ON MATHIEU-BERG’S INEQUALITY

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ABSTRACT. In this paper, by using Euler-Maclaurin’s summation formula, we give a new improvement of Mathieu-Berg’s inequality.

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1. INTRODUCTION

Let \( s(c) \) be the Mathieu’s series as (see [1])

\[
s(c) := \sum_{n=1}^{\infty} \frac{2n}{(n^2 + c^2)^2} \quad (c \in \mathbb{R}).
\] (1.1)

The inequality of the form

\[
\frac{1}{c^2} + \frac{1}{2} < s(c) < \frac{1}{c^2} \quad (c > 0)
\] (1.2)

is usually called Mathieu-Berg’s inequality, which was guessed by E. Mathieu in 1890 and proved by L. Berg in 1952 (see [2]). Since then, a good few mathematicians have had various refinements of it. The representative work of them is the result of the form as (see [3]):

\[
s(c) = \sum_{i=0}^{k-1} (-1)^i \frac{B_{2i}}{2i+2} + \delta_k(c) \quad (c > 0),
\] (1.3)

where \( \delta_k(c) = (-1)^k \frac{2^{2k-1} - 1}{2^{2k}} \left[ 1 + \theta_k \frac{2^{2k} + 1}{2k} \right] \frac{B_{2k}}{2k} \) \( (|\theta_k| < 1) \), \( B_k \)' s are Bernoulli’s numbers, \( B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, \cdots \). In particular, for \( k = 1, 2 \), reducing (1.3), we have

\[
\frac{1}{c^2} - \frac{1}{6c^4} - \frac{23}{256c^6} < s(c) < \frac{1}{c^2} - \frac{1}{6c^4} + \frac{7}{256c^6}.
\] (1.4)

\[
\frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} - \frac{153}{2048c^8} < s(c) < \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} + \frac{57}{2048c^8}.
\] (1.5)

For the remainder term \( \delta_k(c) \), D.C. Russell [4] gave the following integral form:

\[
\delta_k(c) = \frac{(-1)^{k-1}}{c^{2k}} \int_0^\infty \left( \frac{x}{c^2 - x^2} \right)^{2k-1} \cos(cx) dx.
\] (1.6)

In this paper, by using (1.4), (1.5) and Euler-Maclaurin’s summation formula, a sharp result of Mathieu-Berg’s inequality is obtained which is an improvement of (1.2).
2. SOME LEMMAS

Lemma 2.1. If \( f(x) \in C^2[2, \infty) \), \( f'(x) < 0 \), and \( f(\infty) = 0 \), then we have
\[
0 < \int_{2}^{\infty} P_3(x)f(x)dx < \frac{1}{64} f(2),
\]
where \( P_j(x) \ (j = 1, 2, \cdots) \) are Bernoulli’s function(see[5]).

