ON WEITZENBOECK’S INEQUALITY AND ITS GENERALIZATIONS

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Abstract. In this paper, a new proof of the equivalence for Weitzenboeck’s inequality and Finsler-Hadwiger’s inequality is given, and some generalizations of Weitzenboeck’s inequality for only one triangle is proved.

1. Introduction

Throughout the paper we assume \(a, b, c\) denote the opposite sides of triangle \(ABC\), \(A, B, C\) the angles, \(s\) the semi-perimeter, \(\Delta\) the area and \(R\) the circumradius. Moreover, we will customarily use the cyclic sum symbol and cyclic product symbol, that is: \(\sum f(a) = f(a) + f(b) + f(c)\), \(\sum f(a, b) = f(a, b) + f(b, c) + f(c, a)\) and \(\prod f(a) = f(a)f(b)f(c)\), similarly, one defines others.

In 1919, Weitzenboeck \([1]\) obtained the following interesting inequality for sides and area of the triangle

\[
\sum a^2 \geq 4\sqrt{3}\Delta
\]

with equality holding if and only if the triangle \(ABC\) is the equilateral one.

Inequality (1.1) is called Weitzenboeck’s inequality. In the theory of geometric inequality, Weitzenboeck’s inequality and its generalizations often play fundamental role, these results are interesting and useful. For example:

In 1937, P.Finsler and H.Hadwiger \([2]\) first studied the generalization of Weitzenboeck’s inequality, their got:

\[
\sum a^2 \geq 4\sqrt{3}\Delta + \sum (a-b)^2
\]

with equality holding if only if the triangle \(ABC\) is equilateral.

A weighted representation for Weitzenboeck’s inequality is given in \([3]\), A.George verified that

\[
\sum \left( \frac{\alpha_1}{\alpha_2 + \alpha_3} a^2 \right) \geq 2\sqrt{3}\Delta
\]

where \(\alpha_1, \alpha_2, \alpha_3 > 0\).

A.Oppenheim \([4]\) published the following useful weighted formula:

\[
(\sum \lambda_1 a^2)^2 \geq 16\Delta^2(\sum \lambda_1 \lambda_2)
\]

with equality holding if and only if \(\lambda_1 : \lambda_2 : \lambda_3 = (b^2 + c^2 - a^2) : (c^2 + a^2 - b^2) : (a^2 + b^2 - c^2)\), where \(\lambda_1, \lambda_2, \lambda_3\) are the real numbers.

For two triangles, the generalization of Weitzenboeck’s inequality \([1]\) is the well-known Neuberg-Pedoe inequality:

\[
\sum a_i^2 (b_i^2 + c_i^2 - a_i^2) \geq 16\Delta_1 \Delta_2
\]

with equality holding if and only if \(\Delta A_1 B_1 C_1 \sim \Delta A_2 B_2 C_2\).

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In this paper, a new proof of the equivalence for Weitzenboeck’s inequality (1.1) and Finsler-Hadwiger’s inequality (1.2) is given, and some generalizations of Weitzenboeck’s inequality for only one triangle is proved.

2. A NEW PROOF OF THE EQUIVALENCE FOR WEITZENBOECK’S INEQUALITY (1.1) AND FINSLER-HADWIGER’S INEQUALITY (1.2)

To prove the equivalence for Weitzenboeck’s inequality (1.1) and Finsler-Hadwiger’s inequality (1.2), the following Lemma 2.1 will be used.

Lemma 2.1. In every triangle $ABC$, if

\begin{equation}
(2.1) \quad a_1 = \sqrt{a(b + c - a)}, \quad b_1 = \sqrt{b(c + a - b)}, \quad c_1 = \sqrt{c(a + b - c)}
\end{equation}

and

\begin{equation}
(2.2) \quad A_1 = (\pi - A)/2, \quad B_1 = (\pi - B)/2, \quad C_1 = (\pi - C)/2,
\end{equation}

then the triangle $A'B'C'$ for sides $a_1, b_1, c_1$ is a triangle for angles $A_1, B_1, C_1$, and the area $\Delta_1 = \Delta$.

