SOME STRENGTHENED RESULTS ON GERRETSEN'S INEQUALITIES

SHAN-HE WU AND ZHI-HUA ZHANG

ABSTRACT. In this paper, we give some strengthened results on Gerretsen's inequalities, and establish a parameter form for Gerretsen's inequalities by using power series.

1. Introduction and Notations

In practice we assume that s denotes the semi-perimeter of triangle ABC, R the circumradius, r the inradius, and $(2n-3)!! = 1 \cdot 3 \cdot 5 \cdots (2n-3)$. In addition we have $(-1)!! = 1$.

In 1953, J.C.Gerretsen [1] obtained the following important double inequalities:

Theorem 1.1. *In every triangle we have the double-sided inequality*

$$16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2. \tag{1.1}$$

Gerretsen's inequality (1.1) has broad applications in geometric inequalities, and is a powerul tool of research in geometric inequalities. It is as important to geometric inequality theory as Hölder's inequality is to analytic inequality theory.

The purpose of this note is to present a simple but powerful form of strengthening Gerretsen's inequalities for triangles. The parameter form for Gerretsen's inequalities are established by using power series.

2. The Strengthened Form of Gerretsen's Inequalities

In this paper, the following three lemmas are necessary:

Lemma 2.1. *(Basic inequalities for the triangle* [2]*) In every triangle we have*

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr} \le s^2 \le 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}. \tag{2.1}$$

Lemma 2.2. *Assume $-1 \le x \le 1$ and $0 < \alpha < 1$, we have the following power series expansion*

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n \tag{2.2}$$

and the following Bernoulli's inequality [3]

$$(1+x)^\alpha \le 1 + \alpha x \tag{2.3}$$

Lemma 2.3. *Assume $-1 < x < 1$, the following power series expansion is well-known*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n. \tag{2.4}$$

Date: July 15, 2003.

1991 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Gerretsen's inequalities, Euler's inequality, power series, strengthen, parameter.

The authors would like to thank professor Han-fang Zhang and the anonymous referee for some valuable suggestions which have improved the final version of this paper.

This paper was typeset using $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX.

Theorem 2.1. *In every triangle the following inequalities hold*

$$16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} \le s^2 \le 4R^2 + 4Rr + 3r^2 - \frac{r^2(R-2r)}{R-r}. \tag{2.5}$$

Proof. Since basic inequalities (2.1) for the triangle are equivalent to the following inequalities:

$$\begin{aligned} 16Rr - 5r^2 + 2(R-2r)(R-r-\sqrt{R^2-2Rr}) &\le s^2 \\ \le 4R^2+4Rr+3r^2 - 2(R-2r)(R-r-\sqrt{R^2-2Rr}). \end{aligned} \tag{2.6}$$

From Euler's inequality $R \ge 2r$ and Bernoulli's inequality (2.3), we have

$$R-r>0, \quad 0 < \frac{r}{R-r} \le 1,$$

and

$$R-r-\sqrt{R^2-2Rr} = (R-r)\left[1-\sqrt{\frac{R^2-2Rr}{(R-r)^2}}\right]$$

$$= (R-r)\left[1-\sqrt{1-\frac{r^2}{(R-r)^2}}\right] \ge \frac{1}{2}(R-r)\left(\frac{r}{R-r}\right)^2 = \frac{r^2}{2(R-r)}.$$

According to (2.6), it is easy to obtain (2.5). The proof of Theorem 2.1 is completed. □

The inequalities (2.5) were also proved by Xue-zhi Yang in [4], by the use of appropriate trigonometric inequalities.

Now, we will give a generalized result:

Theorem 2.2. *In every triangle we have the following inequalities*

$$\begin{aligned} 16Rr - 5r^2 + r(R-2r)\sum_{n=1}^{\infty}\frac{(2n-3)!!}{2^{n-1}n!}\left(\frac{r}{R-r}\right)^{2n-1} &\le s^2 \\ \le 4R^2+4Rr+3r^2 - r(R-2r)\sum_{n=1}^{\infty}\frac{(2n-3)!!}{2^{n-1}n!}\left(\frac{r}{R-r}\right)^{2n-1}. \end{aligned} \tag{2.7}$$

