Some strengthened results on Gerretsen’s inequalities

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Abstract. In this paper, we give some strengthened results on Gerretsen’s inequalities, and establish a parameter form for Gerretsen’s inequalities by using power series.

1. Introduction and Notations

In practice we assume that $s$ denotes the semi-perimeter of triangle $ABC$, $R$ the circumradius, $r$ the inradius, and $(2n-3)!! = 1 \cdot 3 \cdot 5 \cdots (2n-3)$. In addition we have $(-1)!! = 1$.

In 1953, J.C. Gerretsen [1] obtained the following important double inequalities:

Theorem 1.1. In every triangle we have the double-sided inequality

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2. \tag{1.1}$$

Gerretsen’s inequality (1.1) has broad applications in geometric inequalities, and is a powerful tool of research in geometric inequalities. It is as important to geometric inequality theory as Hölder’s inequality is to analytic inequality theory.

The purpose of this note is to present a simple but powerful form of strengthening Gerretsen’s inequalities for triangles. The parameter form for Gerretsen’s inequalities are established by using power series.

2. The Strengthened Form of Gerretsen’s Inequalities

In this paper, the following three lemmas are necessary:

Lemma 2.1. (Basic inequalities for the triangle [2]) In every triangle we have

$$2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}. \tag{2.1}$$

Lemma 2.2. Assume $-1 \leq x \leq 1$ and $0 < \alpha < 1$, we have the following power series expansion

$$(1 + x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1)(\alpha - 2)\cdots(\alpha - n + 1)x^n}{n!} \tag{2.2}$$

and the following Bernoulli’s inequality [3]

$$(1 + x)^\alpha \leq 1 + \alpha x \tag{2.3}$$

Lemma 2.3. Assume $-1 < x < 1$, the following power series expansion is well-known

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n. \tag{2.4}$$
Theorem 2.1. In every triangle the following inequalities hold

\[(2.5) \quad 16Rr - 5r^2 + \frac{r^2(R - 2r)}{R - r} \leq s^2 \leq 4R^2 + 4Rr + 3r^2 - \frac{r^2(R - 2r)}{R - r}.\]

Proof. Since basic inequalities \((2.1)\) for the triangle are equivalent to the following inequalities:

\[(2.6) \quad 16Rr - 5r^2 + 2(R - 2r)(R - r - \sqrt{R^2 - 2Rr}) \leq s^2 \leq 4R^2 + 4Rr + 3r^2 - 2(R - 2r)(R - r - \sqrt{R^2 - 2Rr}).\]

From Euler’s inequality \(R \geq 2r\) and Bernoulli’s inequality \((2.3)\), we have

\[R - r > 0, \quad 0 < \frac{r}{R - r} \leq 1,\]

and

\[R - r - \sqrt{R^2 - 2Rr} = (R - r) \left[ 1 - \frac{\sqrt{R^2 - 2Rr}}{(R - r)^2} \right] \geq \frac{1}{2}(R - r) \left( \frac{r}{R - r} \right)^2 = \frac{r^2}{2(R - r)}.\]

According to \((2.6)\), it is easy to obtain \((2.5)\). The proof of Theorem 2.1 is completed. \(\square\)

The inequalities \((2.5)\) were also proved by Xue-zhi Yang in \([4]\), by the use of appropriate trigonometric inequalities.

Now, we will give a generalized result:

Theorem 2.2. In every triangle we have the following inequalities

\[(2.7) \quad 16Rr - 5r^2 + r(R - 2r) \sum_{n=1}^{\infty} \frac{(2n - 3)!!}{2^{n-1} n!} \left( \frac{r}{R - r} \right)^{2n-1} \leq s^2 \leq 4R^2 + 4Rr + 3r^2 - r(R - 2r) \sum_{n=1}^{\infty} \frac{(2n - 3)!!}{2^{n-1} n!} \left( \frac{r}{R - r} \right)^{2n-1}.\]

Proof. In \((2.6)\), we have obtained the following equality:

\[R - r - \sqrt{R^2 - 2Rr} = (R - r) \left[ 1 - \frac{r^2}{(R - r)^2} \right].\]