Proof. Because $a + b - c > 0, b + c > a, c + a - b > 0$, and from the expansion (2.1), we have

\begin{align*}
(a_1 + b_1)^2 &= a(b + c - a) + b(c + a - b) + 2\sqrt{a(b + c - a)b(c + a - b)} \\
&> a(b + c - a) + b(c + a - b) - (a - b)^2 \\
&> c(a + b - c)^2 = c(a + b - c) = c_1^2
\end{align*}

that is

\[ a_1 + b_1 > c_1 \]

similarly, we can obtain $b_1 + c_1 > a_1, c_1 + a_1 > b_1$. Therefore, $a_1, b_1, c_1$ are three sides of a triangle.

According to the expansion (2.2) and $A_1 + B_1 + C_1 = (\pi - A)/2 + (\pi - B)/2 + (\pi - C)/2 = \pi$, we get that $A_1, B_1, C_1$ are three angles of a triangle.

Also, assume a triangle for the sides $a_1, b_1, c_1$ is $\Delta A'B'C'$, then utilizing the fact that

\[ \cos A' = \frac{b_1^2 + c_1^2 - a_1^2}{2b_1c_1} = \frac{b(c + a - b) + c(a + b - c) - a(b + c - a)}{2\sqrt{b(c + a - b)c(a + b - c)}} = \frac{\sin A}{2} = \frac{\cos \frac{\pi - A}{2}}{\cos A_1}, \]

we immediately get $A' = A_1$. Similarly, we can obtain $B' = B_1, C' = C_1$. That is the triangle $A'B'C'$ for sides $a_1, b_1, c_1$ is a triangle for angles $A_1, B_1, C_1$.

Finally, we have

\[ 16\Delta_1^2 = 2 \sum a_1^2b_1^2 - \sum a_1^4 = 2 \sum a^2b^2 - \sum a^4 = 16\Delta^2, \]

i.e., $\Delta_1 = \Delta$. The proof of Lemma 2.1 is completed.

Theorem 2.1. Weitzenboeck’s inequality (1.1) and Finsler-Hadwiger’s inequality (1.2) are equivalence.

Proof. Firstly, Finsler-Hadwiger’s inequality (1.2) $\Rightarrow$ Weitzenboeck’s inequality (1.1) is obvious.

Secondly, to prove Weitzenboeck’s inequality (1.1) $\Rightarrow$ Finsler-Hadwiger’s inequality (1.2).

From Weitzenboeck’s inequality (1.1) and Lemma 2.1, we obtain

\[ \sum a_1^2 \geq 4\sqrt{3}\Delta_1, \]

and

\[ \sum a(b + c - a) \geq 4\sqrt{3}\Delta \]

that is Finsler-Hadwiger’s inequality (1.2). Theorem 2.1 is proved.
Remark 2.1. By the same way, we can prove that the Neuberg-Pedoe inequality [1,5] and the following Zh.-P An's inequality [2,3] (see also [14]) are equivalence:

\[
(2.3) \quad \sum a_1(b_1 + c_1 - a_1)(c_2 + a_2 - b_2)(a_2 + b_2 - c_2) \geq 16\Delta_1\Delta_2
\]

with equality holding if and only if \(\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2\).

By using Lemma 2.1 and Finsler-Hadwiger’s inequality [1,2], we easily prove the following corollary:

Corollary 2.1. In every triangle \(ABC\), we have

\[
(2.4) \quad \sum a^2 \geq 4\sqrt{3}\Delta + \sum (a - b)^2 + \sum (\sqrt{a(b + c - a)} - \sqrt{b(c + a - b)})^2
\]

with equality holding if only if the triangle \(ABC\) is the equilateral triangle.

Theorem 2.2. Assume \(\mu\) is a real number, then in every triangle \(ABC\), the inequality

\[
(2.5) \quad \sum a^2 \geq 4\sqrt{3}\Delta + \mu \sum (a - b)^2
\]

holds that the best possible is the coefficient \(\mu = 1\).