Proof. Basing on the periodicity of Bernoulli’s function and observing that 
\[
|B_4| = \max|P_4(x)|,
\]
we have
\[
\int_{2}^{\infty} (|B_4| - |P_4(x)|)|f'(x)|dx > 0.
\]
Since \( f'(x) < 0 \), and \( f(\infty) = 0 \), then \( f(x) \downarrow 0 \ (x \to \infty) \), and \( f(x) > 0 \). Since \( B_4 < 0 \), we obtain
\[
\int_{2}^{\infty} P_4(x)f'(x)dx \leq \int_{2}^{\infty} |P_4(x)||f'(x)|dx < |B_4| \int_{2}^{\infty} |f'(x)|dx = -B_4 f(2).
\]
By using the relation \( P_4'(x) = 4P_3(x) \), and \( P_4(2) = B_4 \), we have
\[
\int_{2}^{\infty} P_3(x)f(x)dx = \frac{1}{4} \int_{2}^{\infty} f(x)dP_4(x) = \frac{1}{4} \left[ -B_4 f(2) - \int_{2}^{\infty} P_4(x)f'(x)dx \right] > 0.
\]
In view of that fact that(see[1, p.83])
\[
P_3(x) = \sum_{v=0}^{3} \binom{3}{v} B_v x^{3-v} \ (x \in [0, 1]) \text{, and} \int_{k}^{k+1} P_3(x)dx = 0 \ (k = 1, 2, \cdots),
\]
we obtain
\[
\int_{2}^{\infty} P_3(x)f(x)dx = \sum_{k=2}^{\infty} \int_{k}^{k+1} P_3(x)(f(x) - f(k+1))dx
\]
\[
= \sum_{k=2}^{\infty} \left\{ \int_{k}^{k+1/2} P_3(x)(f(x) - f(k+1))dx + \int_{k+1/2}^{k+1} P_3(x)(f(x) - f(k+1))dx \right\}
\]
\[
= \sum_{k=2}^{\infty} \left( \int_{k}^{k+1/2} (f(k) - f(k+1))dx + \int_{k+1/2}^{k+1} P_3(x)dx + \sum_{k=2}^{\infty} \alpha_k \right)
\]
\[
= \sum_{k=2}^{\infty} \left( \int_{k}^{k+1/2} (f(k) - f(k+1))dx + \sum_{k=2}^{\infty} \alpha_k \right)
\]
\[
= f(2) \sum_{v=0}^{3} \binom{3}{v} B_v \int_{0}^{1/2} x^{3-v}dx + \sum_{k=2}^{\infty} \alpha_k = \frac{1}{64} f(2) + \sum_{k=2}^{\infty} \alpha_k,
\]
where \( \alpha_k \) is defined by
\[
\alpha_k := \int_{k}^{k+1/2} P_3(x)(f(x) - f(k))dx - \int_{k+1/2}^{k+1} P_3(x)(f(k+1) - f(x))dx.
\]
Note that \( f(x) \downarrow 0 \ (x \to \infty) \), we have \( \alpha_k < 0 \), and \( \sum_{k=2}^{\infty} \alpha_k < 0 \). Hence we have
\[
\int_{2}^{\infty} P_3(x)f(x)dx < \frac{1}{64} f(2).
\]
In view of (2.2) and (2.3), we have (2.1). The lemma is prove.
Lemma 2.2. Define the function $I(x)$ as
\[
I(x) := \frac{2}{1 + x^2} + \frac{1}{4 + x^2} + \frac{11}{6(4 + x^2)^2} + \frac{47}{12(4 + x^2)^3} - \frac{5}{(4 + x^2)^4} \quad (x \in R). \tag{2.4}
\]
Then, (i) if $|c| > 1$, we have
\[
\frac{1}{c^2} - \frac{1}{6c^2} - \frac{1}{30c^4} + \frac{57}{2048c^8} < \frac{1}{c^2 + 1/6}; \tag{2.5}
\]
(ii) if $|c| \leq 1$, we have
\[
I(c) < \frac{1}{c^2 + 1/6}. \tag{2.6}
\]
**Proof.** (i) When $|c| > 1$, inequality (2.5) is equivalent to the following
\[
1 - \left[\frac{1}{c^2} - \frac{1}{6c^2} - \frac{1}{30c^4} + \frac{57}{2048c^8}\right] \left(c^2 + 1/6\right)
= \left(\frac{11c^4}{180} - \frac{2053c^2}{92160} - \frac{57}{12288}\right) \frac{1}{c^8} > 0. \tag{2.5}'
\]
Obviously (2.5)' is always valid when $|c| > 1$. Hence (2.5) is true.
(ii) If $|c| \leq 1$, replacing $1/(4 + c^2)$ by $y$ in (2.6), we obtain after simplifications,
\[
g(y) := 88 - 1393y + 8010y^2 - 21249y^3 + 12420y^4 < 0, y \in [1/5, 1/4]. \tag{2.6}'
\]
In fact, since $g^{(4)}(y) > 0$, $g^{(3)}(y)$ is increasing in $[1/5, 1/4]$, we can find that $\max \{g^{(3)}(y); 1/5 \leq y \leq 1/4\} = g^{(3)}(1/4) < 0$, whence $g^{(3)}(y) < 0$. It follows that $g^{(2)}(y)$ is decreasing in $[1/5, 1/4]$. We find also that $\max \{g^{(2)}(y); 1/5 \leq y \leq 1/4\} = g^{(2)}(1/5) < 0$, whence $g^{(2)}(y) < 0$. Hence $g'(y)$ is decreasing in $[1/5, 1/4]$. Since $g'(1/5) < 0$, then $g'(y) < 0$, and $g(y)$ is increasing in $[1/5, 1/4]$. At last we find that $g(1/5) < 0$, whence $g(y) < 0$. Consequently (2.6)' is true; so is (2.6). The lemma is proved.

