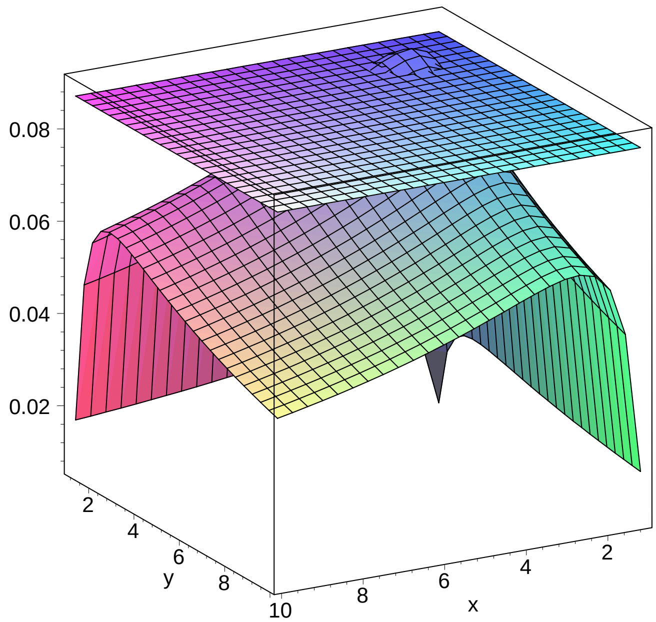

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FIGURE 1. Three dimensional plot of $C(x)C(y)$ and b_A where $C(x)$ is defined in (3.18) and $b_A = 0.08731\ldots$ from (3.19).

Figure 1 shows a plot of the upper bound (3.16) and the best uniform bound b_A as defined in (3.19).

Figure 2 demonstrates the cross-section through $x = y$ showing the small interval for which $b_A < C^2(x)$. The worst upper bound from (3.16) occurs at $x = y = \frac{3+\sqrt{5}}{2}$ and is given as the second upper bound in (3.16). This is represented, by the symbol $+$, in the region $C(x)C(y) = b_A$ shown in Figure 3.

We may state the following corollary given the results above.

Corollary 2. *For $x > 1$ and $y > 1$ we have*

$$0 \leq \frac{1}{xy} - B(x, y) \leq \min\{C(x)C(y), b_A\},$$

where $C(x)$ is defined by (3.18) and b_A by (3.19),

Remark 6. *The upper bound in Theorem 7 taking heed of Remark 4, seems not to be as good as that given in Theorem 8*

4. Bounds for the Zeta Function

The Riemann Zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \operatorname{Re}(s) > 1 \tag{4.1}$$

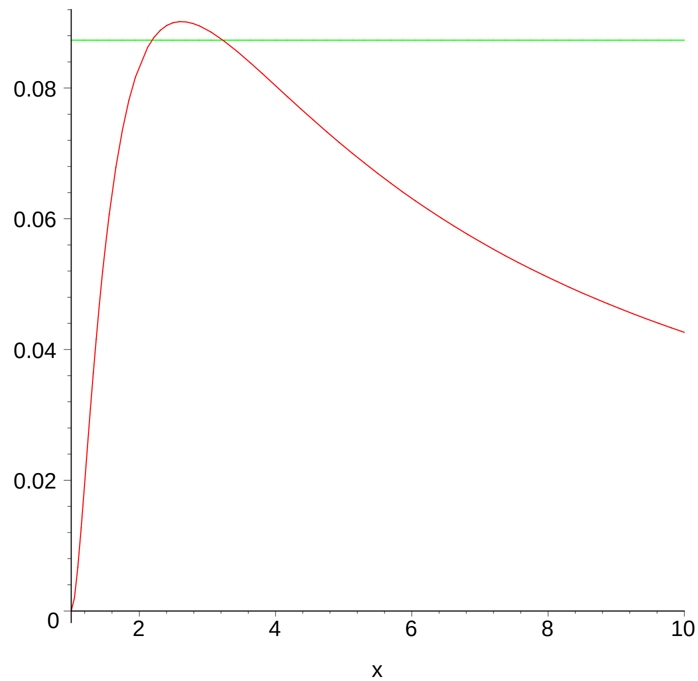

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FIGURE 2. The curve defined by $C^2(x) = \frac{(x-1)^2}{x^2(2x-1)}$ and $b_A = 0.08731\ldots$, from (3.18) and (3.19).