Proof. In (2.5), let \(a = b = 1, c = t\), then we have

\[
2 + t^2 \geq \sqrt{3}t\sqrt{4 - t^2} + 2\mu(1 - t^2)
\]

Set \(t \to 0\), we obtain \(\mu \leq 1\), therefore the coefficient \(\mu = 1\) is the best possible.

3. Some Generalized Results for Triangle

In this section, we will list another generalizations of Weitzenboeck’s inequality for triangle.

Theorem 3.1. If one of \(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3\) and \(\lambda_3 + \lambda_1\) greater then zero, and \(\sum \lambda_1\lambda_2 > 0\), in every triangle \(ABC\) we have

\[
(3.1) \quad \sum \lambda_1a^2 \geq 4\sqrt{\sum \lambda_1\lambda_2} \Delta + (\sqrt{\lambda_3 + \lambda_1} a - \sqrt{\lambda_2 + \lambda_3} b)^2
\]

with equality holding if and only if \(\angle C = \arccos(\lambda_3/\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)})\).

Proof. From one of \(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3\) and \(\lambda_3 + \lambda_1\) greater then zero, and \(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 > 0\), we have another two of \(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3\) and \(\lambda_3 + \lambda_1\) greater then zero, and

\[
\left| \frac{\lambda_3}{\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}} \right| < 1.
\]

Let

\[
\theta = \arccos \frac{\lambda_3}{\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}},
\]

then

\[
\sin \theta = \frac{\sqrt{\sum \lambda_1\lambda_2}}{\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}}, \quad \cos \theta = \frac{\lambda_3}{\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}}.
\]

From \(1 \geq \cos(C - \theta)\), and \(2ab \cos C = a^2 + b^2 - c^2\), \(2ab \sin C = 4\Delta\), we obtain

\[
1 \geq \frac{\lambda_3}{\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}} \cos C + \frac{\sqrt{\sum \lambda_1\lambda_2}}{\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}} \sin C
\]

or

\[
2\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}ab \geq \lambda_3(a^2 + b^2 - c^2) + 4\sqrt{\sum \lambda_1\lambda_2\Delta}
\]
that is inequality (3.1), with the equality holding if and only if $1 = \cos(C - \theta)$, i.e.

$$\angle C = \arccos \frac{\lambda_3}{\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}}.$$

The proof is completed. 

Theorem 3.1 is obtained by K.-Ch. Yang in [5]. We can give some results for Theorem 3.1, its not only among inequality (1.4), but also get some another corollaries:

**Corollary 3.1.** In every triangle, we have

$$(3.2) \quad \sum a^2 \geq 4\sqrt{3} \Delta + 2(a - b)^2$$

with equality holding if and only if $\angle C = \pi/3$.

**Corollary 3.2.** In every triangle $ABC$, we have

$$(3.3) \quad \sum a^2 \geq 4\sqrt{3} \Delta + \sum (a - b)^2 + 2\left(\sqrt{a(b + c - a)} - \sqrt{b(c + a - b)}\right)^2$$

with equality holding if only if $\angle C = \pi/3$.

**Corollary 3.3.** (Pólya-Szegő [6]) In every triangle $ABC$, we have

$$(3.4) \quad \frac{\sqrt{3}}{4} (abc)^2 \geq \Delta$$

with equality holding if and only if the triangle $ABC$ is equilateral.

From Lemma 2.1, we easily obtain

**Lemma 3.1.** In every triangle $ABC$, if the sequence of $\{\Delta A_k B_k C_k\}_{k=0}^n$ that the sides and angles respectively as follow

$$(3.5) \quad a_k = \sqrt{a_{k-1}(b_{k-1} + c_{k-1} - a_{k-1})},$$
$$b_k = \sqrt{b_{k-1}(c_{k-1} + a_{k-1} - b_{k-1})},$$
$$c_k = \sqrt{c_{k-1}(a_{k-1} + b_{k-1} - c_{k-1})}$$