Let \(\frac{r}{R - r} = x \quad (0 < x \leq 1)\),

then

\[R - r - \sqrt{R^2 - 2Rr} = (R - r)(1 - \sqrt{1 - x^2}).\]

From the power series expansion \((2.2)\), we have

\[\sqrt{1 - x^2} = 1 - \frac{1}{2}x^2 - \sum_{n=2}^{\infty} \frac{(2n - 3)!!}{2^n n!} x^{2n} \quad (0 < x \leq 1),\]

or

\[1 - \sqrt{1 - x^2} = \frac{1}{2}x \left[ x + \sum_{n=2}^{\infty} \frac{(2n - 3)!!}{2^n - 1 n!} x^{2n-1} \right] = \frac{r}{2(R - r)} \sum_{n=1}^{\infty} \frac{(2n - 3)!!}{2^{n-1} n!} \left( \frac{r}{R - r} \right)^{2n-1}.\]

Therefore the following equality holds

\[(2.8) \quad R - r - \sqrt{R^2 - 2Rr} = \frac{1}{2} r \sum_{n=1}^{\infty} \frac{(2n - 3)!!}{2^{n-1} n!} \left( \frac{r}{R - r} \right)^{2n-1}.\]
Combining expressions (2.6) and (2.8) we obtain (2.7). Theorem 2.2 is proved.

**Theorem 2.3.** In every triangle we have the following inequalities
\[
16Rr - 5r^2 + r(R - 2r)\zeta \leq s^2 \leq 4R^2 + 4Rr + 3r^2 - r(R - 2r)\zeta,
\]
where
\[
\zeta = \sum_{n=1}^{\infty} \frac{(2n - 3)!!}{2^{n-1}n!} \left( \sum_{m=1}^{\infty} 2^{m-1} \left( \frac{r}{R + r} \right)^m \right)^{2n-1}.
\]

**Proof.** From the power series expansion (2.4), we have
\[
\frac{r}{R - r} = \frac{r}{R + r} \left(1 - \frac{2r}{R + r}\right)^{-1} = \frac{r}{R + r} \sum_{m=0}^{\infty} \left( \frac{2r}{R + r} \right)^m = \sum_{m=1}^{\infty} 2^{m-1} \left( \frac{r}{R + r} \right)^m.
\]
Combining expression (2.7) and (2.10) we immediately get (2.9). Theorem 2.3 is proved.

3. The Parameter Form of Gerretsen’s Inequalities

In this section, we will establish a parameterised form of Gerretsen’s inequalities.

**Theorem 3.1.** Let \( \lambda \) be a nonzero real number, in every triangle we have the following inequalities
\[
-(\lambda - 1)^2 R^2 + 2(\lambda^2 + 5\lambda + 2)Rr - (4 + \lambda)r^2 + \frac{1}{2}(R - 2r) \mid (1 - \lambda^2)R - 2r \mid \varepsilon \leq \lambda s^2 \leq
\]
\[
(\lambda + 1)^2 R^2 - 2(\lambda^2 - 5\lambda + 2)Rr + (4 - \lambda)r^2 - \frac{1}{2}(R - 2r) \mid (1 - \lambda^2)R - 2r \mid \varepsilon,
\]
where
\[
\varepsilon = \sum_{n=1}^{\infty} \frac{(2n - 3)!!}{2^{n-1}n!} \left| \frac{(1 - \lambda^2)R - 2r}{(1 + \lambda^2)R - 2r} \right|^{2n-1}.
\]

