THE GENERALIZATION ON A SPECIAL CASE OF OPPENHIM THEOREM

ZHI-HUA ZHANG AND SHAN-HE WU

Abstract. In this paper, we give the following generalization on a special case of Oppenhim theorem: Let $1 \leq m$, and $a_2 = (b_1^m + c_1^m)^{\frac{1}{m}}$, $b_2 = (c_1^m + a_1^m)^{\frac{1}{m}}$, $c_2 = (a_1^m + b_1^m)^{\frac{1}{m}}$, then the inequalities hold $p_2 \geq 2^\frac{1}{m}p_1$, and $\Delta_2 \geq 2^\frac{2}{m}\Delta_1$.

1. Introduction and Notation

Throughout the paper we assume $p_1$ denote the semi-perimeter of triangle $A_1B_1C_1$, $a_1, b_1, c_1$ the opposite sides, $R_1$ the circumradius, $r_1$ the inradius and $\Delta_1$ the area. Similarly, one defines triangle $A_2B_2C_2$ and triangle $A_3B_3C_3$.

In 1963, A. Oppenheim [1] obtained the following theorem:

Theorem 1.1. Let $1 \leq m \leq 4$, and

\begin{equation}
  a_3 = (a_1^m + a_2^m)^{\frac{1}{m}}, b_3 = (b_1^m + b_2^m)^{\frac{1}{m}}, c_3 = (c_1^m + c_2^m)^{\frac{1}{m}},
\end{equation}

then the following inequalities hold

\begin{equation}
  p_3 \geq 2^\frac{1}{m}-1(p_1 + p_2),
\end{equation}

and

\begin{equation}
  \Delta_3 \geq 2^\frac{2}{m}-1(\Delta_1 + \Delta_2).
\end{equation}

When $m > 4$, a negation of the inequalities (1.2) and (1.3) was obtained in [2].

In this paper, we give a generalization for a special case of Theorem 1.1.

2. Main Result and Lemma

Theorem 2.1. If $m$ be a real number for $1 \leq m$, and

\begin{equation}
  a_2 = (b_1^m + c_1^m)^{\frac{1}{m}}, b_2 = (c_1^m + a_1^m)^{\frac{1}{m}}, c_2 = (a_1^m + b_1^m)^{\frac{1}{m}}
\end{equation}

then the following inequalities hold

\begin{equation}
  p_2 \geq 2^\frac{1}{m}p_1
\end{equation}

and

\begin{equation}
  \Delta_2 \geq 2^\frac{2}{m}\Delta_1.
\end{equation}

To prove Theorem 2.1, we will use the following lemma:

Lemma 2.1. If $0 < u \leq 1, v > 0$, and $\lambda < 1$, then

\begin{equation}
  (v + 1)^\lambda - (v + u)^\lambda \leq (4v)^\frac{\lambda+1}{2}(1 - u^\frac{\lambda+1}{2})
\end{equation}

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Proof. Set a function
\[ f(u) = (u + v)^\lambda + (4v)^{\frac{\lambda - 1}{2}}(1 - u)^{\frac{\lambda + 1}{2}} - (1 + v)^\lambda, \quad (0 < u \leq 1, v > 0, \lambda < 1) \]
We have
\[ \frac{\partial f(u)}{\partial u} = \lambda (u + v)^{\lambda - 1} - \frac{\lambda + 1}{2}(4uv)^{\frac{\lambda - 1}{2}} \]
By using the arithmetic-geometric mean inequality, we obtain
\[ \frac{\partial f(u)}{\partial u} \leq \lambda (2\sqrt{uv})^{\lambda - 1} - \frac{\lambda + 1}{2}(4uv)^{\frac{\lambda - 1}{2}} = \left(\frac{\lambda - 1}{2}\right)(4uv)^{\frac{\lambda - 1}{2}} < 0, \]
since
\[ \lambda - 1 < 0, u > 0, v > 0. \]
Therefore, the function \( f \) with fixation \( v (v > 0) \) be monotonically decreasing in \( u \in (0, 1] \), that is
\[ f(u) \geq f(1) = 0 \]
i.e., the inequality (2.4) is true. This completes the proof. \( \blacksquare \)