and

$$(3.6) \quad A_k = (\pi - A_{k-1})/2, B_k = (\pi - B_{k-1})/2, C_k = (\pi - C_{k-1})/2,$$

where $\Delta A_0 B_0 C_0$ is the triangle $ABC$. Then the triangle for sides $a_k, b_k, c_k$ is a triangle for angles $A_k, B_k, C_k$, and we have

$$(3.7) \quad A_k = [(2^k - (-1)^k)(\pi/3) + (-1)^k A]/2^k$$

similarly, one defines $B_k, C_k$, $k = 0, 1, 2, \cdots$;

$$(3.8) \quad \lim_{k \to \infty} A_k = \lim_{k \to \infty} B_k = \lim_{k \to \infty} C_k = \pi/3;$$

and $\Delta_1 = \Delta_2 = \cdots = \Delta_k = \cdots = \Delta$.

**Theorem 3.2.** In every triangle $ABC$, if the sequence of $\{\Delta A_k B_k C_k\}_{k=0}^n$ that the sides and angles respectively define as (3.5) and (3.6), then we have

$$(3.9) \quad \sum a^2 \geq 4\sqrt{3} \Delta + \sum_{k=0}^n \left[\sum (a_k - b_k)^2\right]$$
Proof. In $\Delta A_{k-1}B_{k-1}C_{k-1}$, utilizing the fact that
$$a_{k-1}^2 = 4\Delta_{k-1} \tan \frac{A_{k-1}}{2} + (b_{k-1} - c_{k-1})^2,$$
we have
$$\sum a_{k-1}^2 = 4\Delta_{k-1} \sum \tan \frac{A_{k-1}}{2} + \sum (b_{k-1} - c_{k-1})^2. \quad (3.10)$$
From (3.6), then (3.10) become
$$\sum a_{k-1}^2 = 4\Delta_{k-1} \sum \cot A_k + \sum (b_{k-1} - c_{k-1})^2. \quad (3.11)$$
Also, by using the laws of cosines, we obtain
$$\sum a_{k-1}^2 = 4\Delta_{k-1} \sum \cot A_k - \sum (b_{k-1} - c_{k-1})^2. \quad (3.12)$$
Combining expansion (3.11) and (3.12), we have
$$4\Delta_{k-1} \sum \cot A_k = 4\Delta_{k-1} \sum \cot A_k + \sum (b_{k-1} - c_{k-1})^2. \quad (3.13)$$
From $\Delta = \Delta_0 = \Delta_1 = \Delta_2 = \cdots = \Delta_n$, we get
$$4\Delta \sum \cot A_{k-1} = 4\Delta \sum \cot A_k + \sum (b_{k-1} - c_{k-1})^2. \quad (3.14)$$
In (3.14), summing from 1 to $n + 1$ for $k$, we obtain
$$4\Delta \sum \cot A = 4\Delta \sum \cot A_{n+1} + \sum_{k=1}^{n+1} (b_{k-1} - c_{k-1})^2. \quad (3.15)$$
Utilizing the fact that
$$\sum \cot A_{n+1} \geq \sqrt{3}$$
and combining expansion (3.15), (3.12) and Lemma 3.1, the proof of inequality (3.2) is completed.

4. The Index Generalizations for Weitzenboeck’s Type Inequality

Theorem 4.1. Let $\lambda \geq 1$, and $n \in \mathbb{N}$, in every $\Delta ABC$, we have
$$\sum a_{2\lambda}^2 \geq 4\lambda^3 \Delta^2 + \sum_{k=0}^{n} |b_k - c_k|^2 \lambda \quad (4.1)$$
where $a_k, b_k, c_k$ define as Lemma 3.1.

Proof. When $\lambda \geq 1$, using the fact that [7]
$$\sum a_{2\lambda}^2 \geq 3^{1-\lambda} (\sum a^2)^\lambda \quad (4.2)$$
and
$$\left( \sum_{k=1}^{m} x_k \right)^\lambda \geq \sum_{k=1}^{m} x_k^\lambda \quad (4.3)$$
where $x_i \geq 0 (i = 1, 2, \cdots, n)$ and combining Theorem 3.2, we obtain inequality (4.1). The Theorem 4.1 is proved.

