# SOME STRENGTHENED RESULTS ON EULER'S INEQUALITY

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Abstract. In this short note, we give some strengthened results on Euler's inequality.

## 1. Introduction

One of the oldest inequalities about triangles is that relating the radii of the circumcircle and incircle. It was proved by Euler and is contained in the following theorems.

**Theorem 1.1.** *(Euler 1765) Let $O$ and $I$ be the circumcenter and incenter, respectively, of a triangle with circumradius $R$ and inradius $r$; let $d$ be the distance $OI$. Then*

$$d^2 = R^2 - 2Rr. \tag{1.1}$$

**Theorem 1.2.** *In a triangle with circumradius $R$ and inradius $r$, the inequality holds*

$$R \geq 2r \tag{1.2}$$

*with equality holding if and only if the triangle $ABC$ is the equilateral triangle.*

Inequality (1.2) is called Euler's inequality [1].

Let $s$ denote the semiperimeter of triangle ABC, $A, B, C$ the angles, $a, b, c$ the opposite sides, $R$ the circumradius and $r$ the inradius. Similarly define triangle $A'B'C'$. In this short note, we give some strengthened results on Euler's inequality.

## 2. Main Results

In order to prove Theorem 2.1 below, we require the following lemma (see [2]):

**Lemma 2.1.** *(Oppenheim) Let $x, y, z \geq 0$, and $A, B, C$ denote the angles of triangle $ABC$, then*

$$(x+y+z)^2 \geq 2\sqrt{3}(yx\sin A + zx\sin B + xy\sin C) \tag{2.1}$$

*with equality holding if and only if $x = y = z$, and the triangle $ABC$ is the equilateral triangle.*

**Theorem 2.1.** *We have*

$$\frac{R}{r'} \geq \frac{2}{3}\left(\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\right) \tag{2.2}$$

*with equality holding if and only if both triangles are equilateral.*

Let $a = a', b = b', c = c'$, then the inequality (1.2) becomes (2.2), therefore the inequality (2.2) is called the generalized Euler's inequality.

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*Proof.* Set $x = \sin A', y = \sin B', z = \sin C'$ in (2.1), from the known result

$$\sin A' + \sin B' + \sin C' \leq \frac{3\sqrt{3}}{2} \tag{2.3}$$

and law of sines, and $8R'r's' = a'b'c'$, we obtain inequality (2.2), with equality holding if and only if $x = y = z$, and the triangle $ABC$ is an equilateral one. This completes the proof. □

## 3. Applications

**Corollary 3.1.** *In every triangle, the following inequality holds*

$$\frac{R}{r} \geq \frac{\sqrt{3}}{3}\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right) \tag{3.1}$$

*with equality holding if and only if the triangle is equilateral.*

*Proof.* Let $a = b = c$, and alter $a' \to a, b' \to b, c' \to c$ in Theorem 2.1, we obtain (3.1). □

**Corollary 3.2.** *In a triangle with circumradius $R$, inradius $r$, we have*

$$\frac{3R}{2r} - 1 \geq \frac{b}{c} + \frac{c}{b} \tag{3.2}$$

*with equality holding if and only if the triangles are the equilateral.*

*Proof.* Let $a = a', b = c', c = b'$ in Theorem 2.1, This completes the proof. □

**Corollary 3.3.** *In a triangle with circumradius $R$, inradius $r$, and the sides $a, b, c$, the following inequality holds*

$$\frac{R}{r} \geq \frac{2}{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \tag{3.3}$$

*with equality holding if and only if the triangles are equilateral.*

*Proof.* Let $a = c', b = a', c = b'$ in Theorem 2.1, we obtain (3.3). □

**Corollary 3.4.** *In every triangle, we have*

$$\frac{R}{r} \geq \frac{2}{9}(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \tag{3.4}$$

*with equality holding if and only if the triangle are the equilateral triangle.*

*Proof.* From Corollary 3.3, and Klamkin inequality

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{1}{3}(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

we obtain (3.4). This completes the proof. □

**Corollary 3.5.** *In every triangle, we have*

$$\frac{R}{r} \geq \frac{1}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) \tag{3.5}$$

*with equality holding if and only if the triangles are equilateral.*

The proof of Corollary 3.5 will be left to the readers.

**Corollary 3.6.** *In every triangle, we have*

$$\frac{R}{r} \geq \frac{8}{9}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2 \tag{3.6}$$

*with equality holding if and only if the triangles are equilateral.*

*Proof.* If $a' = b + c, b' = c + a, c' = a + b$, then $s'r' = 2s\sqrt{2Rr}$ (see [3]), from Theorem 2.1, we get inequality (3.6). □

**Corollary 3.7.** *In a triangle with circumradius $R$, inradius $r$, we have*

$$\frac{R}{r} \geq \frac{\sqrt{2}}{3}\left(\sqrt{\frac{a}{s-a}} + \sqrt{\frac{b}{s-b}} + \sqrt{\frac{c}{s-c}}\right) \tag{3.7}$$

*with equality holding if and only if the triangles are both equilateral.*

*Proof.* If $a' = \sqrt{a(s-a)}, b' = \sqrt{b(s-b)}, c' = \sqrt{c(s-c)}$, then $2s'r' = sr$ (see [3]), from Theorem 2.1, we have inequality (3.7). This completes the proof. □

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