Proof. In (2.6), we have obtained the following equality:

$$R-r-\sqrt{R^2-2Rr} = (R-r)\left[1-\sqrt{1-\frac{r^2}{(R-r)^2}}\right].$$

Let

$$\frac{r}{R-r} = x \quad (0 < x \le 1),$$

then

$$R-r-\sqrt{R^2-2Rr} = (R-r)(1-\sqrt{1-x^2}).$$

From the power series expansion (2.2), we have

$$\sqrt{1-x^2} = 1 - \frac{1}{2}x^2 - \sum_{n=2}^{\infty}\frac{(2n-3)!!}{2^n n!}x^{2n} \quad (0 < x \le 1),$$

or

$$1-\sqrt{1-x^2} = \frac{1}{2}x\left[x + \sum_{n=2}^{\infty}\frac{(2n-3)!!}{2^{n-1}n!}x^{2n-1}\right] = \frac{r}{2(R-r)}\sum_{n=1}^{\infty}\frac{(2n-3)!!}{2^{n-1}n!}\left(\frac{r}{R-r}\right)^{2n-1}.$$

Therefore the following equality holds

$$R-r-\sqrt{R^2-2Rr} = \frac{1}{2}r\sum_{n=1}^{\infty}\frac{(2n-3)!!}{2^{n-1}n!}\left(\frac{r}{R-r}\right)^{2n-1}. \tag{2.8}$$

Combining expressions (2.6) and (2.8) we obtain (2.7). Theorem 2.2 is proved. □

Theorem 2.3. *In every triangle we have the following inequalities*

$$16Rr - 5r^2 + r(R-2r)\varsigma \leq s^2 \leq 4R^2 + 4Rr + 3r^2 - r(R-2r)\varsigma, \tag{2.9}$$

where

$$\varsigma = \sum_{n=1}^{\infty} \frac{(2n-3)\,!!}{2^{n\,-\,1} n\,!} \left[\sum_{m=1}^{\infty} 2^{m-1} \left(\frac{r}{R+r} \right)^m \right]^{2n-1}.$$

Proof. From the power series expansion (2.4), we have

$$\frac{r}{R-r} = \frac{r}{R+r}\left(1 - \frac{2r}{R+r}\right)^{-1} = \frac{r}{R+r}\sum_{m=0}^{\infty}\left(\frac{2r}{R+r}\right)^m = \sum_{m=1}^{\infty} 2^{m\,-\,1}\left(\frac{r}{R+r}\right)^m. \tag{2.10}$$

Combining expression (2.7) and (2.10) we immediately get (2.9). Theorem 2.3 is proved. □

3. The Parameter Form of Gerretsen's Inequalities

In this section, we will establish a parameterised form of Gerretsen's inequalities.

Theorem 3.1. *Let λ be a nonzero real number, in every triangle we have the following inequalities*

$$\begin{aligned} -(\lambda-1)^2R^2 + 2(\lambda^2+5\lambda+2)Rr - (4+\lambda)r^2 + \frac{1}{2}(R-2r)\left|\,(1-\lambda^2)R - 2r\,\right|\varepsilon \leq \lambda s^2 \leq \\ (\lambda+1)^2R^2 - 2(\lambda^2-5\lambda+2)Rr + (4-\lambda)r^2 - \frac{1}{2}(R-2r)\left|\,(1-\lambda^2)R - 2r\,\right|\varepsilon, \end{aligned} \tag{3.1}$$

where

$$\varepsilon = \sum_{n=1}^{\infty} \frac{(\,2n-3\,)\,!!}{2^{n\,-\,1} n\,!} \left| \frac{(1-\lambda^2)R-2r}{(1+\lambda^2)R-2r} \right|^{2n-1}.$$

Proof. When $\lambda > 0$, from inequality (2.1), we have

$$\begin{aligned} 2\lambda R^2 + 10\lambda Rr - \lambda r^2 - 2\lambda(R-2r)\sqrt{R^2-2Rr} \leq \lambda s^2 \leq \\ 2\lambda R^2 + 10\lambda Rr - \lambda r^2 + 2\lambda(R-2r)\sqrt{R^2-2Rr}, \end{aligned}$$

or

$$\begin{aligned} &-(\lambda-1)^2R^2 + 2(\lambda^2+5\lambda+2)Rr - (4+\lambda)r^2 \\ &+(R-2r)[(\lambda^2+1)R - 2r - 2\lambda\sqrt{R^2-2Rr}] \leq \lambda s^2 \leq \\ &(\lambda+1)^2R^2 - 2(\lambda^2-5\lambda+2)Rr + (4-\lambda)r^2 - (R-2r)[(\lambda^2+1)R - 2r - 2\lambda\sqrt{R^2-2Rr}], \end{aligned} \tag{3.2}$$