To prove the next theorem, the following lemmas [8] are necessary:

Lemma 4.1. If $k \leq (\ln 9 - \ln 4)/(\ln 4 - \ln 3)$, then in every $\Delta ABC$ we have
$$\left( \frac{1}{3} \sum a^k \right) \frac{1}{k} \leq \sqrt{3}R \quad (4.4)$$
Theorem 4.2. Let $0 < \lambda < (\ln 9 - \ln 4)/(\ln 4 - \ln 3)$, and $n \in \mathbb{N}$, in every $\Delta ABC$, we have

\[(4.5) \sum \frac{1}{a^{2\lambda}} \leq \frac{3^{1+\frac{\lambda}{2}}}{(4\Delta)^{\lambda}} + \frac{1}{2} \sum \left( \frac{1}{a^\lambda} - \frac{1}{b^\lambda} \right)^2\]

Proof. When $0 < \lambda < (\ln 9 - \ln 4)/(\ln 4 - \ln 3)$, from Lemma 4.1, we get

\[\left( \frac{1}{3} \sum a^\lambda \right)^{\frac{1}{\lambda}} \leq \sqrt{3}R\]

i.e.,

\[(4.6) \sum a^\lambda \leq 3^{1+\frac{\lambda}{2}}R^\lambda.\]

Combining expansion $R = abc/(2\Delta)$, inequality (4.6) becomes (4.5). The Theorem 4.2 is proved. □

Theorem 4.2 is obtained by J. Chen in [13].

Corollary 4.1. In every $\Delta ABC$, we have

\[(4.7) \sum \frac{1}{a^2} \leq \frac{3\sqrt{3}}{4\Delta} + \frac{1}{2} \sum \left( \frac{1}{a} - \frac{1}{b} \right)^2\]

and the coefficient $1/2$ is the best possible.

Proof. Let $\lambda = 1$ for (4.5), we easily get inequality (4.7). Let $a = b = 1, c = t$, then inequality \[\sum \frac{1}{a^\lambda} \leq \frac{3\sqrt{3}}{4\Delta} + \mu \sum \left( \frac{1}{a} - \frac{1}{b} \right)^2\]
becomes

\[\frac{1}{t^2} \left[ 2t^2 + 1 - 2\mu(1 - t)^2 - \frac{3\sqrt{3}t}{\sqrt{4 - t^2}} \right] \leq 0\]

Set $t \to 0$, we can obtain $\mu \geq \frac{1}{2}$, therefore the coefficient $\mu = \frac{1}{2}$ is the best possible. □

5. Weitzenboeck’s Type Inequalities of the Planar Convex Polygon

The next Lemma 5.1 is a preliminary election problem of the 29th IMO.

Lemma 5.1. If $\alpha_k > 0, \beta_k$ are real numbers $(k = 1, 2, \cdots, n)$, and \[\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = \pi,\] then we have

\[(5.1) \sum_{k=1}^{n} \frac{\cos \beta_k}{\sin \alpha_k} \leq \sum_{k=1}^{n} \cot \alpha_k\]

with equality holding if and only if $\alpha_k = \beta_k$ $(k = 1, 2, \cdots, n)$.

Theorem 5.1. Assume $a_k$ $(k = 1, 2, \cdots, n)$ denote the sides of a planar convex polygon $A_1A_2 \cdots A_n$ and $F$ the area. If $\alpha_k > 0$ $(k = 1, 2, \cdots, n)$ for \[\sum_{k=1}^{n} \alpha_k = \pi,\] then in every planar convex polygon the following inequality holds

\[(5.2) \sum_{k=1}^{n} a_k^2 \cot \alpha_k \geq 4F\]