and is related to the Gamma function via the relation

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}dx, \qquad \mathrm{Re}(s) > 1. \tag{4.2}$$

The Zeta function seems to be known explicitly only for $s = 2m$ where m is a positive integer. Euler showed that for $m \in \mathbb{N}$

$$\zeta(2m) = (-1)^{m-1} \cdot \frac{2^{2m-1}}{(2m)!}B_{2m} \cdot \pi^{2m},$$

where B_{2m} are the Bernoulli numbers satisfying the relation

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!}B_k, \qquad |t| < 2\pi.$$

Theorem 9. *For $\alpha > 1$, the Zeta function satisfies the inequality*

$$\left|\zeta(\alpha) - 2^{\alpha-1} \cdot \frac{\pi^2}{6}\right| \le \kappa \cdot 2^{\alpha-1}\left[\frac{2\Gamma(2\alpha-1)}{\Gamma^2(\alpha)} - 1\right]^{\frac{1}{2}}, \tag{4.3}$$

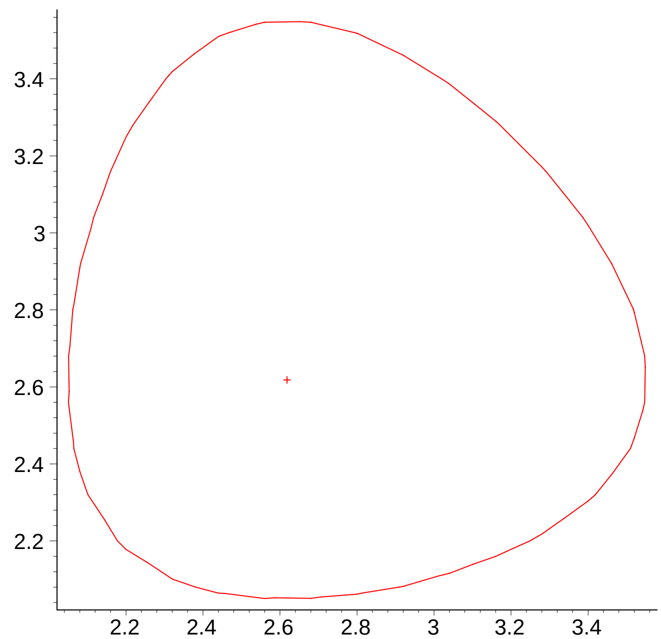

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FIGURE 3. Region over which $C(x)C(y) > b_A$ where $C(x)$ is as defined in (3.18) and b_A is the best uniform bound of Alzer given by (3.19).

where

$$\kappa = \left[\pi^2\left(1-\frac{\pi^2}{72}\right) - 7\zeta(3)\right]^{\frac{1}{2}} = 0.319846901\ldots. \tag{4.4}$$

Proof. Let

$$\tau(\alpha) = \int_0^\infty \frac{x^\alpha}{e^x-1}dx = \int_0^\infty e^{-\frac{x}{2}} \cdot \frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} \cdot x^{\alpha-1}dx, \quad \alpha > 1 \tag{4.5}$$

and make the associations

$$p(x) = e^{-\frac{x}{2}}, \quad f(x) = x^{\alpha-1}, \quad g(x) = \frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}. \tag{4.6}$$

We then have

$$\begin{cases} P = \int_0^\infty e^{-\frac{x}{2}}dx = 2; \\ \mathcal{M}(f;p) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}}x^{\alpha-1}dx = 2^{\alpha-1}\Gamma(\alpha) \quad \text{and} \\ \mathcal{M}(g;p) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}} \cdot \frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}dx = \frac{\zeta(2)}{2} = \frac{1}{2}\cdot\frac{\pi^2}{6}. \end{cases} \tag{4.7}$$

Thus, from (1.15) we have

$$P\cdot T(f,g;p) = \tau(\alpha) - 2^{\alpha-1}\Gamma(\alpha)\cdot\frac{\pi^2}{6} = \int_0^\infty e^{-\frac{x}{2}}\left(x^{\alpha-1}-\gamma\right)\left(\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}-\frac{\pi^2}{12}\right)dx \tag{4.8}$$

and so taking the modulus of (4.8) gives on using the Euclidean norm

$$\left|\tau(\alpha) - 2^{\alpha-1}\Gamma(\alpha)\cdot\frac{\pi^2}{6}\right| \le \left(\int_0^\infty e^{-\frac{x}{2}}\left(x^{\alpha-1}-1\right)^2 dx\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}-\frac{\pi^2}{12}\right)^2 dx\right)^{\frac{1}{2}}. \tag{4.9}$$

