ON A RESULT OF CARTLIDGE

PENG GAO

Abstract. We prove a result of J.M.Cartlidge on the l^p operator norms of weighted mean matrices by using the method of Redheffer's "recurrent inequalities".

Suppose throughout that $p \neq 0, \frac{1}{p}+\frac{1}{q}=1$. Let l^p be the Banach space of all complex sequences $\mathbf{a}=(a_n)_{n\geq 1}$ with norm

$$||\mathbf{a}|| := (\sum_{n=1}^{\infty}|a_n|^p)^{1/p} < \infty.$$

The celebrated Hardy's inequality([5], Theorem 326) asserts that for $p>1$,

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n}a_k\Big|^p \leq (\frac{p}{p-1})^p\sum_{n=1}^{\infty}|a_n|^p. \tag{1}$$

From now on we will assume $a_n \geq 0$ for $n \geq 1$ and any infinite sum converges. Among the many papers appeared providing new proofs, generalizations and sharpenings of (1), we refer the reader to R.Redheffer's remarkable proof by his method of "recurrent inequalities"[6]. We also note the following result of E.B.Elliot[4](see also T.A.A. Broadbent[2]).

$$\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}a_k)^p \leq (\frac{p}{p-1})\sum_{n=1}^{\infty}a_nA_n^{p-1}, \tag{2}$$

where $A_n=\sum_{k=1}^{n}a_k/n$ and $p>1$.

Inequality (2) implies (1) since by Hölder's inequality, one has

$$\sum_{n=1}^{\infty}a_nA_n^{p-1} \leq (\sum_{n=1}^{\infty}a_n^p)^{1/p}(\sum_{n=1}^{\infty}A_n^p)^{1-1/p}. \tag{3}$$

Hardy's inequality can be regarded as a special case of the following inequality:

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{\infty}c_{n,k}a_k\Big|^p \leq U\sum_{n=1}^{\infty}|a_n|^p,$$

in which $C=(c_{n,k})$ and the parameter p are assumed fixed($p>1$), and the estimate is to hold for all real sequences $\mathbf{a}$. The l^p operator norm of C is then defined as the p-th root of the smallest value of the constant U:

$$||C||_{p,p}=U^{\frac{1}{p}}.$$

Hardy's inequality thus asserts that the Cesáro matrix operator C, given by $c_{n,k}=1/n, k\leq n$ and 0 otherwise, is bounded on l^p and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.)

Date: October 21, 2003.
1991 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. Hardy's inequality, recurrent inequality.

We say a matrix A is a summability matrix if its entries satisfy: $a_{n,k} \geq 0$, $a_{n,k} = 0$ for $k > n$ and $\sum_{k=1}^{n} a_{n,k} = 1$. We say a summability matrix A is a weighted mean matrix if its entries satisfy:

$$a_{n,k} = \lambda_k/\Lambda_n,\ 1 \leq k \leq n; \Lambda_n = \sum_{k=1}^{n} \lambda_k. \tag{4}$$

In an unpublished dissertation[3], J. M. Cartlidge studied weighted mean matrices as operators on l^p spaces and obtained the following result(see also [1], p. 416, Theorem C).

Theorem 1. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (4). *If*

$$L = \sup_n \left(\frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_n}{\lambda_n}\right) < p\ , \tag{5}$$

then $||A||_{p,p} \leq p/(p-L)$.

Motivated by the work of Redheffer, Elliott and Broadbent, we now present a proof of the above result of Cartlidge by obtaining a result similar to (2) using Redheffer's method.

Theorem 2. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (4). *If* (5) *is satisfied, then*

$$\sum_{n=1}^{\infty} A_n^p \leq \left(\frac{p}{p-L}\right) \sum_{n=1}^{\infty} a_n A_n^{p-1}, \tag{6}$$

where $A_n = \sum_{k=1}^{n} \lambda_k a_k/\Lambda_n$. In particular, $||A||_{p,p} \leq p/(p-L)$.

Proof. It suffices to prove the theorem for any finite summation from $n = 1$ to N with $N \geq 1$. If (5) is satisfied then $\lambda_n > 0$ for any n. For $n \geq 2, a_n \geq 0$, let

$$f(a_n) = (\Lambda_n/\lambda_n + p - 1)A_n^p - pa_n A_n^{p-1}.$$

Then

$$f'(a_n) = p(p-1)\lambda_n/\Lambda_n(A_n^{p-1} - a_n A_n^{p-2}).$$

Hence $f'(a_n) = 0$ implies $a_n = A_{n-1}$. It's easy to check $\lim_{a_n \to \infty} f(a_n) \leq 0$ and note also $f'(0) \geq 0$. It follows then

$$(\Lambda_n/\lambda_n + p - 1)A_n^p - pa_n A_n^{p-1} = f(a_n) \leq f(A_{n-1}) = (\Lambda_n/\lambda_n - 1)A_{n-1}^p. \tag{7}$$

By defining $A_0 = 0$ the above inequality also holds for $n = 1$.

Summing (7) from $n = 1$ to N gives

$$(\Lambda_{N+1}/\lambda_{N+1} - 1)A_N^p + \sum_{n=1}^{N} (\Lambda_n/\lambda_n - \Lambda_{n+1}/\lambda_{n+1} + p)A_n^p \leq p \sum_{n=1}^{N} a_n A_n^{p-1}. \tag{8}$$

By condition (5), $\Lambda_n/\lambda_n - \Lambda_{n+1}/\lambda_{n+1} + p \geq p - L$. Hence inequality (6) follows from (8) and this completes the proof. □

References

[1] G.Bennett, Some elementary inequalities, *Quart. J. Math. Oxford Ser. (2)* **38** (1987), 401–425.
[2] T.A.A.Broadbent, A proof of Hardy's convergence theorem, *J. London Math. Soc.* **3** (1928), 242-243.
[3] J.M.Cartlidge, Weighted mean matrices as operators on l^p, Ph.D. thesis, Indiana University.
[4] E.B. Elliott, A simple exposition of some recently proved facts as to convergency, *J. London Math. Soc.* **1** (1926), 93–96.
[5] G.H.Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1952.
[6] R.M.Redheffer, Recurrent inequalities, *Proc. London Math. Soc. (3)* **17** (1967), 683–699.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: penggao@umich.edu