Proof. To prove planar convex polygon $A_1A_2 \cdots A_n$ inscribed in a circle, because its area is maximal. Set $\beta_k = 2\gamma_k - \alpha_k$, where $2\gamma_k$ is the angle at the centre of the side $a_k$, $k = 1, 2, \cdots, n$. From Lemma 5.1 we have

\[\sum_{k=1}^{n} \frac{\cos(2\gamma_k - \alpha_k)}{\sin \alpha_k} \leq \sum_{k=1}^{n} \cot \alpha_k\]
that is

\[ \sum_{k=1}^{n} \sin^2 \gamma_k \cdot \cot \alpha_k \geq \sum_{k=1}^{n} \sin 2\gamma_k \]  

Utilizing the fact that \( a_k = 2R \sin \gamma_k \) \( (k = 1, 2, \cdots, n) \), and \( F = \frac{1}{2} R^2 \sum_{k=1}^{n} \sin 2\gamma_k \), inequalities (5.3) becomes (5.2). Lemma 5.1 is proved.

From Theorem 5.1, the following Corollary 5.1 is obvious.

**Corollary 5.1.** Assume \( a_k \) \( (k = 1, 2, \cdots, n) \) denote the sides of a planar convex polygon \( A_1A_2 \cdots A_n \) and \( F \) the area, then in every convex polygon the following inequality holds

\[ \sum_{k=1}^{n} a_k^2 \geq 4F \tan \frac{\pi}{n} \]  

**Corollary 5.2.** Assume \( a_k \) \( (k = 1, 2, 3, 4) \) denote the sides of quadrilateral \( ABCD \) and \( F \) the area. If \( \lambda_k \) \( (k = 1, 2, 3, 4) \) are the real numbers for

\[ \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_3 > 0, \]

and

\[ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > 0, \]

then in every quadrilateral we have

\[ \sum_{k=1}^{4} \lambda_k a_k^2 \geq 4F \sqrt{\frac{\lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}}, \]

**Proof.** Set \( t\lambda_k = \cot \alpha_k \) \( (k = 1, 2, 3, 4) \), \( t \) is a real constant. From Theorem 5.1 for \( n = 4 \), we have

\[ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi, \]

and

\[ \tan(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0. \]

Using the fact that

\[ \tan(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \frac{\sum_{1 \leq i < j < k \leq 4} \cot \alpha_i \cot \alpha_j \cot \alpha_k - \sum_{k=1}^{4} \cot \alpha_k}{\prod_{k=1}^{4} \cot \alpha_k - \sum_{1 \leq j < k \leq 4} \cot \alpha_j \cot \alpha_k + 1}, \]

we can obtain

\[ t^2 = \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{\lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_3}, \]

the proof of Corollary 5.2 is completed.

**Corollary 5.3.** Assume \( a_k \) \( (k = 1, 2, 3, 4) \) denote the sides of quadrilateral \( ABCD \) and \( F \) the area, in every quadrilateral we have

\[ \prod_{k=1}^{4} a_k \sum_{k=1}^{4} a_k \sum_{k=1}^{4} a_k^{-1} \geq 16F^2 \]

**Proof.** Let \( \lambda_1a_1 = \lambda_2a_2 = \lambda_3a_3 = \lambda_4a_4 \) in Corollary 5.2 then inequality (5.5) become (5.6). Corollary 5.3 is proved.

**Remark 5.1.** The results of this section were obtained by X.-Zh. Yang in [9].
6. The Reverse Weitzenboeck’s Inequality

**Theorem 6.1.** In every \(\Delta ABC\), we have
\[
\sum a^2 \leq 4\sqrt{3}\Delta + 3 \sum (a - b)^2
\]
and the coefficient 3 is the best possible.

*Proof.* Utilizing the fact that
\[
\prod \sin A = sr/2R^2,
\]
(6.2)
\[
\sum \sin^2 A = (s^2 - 4Rr - r^2)/2R^2,
\]
and
\[
\sum \sin B \sin C = (s^2 + 4Rr + r^2)/4R^2,
\]
then (6.1) \(\iff\) \(2\sqrt{3}\prod \sin A + 5 \sum \sin^2 A - 6 \sum \sin B \sin C \geq 0 \iff \sqrt{3}sr + s^2 - 16Rr - 4r^2 \geq 0 \iff (6.5)
\[
\sqrt{3}r(s - 3\sqrt{3}r) + s^2 - 16Rr + 5r^2 \geq 0
\]
From Gerretsen’s inequalities (6.6) and Euler inequality \(R \geq 2r\), inequality (6.5) or (6.1) is true.