**Proof.** When \( \lambda > 0 \), from inequality (2.1), we have
\[
2\lambda R^2 + 10\lambda Rr - \lambda r^2 - 2\lambda(R - 2r)\sqrt{R^2 - 2Rr} \leq \lambda s^2 \leq
\]
\[
2\lambda R^2 + 10\lambda Rr - \lambda r^2 + 2\lambda(R - 2r)\sqrt{R^2 - 2Rr},
\]
or
\[
-(\lambda - 1)^2 R^2 + 2(\lambda^2 + 5\lambda + 2)Rr - (4 + \lambda)r^2
\]
\[
+(R - 2r)(\lambda^2 + 1)R - 2r - 2\lambda\sqrt{R^2 - 2Rr} \leq \lambda s^2 \leq
\]
\[
(\lambda + 1)^2 R^2 - 2(\lambda^2 - 5\lambda + 2)Rr + (4 - \lambda)r^2 - (R - 2r)((\lambda^2 + 1)R - 2r - 2\lambda\sqrt{R^2 - 2Rr}].
\]
From Euler’s inequality \( R \geq 2r \), we obtain \((\lambda^2 + 1)R - 2r > 0\), and
\[
(\lambda^2 + 1)R - 2r - 2\lambda\sqrt{R^2 - 2Rr} = [(\lambda^2 + 1)R - 2r] \left[ 1 - \sqrt{\frac{4\lambda^2(R^2 - 2Rr)}{(1 + \lambda^2)R - 2r]^2} \right]
\]
\[
= [(\lambda^2 + 1)R - 2r] \left[ 1 - \sqrt{1 - \frac{(1 - \lambda^2)R - 2r}{(1 + \lambda^2)R - 2r}^2} \right].
\]
Let
\[
\frac{(1 - \lambda^2)R - 2r}{(1 + \lambda^2)R - 2r} = x(0 < x \leq 1),
\]
then
\[
(\lambda^2 + 1)R - 2r - 2\lambda\sqrt{R^2 - 2Rr} = [(\lambda^2 + 1)R - 2r](1 - \sqrt{1 - x^2}).
\]
From the power series expansion (2.2), we have

\[ 1 - \sqrt{1 - x^2} = \frac{1}{2} x \sum_{n=1}^{\infty} \frac{(2n-3)!!}{2^{n-1} n!} x^{2n-1}, \]

and

\[ (3.3) \quad (\lambda^2 + 1)R - 2r - 2\lambda \sqrt{R^2 - 2Rr} = \frac{1}{2} \left| (1 - \lambda^2)R - 2r \right| \sum_{n=1}^{\infty} \frac{(2n-3)!!}{(1 + \lambda^2)^n} \left| \frac{(1 - \lambda^2)R - 2r}{(1 + \lambda^2)^n} \right|^{2n-1}. \]

Combining expression (3.2) and (3.3), the inequality (3.1) is proved.

If \( \lambda < 0 \), then \( -\lambda > 0 \), applying the above result, we have

\[ -(-\lambda - 1)^2 R^2 + 2(\lambda^2 - 5\lambda + 2)Rr - (4 - \lambda) r^2 + \frac{1}{2} (R - 2r) \left| (1 - \lambda^2)R - 2r \right| \varepsilon \leq \lambda s^2 \leq \]

\[ (-\lambda + 1)^2 R^2 - 2(\lambda^2 + 5\lambda + 2)Rr + (4 + \lambda) r^2 - \frac{1}{2} (R - 2r) \left| (1 - \lambda^2)R - 2r \right| \varepsilon. \]

It is easy to see that above inequalities are equivalent to inequalities (3.1). The proof of Theorem 3.1 is completed.

**Theorem 3.2.** Let \( \lambda, t \) be real number, and \( \lambda \neq 2t \), in every triangle we have the following inequalities

\[ (3.4) \quad -(t - 1)^2 R^3 + 2 [t^2 + (\lambda + 5)t - \lambda + 2] R^2 r - [(4\lambda + 1)t + \lambda^2 + 10\lambda + 4] Rr^2 + (2\lambda^2 + \lambda) r^3 \]

\[ + \frac{1}{2} (R - 2r) \left| (t R - \lambda^2) - R(R - 2r) \right| \varepsilon \leq (t R - \lambda r) s^2 \leq \]

\[ (t + 1)^2 R^3 - 2[t^2 + (\lambda - 5)t + \lambda + 2] R^2 r + [(4\lambda - 1)t + \lambda^2 - 10\lambda + 4] Rr^2 - (2\lambda^2 - \lambda) r^3 \]

\[ - \frac{1}{2} (R - 2r) \left| (t R - \lambda^2) - R(R - 2r) \right| \varepsilon, \]

where

\[ \varepsilon = \sum_{n=1}^{\infty} \frac{(2n-3)!!}{2^{n-1} n!} \left| \frac{(t R - \lambda^2) - R(R - 2r)}{(t R - \lambda^2)^2 + R(R - 2r)} \right|^{2n-1}. \]