3. THE PROOF OF THEOREM 2.1

Proof. From \( a_2, b_2, c_2 > 0, m - 1 \geq 0 \), we have
\[ (a_2 + b_2)^m = (a_2 + b_2)(a_2 + b_2)^{m-1} = a_2(a_2 + b_2)^{m-1} + b_2(a_2 + b_2)^{m-1} \]
\[ a_2(a_2)^{m-1} + b_2(b_2)^{m-1} = (a_2)^m + (b_2)^m \]
\[ = b_1^m + c_1^m + a_1^m > a_1^m + c_1^m = b_2^m \]
therefore \( a_2 + b_2 > c_2 \). Similarly, we obtain \( b_2 + c_2 > a_2 \) and \( c_2 + a_2 > b_2 \). That is to say, \( a_2, b_2, c_2 \) are three sides of the triangle \( \triangle A_2B_2C_2 \).
By using the power mean inequality, we have
\[ p_2 = \frac{1}{2}[(a_1^m + b_1^m)^{\frac{1}{m}} + (b_1^m + c_1^m)^{\frac{1}{m}} + (c_1^m + a_1^m)^{\frac{1}{m}}] \]
\[ \geq 2^{\frac{1}{m}-2}(a_1 + b_1) + 2^{\frac{1}{m}-2}(b_1 + c_1) + 2^{\frac{1}{m}-2}(c_1 + a_1) = 2^{\frac{1}{m}}p_1 \]
where \( m \geq 1 \). This completes the proof of inequality (2.2).

In order to prove inequality (2.3). We first prove the following inequality:
\[ (3.1) \quad (a_1^m + b_1^m)^{\frac{2}{m}} + (b_1^m + c_1^m)^{\frac{2}{m}} - (c_1^m + a_1^m)^{\frac{2}{m}} \leq 2^{\frac{2-m}{m}}(b_1^2 + a_1b_1 + b_1c_1 - c_1a_1) \]
Let \( \alpha_{b_1} = x^{\frac{1}{m}} \quad (x \geq 1) \), and \( \alpha_{b_1} = t^{\frac{1}{m}} \quad (0 < t \leq 1) \), then
\[ b_1^{-2}\left[(a_1^m + b_1^m)^{\frac{2}{m}} + (b_1^m + c_1^m)^{\frac{2}{m}} - (c_1^m + a_1^m)^{\frac{2}{m}} - 2^{\frac{2-m}{m}}(b_1^2 + a_1b_1 + b_1c_1 - c_1a_1)\right] \]
\[ = (x + 1)^{\frac{2}{m}} + (t + 1)^{\frac{2}{m}} - (x + t)^{\frac{2}{m}} - 2^{\frac{2-m}{m}}[x^{\frac{1}{m}}(1-t^{\frac{1}{m}}) + t^{\frac{1}{m}}] \]
Set
\[ g(x) = (x + 1)^{\frac{2}{m}} + (t + 1)^{\frac{2}{m}} - (x + t)^{\frac{2}{m}} - 2^{\frac{2-m}{m}}[x^{\frac{1}{m}}(1-t^{\frac{1}{m}}) + t^{\frac{1}{m}} + 1] \]
we have
\[ \frac{\partial g(x)}{\partial x} = \frac{2}{m}[x^{\frac{2-m}{m}} - (x + t)^{\frac{2-m}{m}} - 4x^{\frac{1-m}{m}}(1-t^{\frac{1}{m}})] \]
From Lemma 2.1 and \( 0 < t \leq 1, x \geq 1, \quad \frac{2-m}{m} < 1 \), we obtain \( \frac{\partial g(x)}{\partial x} < 0 \). Therefore, the function \( g \) with fixation \( t \) \((0 < t \leq 1)\) be monotonically decreasing in \( x \in [1, +\infty) \), that is \( g(x) \leq g(1) = 0 \).
i.e., the inequality (3.1) is true.
 Secondly, to proved inequality (2.3).
By using the power mean inequality, we have
\[ (a_1^m + b_1^m)^{\frac{2}{m}}(b_1^m + c_1^m)^{\frac{2}{m}} \geq 2^{\frac{4-m}{m}}(a_1 + b_1)^2(b_1 + c_1)^2, \quad (m \geq 1) \]
From Heron’s formula \(3\) and (3.1), we follow