Now, the best value for γ is $2^{\alpha-1}\Gamma(\alpha)$, the integral mean, so that

$$\int_0^\infty e^{-\frac{x}{2}}\left(x^{\alpha-1}-\gamma\right)^2 dx = \int_0^\infty e^{-\frac{x}{2}}x^{2\alpha-2}dx - 2^{2\alpha-2}\Gamma^2(\alpha) = 2^{2\alpha-2}\left(2\Gamma(2\alpha-1)-\Gamma^2(\alpha)\right), \tag{4.10}$$

where we have used the fact that

$$\int_0^\infty e^{-ax}x^s dx = \frac{\Gamma(s+1)}{a^{s+1}}. \tag{4.11}$$

Further, from (1.27) with the associations (4.7),

$$\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}-\frac{\pi^2}{12}\right)^2 dx = \int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\right)^2 dx - 2\cdot\left(\frac{\pi^2}{12}\right)^2. \tag{4.12}$$

To calculate the above integral we have

$$\begin{aligned}\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\right)^2 dx &= \int_0^\infty e^{-\frac{3}{2}x}\cdot x^2\left(1-e^{-\frac{x}{2}}\right)^{-2}dx \\ &= \sum_{n=1}^\infty n\int_0^\infty e^{-\left(n+\frac{1}{2}\right)x}x^2dx \\ &= \sum_{n=1}^\infty \frac{n\Gamma(3)}{\left(n+\frac{1}{2}\right)^3} \\ &= 2\sum_{n=1}^\infty\frac{1}{\left(n+\frac{1}{2}\right)^2} - \sum_{n=1}^\infty\frac{1}{\left(n+\frac{1}{2}\right)^3} \\ &= \pi^2 - 7\zeta(3).\end{aligned} \tag{4.13}$$

In the above we have undertaken the permissible interchange of summation and integration and used (4.11).

From (4.10) – (4.13) we have on substitution in (4.9)

$$\left|\tau(\alpha) - 2^{\alpha-1}\Gamma(\alpha)\cdot\frac{\pi^2}{6}\right| \le 2^{\alpha-1}\left[2\Gamma(2\alpha-1)-\Gamma^2(\alpha)\right]^{\frac{1}{2}}\cdot\left[\pi^2\left(1-\frac{\pi^2}{72}\right)-7\zeta(3)\right]^{\frac{1}{2}} \tag{4.14}$$

Finally, from (4.2) and (4.5) we readily obtain the stated result (4.3) via (4.14). $\square$

Remark 7. *A bonus is obtained from Theorem 9 giving, since from (4.4) $\kappa > 0$, that*

$$\zeta(3) < \frac{\pi^2}{7}\left(1 - \frac{\pi^2}{72}\right) = 1.216671471\ldots. \tag{4.15}$$

We note that Guo [18] *obtains*

$$\zeta(3) < \frac{\pi^4}{72} = 1.35290404\ldots$$

and Luo, Wei and Qi [20] *using a refinement of the well known Jordan inequality in the expression*

$$\zeta(3) = \frac{8}{7}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3} = \frac{2}{7}\int_0^{\frac{\pi}{2}}\frac{x(\pi - x)}{\sin x}dx, \tag{4.16}$$

obtain the bounds $0.201\cdots \le \zeta(3) \le 1.217\ldots$, which are to be compared with the numerical approximation of $1.2020569032\ldots$. The upper bound in (4.15) is better than that obtained by Guo and also marginally better than the result of Luo et al. [20].

5. Concluding Remarks

In the paper the usefulness of some recent results in the analysis of inequalities, has been demonstrated through application to some special functions. Although these techniques have been applied in a variety of areas of applied mathematics, their application to special functions does not seem to have received much attention, if any, to date. There are many special functions which may be represented as the integral of products of functions. The investigation in the current article has restricted itself to the investigation of the Bessel function of the first kind, the Beta function and the Zeta function.

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