**Lemma 2.3.** Define the function $J(x)$ as
\[
J(x) := \frac{2}{1 + x^2} + \frac{1}{4 + x^2} + \frac{11}{6(4 + x^2)^2} + \frac{65}{48(4 + x^2)^3} + \frac{8}{(4 + x^2)^4} - \frac{16}{(4 + x^2)^5} \quad (x \in R). \tag{2.7}
\]
Then, (i) if $c^2 > 5/3$, we have
\[
\frac{1}{c^2 + 12/5} > \frac{1}{c^2} - \frac{1}{6c^4} - \frac{23}{56c^6}; \tag{2.8}
\]
(ii) if $c^2 \leq 5/3$, we have
\[
J(c) > \frac{1}{c^2 + 12/5}. \tag{2.9}
\]
**Proof.** (i) When $c^2 > 5/3$, inequality (2.8) is equivalent to the following
\[
c^4 - \frac{367c^2}{576} - \frac{115}{768} > 0. \tag{2.8}'
\]
Obviously (2.8)' is valid for all $c$ ($c^2 > 5/3$). Hence (2.8) is true.
Let \( s(c) \) be Mathieu’s series defined by (1.1). Then we have

\[
h(y) := 3 - \frac{1084 y}{48} + \frac{10765 y^2}{48} - \frac{63642 y^3}{48} + \frac{203709 y^4}{48} - 8952 y^5 + 6192 y^6 > 0 \quad (y \in \left[ \frac{3}{17}, \frac{1}{4} \right]). \tag{2.9}'
\]

In fact, \( h^{(4)}(y) = 203709/2 - 1074240y + 2229120y^2 \), \( h^{(4)}(3/17) = -18298.3027 < 0 \), and \( h^{(4)}(1/4) = -27385.5 < 0 \), then \( h^{(4)}(y) < 0 \), and \( h^{(4)} \) is decreasing in \([3/17, 1/4] \). We can find that \( \max \{ h^{(3)}(y) ; y \in [3/17, 1/4] \} = h^{(3)}(3/17) = -2624.375853 < 0 \), then \( h^{(3)}(y) < 0 \), and \( h^{(3)}(y) \) is decreasing in \([3/17, 1/4] \). We find that \( h^{(2)}(3/17) = -397.41057 < 0 \), then \( h^{(2)}(y) < 0 \), and \( h'(y) \) is decreasing in \([3/17, 1/4] \). Since \( h'(3/17) = -11.0578809 < 0 \), then \( h'(y) < 0 \), and \( h(y) \) is increasing in \([3/17, 1/4] \). At last, we find that \( h(1/4) = 18.64689127 > 0 \), whence \( h(y) > 0 \) and (2.9)' is true; so is (2.9). The lemma is proved.

3. MAIN RESULT AND REMARK

**Theorem 3.1.** Let \( s(c) \) be Mathieu’s series defined by (1.1). Then we have

\[
\frac{1}{c^2 + 5/12} < s(c) < \frac{1}{c^2 + 1/6} \quad (c \in R), \tag{3.1}
\]

where the constant \( 1/6 \) is the best possible.

**Proof.** Let \( f(x) \) be a function which is continuous and qth differential. Using Euler-Maclaurin’s formula [see [5]], we have

\[
\sum_{k=n+1}^{m} f(k) = \int_{n}^{m} f(x) \, dx + \frac{1}{2} f(2) - \frac{1}{12} f'(2) + \frac{1}{6} \int_{2}^{\infty} P_{3}(x) f^{(3)}(x) \, dx.
\]

where \( m > n, n, m \in N \). Let \( q=3, n=2, \) and \( m \to \infty \). we can obtain

\[
\sum_{k=2}^{\infty} f(k) = \int_{2}^{\infty} f(x) \, dx + \frac{1}{2} f(2) - \frac{1}{12} f'(2) + \frac{1}{6} \int_{2}^{\infty} P_{3}(x) f^{(3)}(x) \, dx.
\]