\[
\Delta_2^2 = \frac{1}{4} [c_2^2 a_2^2 - \frac{1}{4} (c_2^2 + a_2^2 - b_2^2)^2]
\]
\[
= \frac{1}{4} (a_1^m + b_1^m) \frac{2}{m} (b_1^m + c_1^m) \frac{2}{m} - \frac{1}{16} \left[ (a_1^m + b_1^m)^\frac{2}{m} + (b_1^m + c_1^m)^\frac{2}{m} - (c_1^m + a_1^m)^\frac{2}{m} \right]^2
\]
\[
\geq \frac{1}{4} \left[ 2^{\frac{4-m}{m}} (a_1 + b_1)^2 (b_1 + c_1)^2 \right] - \frac{1}{16} \left[ 2^{\frac{2-m}{m}} (a_1 b_1 + b_1 c_1 - c_1 a_1) \right]^2
\]
\[
= 2^{\frac{4-m}{m}} [(a_1 + b_1)^2 (b_1 + c_1)^2 - (b_1^2 + a_1 b_1 + b_1 c_1 - c_1 a_1)^2]
\]
\[
= 2^{\frac{4-m}{m}} a_1 b_1 c_1 (a_1 + b_1 + c_1)
\]

According to the expansion \(a_1 b_1 c_1 = 4R_1 \Delta_1\), \(a_1 + b_1 + c_1 = \frac{2 \Delta_1}{r_1}\) for triangle, and Euler’s inequality \(R_1 \geq 2r_1\), we have

\[
\Delta_2^2 \geq 2^{\frac{4-m}{m}} a_1 b_1 c_1 (a_1 + b_1 + c_1) = 2^{\frac{4-m}{m}} \Delta_1^2 \left( \frac{R_1}{2r_1} \right) \geq (2^\frac{m}{2} \Delta_1)^2.
\]

Rearranging we obtain inequality (2.3). This completes the proof.

4. The Analogue of Oppenheim’s Inequality and an Open Problem

**Theorem 4.1.** Let \(a_2 = (b_1^2 + c_1^2)^\frac{1}{2}\), \(b_2 = (c_1^2 + a_1^2)^\frac{1}{2}\), \(c_2 = (a_1^2 + b_1^2)^\frac{1}{2}\), then the following inequality holds

\[
r_2 \geq \sqrt{2} r_1
\]

**Proof.** From Heron’s formula, we have

\[
\Delta_2^2 = \frac{1}{4} [c_2^2 a_2^2 - \frac{1}{4} (c_2^2 + a_2^2 - b_2^2)^2]
\]
\[
= \frac{1}{4} (a_1^m + b_1^m) \frac{2}{m} (b_1^m + c_1^m) \frac{2}{m} - \frac{1}{16} \left[ (a_1^m + b_1^m)^\frac{2}{m} + (b_1^m + c_1^m)^\frac{2}{m} - (c_1^m + a_1^m)^\frac{2}{m} \right]^2
\]
\[
= \frac{1}{4} (a_1^m b_1^2 + b_1^m c_1^2 + c_1^m a_1^2),
\]

or

\[
\Delta_2 = \frac{1}{2} \sqrt{a_1^m b_1^2 + b_1^m c_1^2 + c_1^m a_1^2},
\]