**Proof.** From inequality (2.1), we have

\[ (3.5) \quad \left| t R - \lambda r \right| \left| s^2 - 2R^2 - 10Rr + r^2 \right| - 2(R - 2r) \sqrt{R^2 - 2Rr} \leq 0. \]

Since (3.5) \( \Leftrightarrow \) \( \left| (t R - \lambda^2)(s^2 - 2R^2 - 10Rr + r^2) \right| \leq 2 \left| t R - \lambda r \right| (R - 2r) \sqrt{R^2 - 2Rr} \)

\[ \Leftrightarrow (t R - \lambda^2)(2R^2 + 10Rr - r^2) - 2 \left| t R - \lambda r \right| (R - 2r) \sqrt{R^2 - 2Rr} \leq (t R - \lambda r)s^2 \leq \]

\[ (t R - \lambda^2)(2R^2 + 10Rr - r^2) + 2 \left| t R - \lambda r \right| (R - 2r) \sqrt{R^2 - 2Rr}, \]

that is

\[ (3.6) \quad -(t - 1)^2 R^3 + 2 [t^2 + (\lambda + 5)t - \lambda + 2] R^2 r - [(4\lambda + 1)t + \lambda^2 + 10\lambda + 4] Rr^2 + (2\lambda^2 + \lambda) r^3 \]

\[ + (R - 2r)[(t R - \lambda^2)^2 + R(R - 2r) - 2 \left| t R - \lambda r \right| \sqrt{R^2 - 2Rr}] \leq (t R - \lambda r)s^2 \leq \]

\[ (t + 1)^2 R^3 - 2[t^2 + (\lambda - 5)t + \lambda + 2] R^2 r + [(4\lambda - 1)t + \lambda^2 - 10\lambda + 4] Rr^2 - (2\lambda^2 - \lambda) r^3 \]

\[ -(R - 2r)[(t R - \lambda^2)^2 + R(R - 2r) - 2 \left| t R - \lambda r \right| \sqrt{R^2 - 2Rr}], \]

According to \( \lambda \neq 2t \) and Euler’s inequality \( R \geq 2r \), we obtain \( (t R - \lambda r)^2 + R(R - 2r) > 0 \), and

\[ (t R - \lambda^2)^2 + R(R - 2r) - 2 \left| t R - \lambda r \right| \sqrt{R^2 - 2Rr} = [(t R - \lambda^2)^2 + R(R - 2r)] \left[ 1 - \sqrt{\frac{4(t R - \lambda r)^2(R^2 - 2Rr)}{[(t R - \lambda^2)^2 + R(R - 2r)]^2}} \right] \]
\[(tR - λr)^2 + R(R - 2r) = (tR - λr)^2 - R(R - 2r) + R(R - 2r)\]

Let \[x = \left| \frac{(tR - λr)^2 - R(R - 2r)}{(tR - λr)^2 + R(R - 2r)} \right| = x(0 < x \leq 1),\]
then
\[(tR - λr)^2 + R(R - 2r) - 2|tR - λr|\sqrt{R^2 - 2Rr} = [(tR - λr)^2 + R(R - 2r)](1 - \sqrt{1 - x^2}),\]
From the power series expansion (2.2), we have
\[1 - \sqrt{1 - x^2} = \frac{1}{2}x \sum_{n=1}^{\infty} \frac{(2n - 3)!!}{2^n - 1} n! x^{2n - 1},\]
therefore
\[(tR - λr)^2 + R(R - 2r) - 2|tR - λr|\sqrt{R^2 - 2Rr} = \frac{1}{2}[(tR - λr)^2 - R(R - 2r)] \sum_{n=1}^{\infty} \frac{(2n - 3)!!}{2^n - 1} n! \frac{(tR - λr)^2 - R(R - 2r)}{(tR - λr)^2 + R(R - 2r)} 2^{n-1}.\]
Combining expression (3.6) and (3.7) we can get the inequalities (3.4). Theorem 3.2 is proved.