From Euler's inequality $R \geq 2r$, we obtain $(\lambda^2+1)R - 2r > 0$, and

$$\begin{aligned} (\lambda^2+1)R - 2r - 2\lambda\sqrt{R^2-2Rr} &= [(\lambda^2+1)R-2r]\left[1 - \sqrt{\frac{4\lambda^2(R^2-2Rr)}{[(1+\lambda^2)R-2r]^2}}\right] \\ &= [(\lambda^2+1)R-2r]\left[1 - \sqrt{1 - \left[\frac{(1-\lambda^2)R-2r}{(1+\lambda^2)R-2r}\right]^2}\right]. \end{aligned}$$

Let

$$\left| \frac{(1-\lambda^2)R-2r}{(1+\lambda^2)R-2r} \right| = x(0 < x \leq 1),$$

then

$$(\lambda^2+1)R - 2r - 2\lambda\sqrt{R^2-2Rr} = [(\lambda^2+1)R-2r](1-\sqrt{1-x^2}).$$

From the power series expansion (2.2), we have

$$1 - \sqrt{1-x^2} = \frac{1}{2}x\sum_{n=1}^{\infty}\frac{(2n-3)\,!!}{2^{n-1}n\,!}x^{2n-1},$$

and

$$(3.3)\quad (\lambda^2+1)R-2r-2\lambda\sqrt{R^2-2Rr} = \frac{1}{2}\left|\,(1-\lambda^2)R-2r\,\right|\sum_{n=1}^{\infty}\frac{(2n-3)\,!!}{2^{n-1}n\,!}\left|\frac{(1-\lambda^2)R-2r}{(1+\lambda^2)R-2r}\right|^{2n-1}.$$

Combining expression (3.2) and (3.3), the inequality (3.1) is proved.

If $\lambda<0$, then $-\lambda>0$, applying the above result, we have

$$-(-\lambda-1)^2R^2+2(\lambda^2-5\lambda+2)Rr-(4-\lambda)r^2+\frac{1}{2}(R-2r)\left|\,(1-\lambda^2)R-2r\,\right|\varepsilon \le \lambda s^2 \le$$
$$(-\lambda+1)^2R^2-2(\lambda^2+5\lambda+2)Rr+(4+\lambda)r^2-\frac{1}{2}(R-2r)\left|\,(1-\lambda^2)R-2r\,\right|\varepsilon.$$

It is easy to see that above inequalities are equivalent to inequalities (3.1). The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Let λ, t be real number, and $\lambda \neq 2t$, in every triangle we have the following inequalities*

$$(3.4)\quad -(t-1)^2R^3+2\left[t^2+(\lambda+5)t-\lambda+2\right]R^2r-\left[(4\lambda+1)t+\lambda^2+10\lambda+4\right]Rr^2+(2\lambda^2+\lambda)r^3$$
$$+\frac{1}{2}(R-2r)\left|(\,tR-\lambda r)^2-R(R-2r\,)\right|\varepsilon \le (tR-\lambda r)s^2 \le$$
$$(t+1)^2R^3-2[t^2+(\lambda-5)t+\lambda+2]R^2r+[(4\lambda-1)t+\lambda^2-10\lambda+4]Rr^2-(2\lambda^2-\lambda)r^3$$
$$-\frac{1}{2}(R-2r)\left|(\,tR-\lambda r)^2-R(R-2r\,)\right|\varepsilon,$$

where

$$\varepsilon=\sum_{n=1}^{\infty}\frac{(2n-3)\,!!}{2^{n-1}n\,!}\left|\frac{(tR-\lambda r)^2-R(R-2r)}{(tR-\lambda r)^2+R(R-2r)}\right|^{2n-1}.$$

Proof. From inequality (2.1), we have

$$(3.5)\qquad |\,tR-\lambda r\,|\ \left[\left|\,s^2-2R^2-10Rr+r^2\right|-2(R-2r)\sqrt{R^2-2Rr}\,\right]\le 0.$$