Assume that \( f(x) := 2x/(x^2 + c^2)^2 \). By computation, we find the following results:

\[
\int_{2}^{\infty} f(x) \, dx = \frac{1}{4 + c^2} \quad f(2) = \frac{4}{(4 + c^2)^2} \quad f'(2) = \frac{2}{(4 + c^2)^2} - \frac{32}{(4 + c^2)^3};
\]

\[
f^{(3)}(x) = \frac{480 c^2}{(x^2 + c^2)^4} = \frac{120}{(x^2 + c^2)^3} + \frac{384 c^4}{(x^2 + c^2)^5}.
\]

Hence by (3.3), we have

\[
s(c) = -\frac{1}{6} \int_{2}^{\infty} P_{3}(x) \left[ \frac{120}{(x^2 + c^2)^3} + \frac{384 c^4}{(x^2 + c^2)^5} \right] \, dx + \frac{1}{6} \int_{2}^{\infty} P_{3}(x) \frac{480 c^2}{(x^2 + c^2)^4} \, dx. \tag{3.4}
\]
By Lemma 2.1, we have
\[
0 < \int_2^\infty P_3(x) \frac{480c^2}{(x^2 + c^2)^4} \, dx < \frac{480c^2}{64(4 + c^2)^4} = \frac{30}{4(4 + c^2)^3} - \frac{30}{(1 + c^2)^4};
\]
\[
- \frac{1}{64} \left( \frac{120}{(4 + c^2)^3} + \frac{384c^4}{(4 + c^2)^5} \right) < - \int_2^\infty P_3(x) \frac{120}{(x^2 + c^2)^4} + \frac{384c^4}{(x^2 + c^2)^5} \, dx < 0.
\]
Consequently, (3.4) can be reduced to \(J(c) < s(c) < I(c)\), where \(I(x)\) and \(J(x)\) are defined respectively by (2.4) and (2.7). In virtue of (2.8) and (2.9), and in view of the left-hand side of (1.4), it follows that the inequality of the left-hand side of (3.1) is valid. It is known from the right-hand side of (1.5), (2.5) and (2.6) that the right-hand side of (3.1) is also valid.

We now show that for any \(\epsilon > 0\), there exists \(c_0 > 0\), such for \(c > c_0\), that
\[
s(c) \geq \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1 + \epsilon}{30c^6} + \frac{57}{2048c^8}.
\]
(3.5)

Otherwise, by (1.5), we may get an increasing sequence \(\{c_n; c_n \uparrow \infty\}\), which provides the following inequality
\[
\frac{1}{c_n^2} - \frac{1}{6c_n^4} - \frac{1}{30c_n^6} - \frac{153}{2048c_n^8} < s(c_n) < \frac{1}{c_n^2} - \frac{1}{6c_n^4} - \frac{1 + \epsilon}{30c_n^6} + \frac{57}{2048c_n^8};
\]
\[
\left( \frac{\epsilon}{30} \frac{c_n^2}{2048} \right) \frac{1}{c_n^2} < s(c_n) - \left( \frac{1}{c_n^2} \frac{1}{6c_n^4} - \frac{1 + \epsilon}{30c_n^6} + \frac{57}{2048c_n^8} \right) < 0.
\]
When \(n\) is large enough, we have \(\left( \frac{\epsilon}{30} c_n^2 \frac{2048}{2048} \right) \frac{1}{c_n^2} > 0\). This is a contradictory. Then (3.5) is valid. Hence we may find a real number \(c\), such \(c > c_0\) is larger enough, that
\[
1 - s(c) \left( c^2 + \frac{6}{c} + \epsilon \right) \leq 1 - \left[ \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1 + \epsilon}{30c^6} + \frac{57}{2048c^8} \right] \left( c^2 + \frac{6}{c} + \epsilon \right)
\]
\[
= - \left( \epsilon - \frac{11 + 36\epsilon}{180c^2} - \cdots \right) \frac{1}{c^2}.
\]
(3.6)

In virtue of (3.6), we obtain that \(s(c) > \frac{1}{c^2 + \frac{6}{c} + \epsilon}\), for this \(c\). Hence the constant 1/6 in (3.1) is the best possible. Thus we complete the proof of the theorem.

**Remark 3.2.** New inequality (3.1) is always valid for any real number \(c\). Evidently the obtained result (3.1) is a bilateral improvement of (1.2). Moreover it is more succinct than (1.3). Particularly we point out that it is better than (1.3) when \(|c| < 1\). At last we mention that it is impossible to improve the left-hand side of (3.1) any more, because of the constant 1/6 being the best possible.

**References**