Now, we give some corollaries from Theorem 3.1 and Theorem 3.2.

**Corollary 3.1.** Let λ, t be real numbers, respectively, and λ \( \neq 2t \), in every triangle we have the following inequalities
\[(t - 1)^2R^2 + 2[t^2 + (λ + 5)t - λ + 2]R^2r - [(4λ + 1)t + λ^2 + 10λ + 4]Rr^2 + (2λ^2 + λ)r^3 \leq (tR - λr)s^2\]
\[≤ (t + 1)^2R^2 - 2[t^2 + (λ - 5)t + λ + 2]R^2r + [(4λ - 1)t + λ^2 - 10λ + 4]Rr^2 - (2λ^2 - λ)r^3.\]

**Corollary 3.2.** Let λ be a nonzero real numbers, in every triangle we have the following double inequality
\[-(λ - 1)^2R^2 + 2(λ^2 + 5λ + 2)Rr - (4 + λ)r^2 ≤ λs^2 ≤ (λ + 1)^2R^2 - 2(λ^2 - 5λ + 2)Rr + (4 - λ)r^2.\]
Inequalities (3.8) and (3.9) include Gerretsen’s inequalities and a lot of new geometric inequalities.

**Corollary 3.3.** Let λ, t be real numbers, respectively, and λ \( \neq 2t \), in every triangle we have the following double inequality
\[(t - 1)^2R^2 + 2[t^2 + (λ + 5)t - λ + 2]R^2r - [(4λ + 1)t + λ^2 + 10λ + 4]Rr^2 + (2λ^2 + λ)r^3\]
\[+ \frac{1}{2}(R - 2r) - R(R - 2r) |tR - λr|\sqrt{R^2 - 2Rr} \leq (tR - λr)s^2 \leq\]
\[(t + 1)^2R^2 - 2[t^2 + (λ - 5)t + λ + 2]R^2r + [(4λ - 1)t + λ^2 - 10λ + 4]Rr^2 - (2λ^2 - λ)r^3 - \frac{1}{2}(R - 2r) - R(R - 2r) |tR - λr|^2 - R(R - 2r) |tR - λr|^2 - R(R - 2r) |tR - λr|^2,\]
where
\[ς = \sum_{m=1}^{\infty} \frac{(2m - 3)!!}{2^m - 1} m! \left[ \frac{1}{2Rr} \left| (tR - λr)^2 - R(R - 2r) \right| \sum_{n=1}^{\infty} 2^{n-1} \left( \frac{2Rr}{(tR - λr)^2 + R(R + 2r)} \right)^n \right]^{2m-1}.\]
Proof. From Euler’s inequality $R \geq 2r$, we obtain

$$0 < \frac{4Rr}{(tR - \lambda r)^2 + R(R + 2r)} \leq \frac{4Rr}{(tR - \lambda r)^2 + 4Rr} < 1.$$ 

Using the power series expansion (2.4), we have

$$\left| \frac{(tR - \lambda r)^2 - R(R - 2r)}{(tR - \lambda r)^2 + R(R + 2r)} \right| = \left| \frac{(tR - \lambda r)^2 - R(R - 2r)}{(tR - \lambda r)^2 + R(R + 2r)} \right| \left| \frac{(tR - \lambda r)^2 + R(R + 2r)}{(tR - \lambda r)^2 + R(R - 2r)} \right| \leq \frac{1}{1 - \frac{4Rr}{(tR - \lambda r)^2 + R(R + 2r)}} \left[ \frac{4Rr}{(tR - \lambda r)^2 + R(R + 2r)} \right]^n \leq \frac{1}{2Rr} \left| (tR - \lambda r)^2 - R(R - 2r) \right| \sum_{n=1}^{\infty} 2^{n-1} \left[ \frac{2Rr}{(tR - \lambda r)^2 + R(R + 2r)} \right]^n.$$ 

Combining expression (3.4) and (3.11) we immediately get the inequalities (3.10), and the proof of Corollary 3.3 is completed.

References


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