Since (3.5)$\Leftrightarrow$ $\left|\,(tR-\lambda r)(s^2-2R^2-10Rr+r^2)\,\right|\le 2\,|\,tR-\lambda r\,|\,(R-2r)\sqrt{R^2-2Rr}$

$$\Leftrightarrow (tR-\lambda r)(2R^2+10Rr-r^2)-2\,|\,tR-\lambda r\,|\,(R-2r)\sqrt{R^2-2Rr}\quad \le (tR-\lambda r)s^2 \le$$
$$(tR-\lambda r)(2R^2+10Rr-r^2)+2\,|\,tR-\lambda r\,|\,(R-2r)\sqrt{R^2-2Rr},$$

that is

$$(3.6)\quad -(t-1)^2R^3+2[t^2+(\lambda+5)t-\lambda+2]R^2r-[(4\lambda+1)t+\lambda^2+10\lambda+4]Rr^2+(2\lambda^2+\lambda)r^3$$
$$+(R-2r)[\ (tR-\lambda r)^2+R(R-2r)-2\ |tR-\lambda r|\sqrt{R^2-2Rr}\ ]\ \le (tR-\lambda r)s^2 \le$$
$$(t+1)^2R^3-2[t^2+(\lambda-5)t+\lambda+2]R^2r+[(4\lambda-1)t+\lambda^2-10\lambda+4]Rr^2-(2\lambda^2-\lambda)r^3$$
$$-(R-2r)[(tR-\lambda r)^2+R(R-2r)-2\ |tR-\lambda r|\sqrt{R^2-2Rr}].$$

According to $\lambda\neq 2t$ and Euler's inequality $R\ge 2r$, we obtain $(tR-\lambda r)^2+R(R-2r)>0$, and

$$(tR-\lambda r)^2+R(R-2r)-2\ |tR-\lambda r|\sqrt{R^2-2Rr}$$
$$=[(tR-\lambda r)^2+R(R-2r)]\left[\,1-\sqrt{\frac{4(tR-\lambda r)^2(R^2-2Rr)}{[(tR-\lambda r)^2+R(R-2r)]^2}}\,\right]$$

$$= [(tR-\lambda r)^2 + R(R-2r)]\left[1-\sqrt{1-\left[\frac{(tR-\lambda r)^2-R(R-2r)}{(tR-\lambda r)^2+R(R-2r)}\right]^2}\,\right].$$

Let

$$\left|\frac{(tR-\lambda r)^2-R(R-2r)}{(tR-\lambda r)^2+R(R-2r)}\right| = x(0<x\le 1),$$

then

$$(tR-\lambda r)^2 + R(R-2r) - 2\,|tR-\lambda r|\sqrt{R^2-2Rr} = [(tR-\lambda r)^2+R(R-2r)](1-\sqrt{1-x^2}),$$

From the power series expansion (2.2), we have

$$1-\sqrt{1-x^2} = \frac{1}{2}x\sum_{n=1}^{\infty}\frac{(2n-3)\,!!}{2^{n-1}n\,!}x^{2n-1},$$

therefore

$$(3.7)\qquad (tR-\lambda r)^2 + R(R-2r) - 2\,|tR-\lambda r|\sqrt{R^2-2Rr}$$
$$= \frac{1}{2}[(tR-\lambda r)^2 - R(R-2r)]\sum_{n=1}^{\infty}\frac{(2n-3)\,!!}{2^{n-1}n\,!}\left|\frac{(tR-\lambda r)^2-R(R-2r)}{(tR-\lambda r)^2+R(R-2r)}\right|^{2n-1}.$$

Combining expression (3.6) and (3.7) we can get the inequalities (3.4). Theorem 3.2 is proved. □

Now, we give some corollaries from Theorem 3.1 and Theorem 3.2.

Corollary 3.1. *Let λ, t be real numbers, respectively, and $\lambda \neq 2t$, in every triangle we have the following inequalities*

$$(3.8)\quad -(t-1)^2R^3+2\,[t^2+(\lambda+5)t-\lambda+2]R^2r-[(4\lambda+1)\,t+\lambda^2+10\lambda+4]Rr^2+(2\lambda^2+\lambda)r^3 \le (tR-\lambda r)s^2$$
$$\le (t+1)^2R^3 - 2[t^2+(\lambda-5)t+\lambda+2]R^2r + [(4\lambda-1)t+\lambda^2-10\lambda+4]Rr^2 - (2\lambda^2-\lambda)r^3.$$

Corollary 3.2. *Let λ be a nonzero real numbers, in every triangle we have the following double inequality*

$$(3.9)\qquad -(\lambda-1)^2R^2+2(\lambda^2+5\lambda+2)Rr-(4+\lambda)r^2 \le \lambda s^2 \le$$
$$(\lambda+1)^2R^2-2(\lambda^2-5\lambda+2)Rr+(4-\lambda)r^2.$$

Inequalities (3.8) ,(3.9) include Gerretsen's inequalities and a lot of new geometric inequalities.