and

\[
r_2 = \frac{\Delta_2}{p_2} = \frac{\sqrt{a_1^m b_1^2 + b_1^m c_1^2 + c_1^m a_1^2}}{\sqrt{a_1^2 + b_1^2} + \sqrt{b_1^2 + c_1^2} + \sqrt{c_1^2 + a_1^2}},
\]

therefore inequality (4.1) is equivalent to

\[
\frac{a_1^2 b_1^2 + b_1^2 c_1^2 + c_1^2 a_1^2}{(\sqrt{a_1^2 + b_1^2} + \sqrt{b_1^2 + c_1^2} + \sqrt{c_1^2 + a_1^2})^2} \geq 2r_1^2.
\]

Utilizing the fact that

\[
(\sqrt{a_1^2 + b_1^2} + \sqrt{b_1^2 + c_1^2} + \sqrt{c_1^2 + a_1^2})^2 \leq 6(a_1^2 + b_1^2 + c_1^2),
\]
\[
a_1^2 + b_1^2 + c_1^2 = 2(p_1^2 - 4R_1 r_1 - r_1^2),
\]

and

\[
a_1^2 b_1^2 + b_1^2 c_1^2 + c_1^2 a_1^2 = (p_1^2 - 4R_1 r_1 - r_1^2)^2 + 4p_1^2 r_1^2,
\]
therefore
\[
\frac{a_1^2 b_1^2 + b_1^2 c_1^2 + c_1^2 a_1^2}{\left(\sqrt{a_1^2 + b_1^2} + \sqrt{b_1^2 + c_1^2} + \sqrt{c_1^2 + a_1^2}\right)^2} \\
\geq \frac{a_1^2 b_1^2 + b_1^2 c_1^2 + c_1^2 a_1^2}{6(a_1^2 + b_1^2 + c_1^2)} \\
= \frac{1}{12}(p_1^2 - 4R_1r_1 - r_1^2 + \frac{4p_1^2 r_1^2}{p_1^2 - 4R_1r_1 - r_1^2}).
\]

By using the arithmetic-geometric mean inequality and well known inequality \( p_1 \geq 3\sqrt{3}r_1 \), we obtain
\( \frac{1}{3}(p_1^2 - 4R_1r_1 - r_1^2) + \frac{4p_1^2 r_1^2}{p_1^2 - 4R_1r_1 - r_1^2} \geq \frac{4}{\sqrt{3}}p_1r_1 \geq 12r_1^2. \)

From Gerretsen’s inequality (see [4]) \( p_1^2 \geq 16R_1r_1 - 5r_1^2 \) and Euler’s inequality \( R_1 \geq 2r_1 \), we get
\( \frac{2}{3}(p_1^2 - 4R_1r_1 - r_1^2) \geq \frac{2}{3}(12R_1r_1 - 6r_1^2) \geq 12r_1^2. \)

Combining inequalities (4.3) and (4.4), we have
\( \frac{1}{12}(p_1^2 - 4R_1r_1 - r_1^2 + \frac{4p_1^2 r_1^2}{p_1^2 - 4R_1r_1 - r_1^2}) \geq 2r_1^2, \)
also inequality (4.2) holds. The proof of Theorem 4.1 is completed.

Finally, we propose an open problem:
Prove that: If \( m \) be a real number for \( m \geq 1 \), and 
\( a_2 = (b_1^m + c_1^m)^{1/m} \), \( b_2 = (c_1^m + a_1^m)^{1/m} \), \( c_2 = (a_1^m + b_1^m)^{1/m} \), then the following inequality holds
\( r_2 \geq 2^{1/m}r_1. \)

\[\text{References}\]


(Zh.-H. Zhang) Zixing Educational Research Section, Chenzhou, Hunan 423400, China.
E-mail address: xzgzzh@163.com

(Sh.-H. Wu) Department of Mathematics, Longyan College, Longyan Fujian 364012, P.R. China
E-mail address: wushanhe@yahoo.com.cn