Set \(a = b = 1, c = t\), then inequality \(\sum a^2 \leq 4\sqrt{3}\Delta + 3 \sum (a - b)^2\) becomes
\[
2 + t^2 \leq \sqrt{3}t\sqrt{4 - t^2} + 2\mu(1 - t)^2,
\]
Let \(t \to 2\), we obtain \(\mu \geq 3\), therefore the coefficient \(\mu = 3\) is the best possible. The proof of Theorem 6.1 is completed.

**Theorem 6.2.** In every \(\Delta ABC\), we have
\[
\sum a^2 \leq 4\sqrt{3}\Delta + \frac{3}{2} \sum (a - b)^2 + 2R^2 \sum (\cos B - \cos C)^2
\]

*Proof.* Utilizing the fact that (6.2), (6.3), (6.4) and
\[
\sum \cos B \cos C = (s^2 - 4R^2 + r^2)/4R^2,
\]
then (6.7) \(\iff\) \(2\sqrt{3}\prod \sin A + 5 \sum \sin^2 A - 3 \sum \sin B \sin C - \sum \cos B \cos C + 3 \geq 0 \iff \sqrt{3}sr - s^2 + 8R^2 - 10Rr - 3r^2 \geq 0 \iff (6.9)
\[
\sqrt{3}r(s - 3\sqrt{3}r) + 2R(2R - 2r)(2R - 3r) + 4R^2 + 4Rr + 3r^2 - s^2 \geq 0
\]
From Gerretsen’s inequalities (6.6) and Euler inequality \(R \geq 2r\), inequality (6.9) or (6.7) is proved.

Above two results are obtained by B.-Q. Liu in [11]. The next Lemma 6.10 is proved by A. Oppenheim in [12]:

**Lemma 6.1.** If \(0 < \theta < 1\), then in every \(\Delta ABC\) we have
\[
(\sum a^\theta) \prod (b^\theta + c^\theta - a^\theta) \geq 3^{1-\theta}(4\Delta)^{2\theta}
\]

**Theorem 6.3.** If \(\lambda \geq 2\), in every \(\Delta ABC\) we have
\[
\sum a^{2\lambda} \leq 4^{\lambda}3^{1-2\lambda}\Delta^\lambda + \sum (b^\lambda - c^\lambda)^2
\]
Proof. Since (6.11) is equivalent with
\[(6.12) \quad \left( \sum a^\frac{\lambda}{2} \right) \prod (b^\frac{\lambda}{2} + c^\frac{\lambda}{2} - a^\frac{\lambda}{2}) \leq 4^\lambda 3^{1-\frac{\lambda}{2}} \Delta^\lambda \]
If \( a^\frac{\lambda}{1}, a^\frac{\lambda}{2}, a^\frac{\lambda}{3} \) are not three sides of a triangle, then the expansion (6.12) or (6.11) is true, and if \( a^\frac{\lambda}{1}, a^\frac{\lambda}{2}, a^\frac{\lambda}{3} \) are three sides of a triangle, from Lemma 6.1 and \( 0 < 2/\lambda \leq 1 \), then we can obtain the expansion (6.12) or (6.11). Theorem 6.3 is proved.

Finally, we propose an open question:
Assume \( \lambda > (\ln 9 - \ln 4)/(\ln 4 - \ln 3) \). Solve that the best possible \( \mu \) for the following inequality holds
\[(6.13) \quad \sum \frac{1}{a^{2\lambda}} \leq \frac{3^{1+\frac{\lambda}{2}}}{(4\Delta)^\lambda} + \mu \left( \frac{1}{a^\lambda} - \frac{1}{b^\lambda} \right)^2.

References


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