Corollary 3.3. *Let λ, t be real numbers, respectively, and $\lambda \neq 2t$, in every triangle we have the following double inequality*

$$(3.10)\quad -(t-1)^2R^3+2\,[t^2+(\lambda+5)t-\lambda+2]R^2r-[(4\lambda+1)\,t+\lambda^2+10\lambda+4]Rr^2+(2\lambda^2+\lambda)r^3$$
$$+\frac{1}{2}(R-2r)\,|(\,tR-\lambda r)^2-R(R-2r\,)|\,\varsigma \le (tR-\lambda r)s^2 \le$$
$$(t+1)^2R^3-2[t^2+(\lambda-5)t+\lambda+2]R^2r+[(4\lambda-1)t+\lambda^2-10\lambda+4]Rr^2-(2\lambda^2-\lambda)r^3$$
$$-\frac{1}{2}(R-2r)\,|(\,tR-\lambda r)^2-R(R-2r\,)|\,\varsigma,$$

where

$$\varsigma = \sum_{m=1}^{\infty}\frac{(\,2m-3\,)\,!!}{2^{m-1}m\,!}\,[\,\frac{1}{2Rr}\left|(tR-\lambda r)^2-R(R-2r)\right|\sum_{n=1}^{\infty}2^{n-1}\,(\,\frac{2Rr}{(tR-\lambda r)^2+R(R+2r)}\,)^n\,]^{2m-1}.$$

Proof. From Euler's inequality $R \geq 2r$, we obtain

$$0 < \frac{4Rr}{(tR-\lambda r)^2 + R(R+2r)} \leq \frac{4Rr}{(tR-\lambda r)^2 + 4Rr} < 1.$$

Using the power series expansion (2.4), we have

$$\left|\frac{(tR-\lambda r)^2 - R(R-2r)}{(tR-\lambda r)^2 + R(R-2r)}\right| = \left|\frac{(tR-\lambda r)^2 - R(R-2r)}{(tR-\lambda r)^2 + R(R+2r)}\right| \left|\frac{(tR-\lambda r)^2 + R(R+2r)}{(tR-\lambda r)^2 + R(R-2r)}\right| \tag{3.11}$$

$$= \left|\frac{(tR-\lambda r)^2 - R(R-2r)}{(tR-\lambda r)^2 + R(R+2r)}\right| \frac{1}{1 - \dfrac{4Rr}{(tR-\lambda r)^2 + R(R+2r)}}$$

$$= \left|\frac{(tR-\lambda r)^2 - R(R-2r)}{(tR-\lambda r)^2 + R(R+2r)}\right| \sum_{n=0}^{\infty} \left[\frac{4Rr}{(tR-\lambda r)^2 + R(R+2r)}\right]^n$$

$$= \frac{1}{2Rr} \left|(tR-\lambda r)^2 - R(R-2r)\right| \sum_{n=1}^{\infty} 2^{n-1} \left[\frac{2Rr}{(tR-\lambda r)^2 + R(R+2r)}\right]^n.$$

Combining expression (3.4) and (3.11) we immediately get the inequalities (3.10), and the proof of Corollary 3.3 is completed. □

References

[1] J.C.Gerretsen. *OngelijKheden in the Driehoek*. Nieuw Tijdschr, Wisk, 41(1953). 1–7.

[2] D.S.Mitrinović, J.E.Pečrić and V.Volenec. *Recent Advances in Geometric Inequalities*. Kluwer Academic Publishers, 1989. 2–3.

[3] D. S. Mitrinović, P.M.Vasić. *Analytic Inequalities*. Springer-Verlag, Berlin, 1970. 44–45.

[4] Xue-zhi Yang. *The proof of a class of trigonometric Inequalities*, Mathematics Olympic, Hunan Education Press, Hunan Education Press, Changsha, China, 19(1994).24–40. (in Chinese).

(SH.-H. Wu) Department of Mathematics, Longyan College, Longyan, Fujian 364012, P.R.China
E-mail address: wushanhe@yahoo.com.cn

(Zh.-H. Zhang) Zixing Educational Research Section, Chenzhou, Hunan 423400,P.R.China.
E-mail address: zxzh1234@163.com