Periodic solution for a delay integro-differential equation in biomathematics

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Abstract
Sufficient conditions for the existence and uniqueness of periodic solution of a delay integro-differential equation which arise in biomathematics are given. The results use a bidimensional variant of the Perov’s fixed point theorem.

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1 Introduction

In this paper we consider a model for the spread of certain infections disease with a contact rate that varies seasonally. This model is governed by the following integro-differential equation

\[ x(t) = \int_{t-\tau}^{t} f(s, x(s), x'(s)) \, ds \]  

(1)

where:
(i) \( x(t) \) is the proportion of infectious in population at time \( t \);
(ii) \( \tau > 0 \) is the length of time in which an individual remains infectious;
(iii) \( x'(t) \) is the speed of infectivity;
(iv) $f(t, x(t), x'(t))$ is the proportion of new infections on unit time.

We study the existence and uniqueness of a positive and periodic solution for equation (1).

A similar integral equation which models the same problem

$$x(t) = \int_{t-\tau}^{t} f(s, x(s)) \, ds$$

has been considered in [4], [5], [9], [8], [13] and [10] where sufficient conditions for the existence of nontrivial periodic nonnegative and continuous solutions for this equation are given in the case of a periodic contact rate: $f(t + \omega, x) = f(t, x)$, $\forall t \in \mathbb{R}$. The tools were: Banach fixed point principle in [10], topological fixed point theorems in [4], [5], [8], [13], fixed point index theory in [5] and monotone technique in [5], [8], [9]. Also, a system of integral equations in the form (2) has been studied in [2] and [11] using: the monotone technique in [2] and the Perov’s fixed point theorem for differentiable dependence by the parameter of the solution in [11]. In [1], sufficient conditions for the existence and uniqueness of a positive, continuous solution of the following initial value problem

$$x(t) = \begin{cases} \int_{t-\tau}^{t} f(s, x(s), x'(s)) \, ds, & t \in [0, T] \\ \varphi(t), & t \in [-\tau, 0] \end{cases}$$

are obtained.

In the following, if $X$ is a nonempty set then by a generalized metric $d : X \times X \to \mathbb{R}^n$ which fulfills the following:

$$0_{\mathbb{R}^n} \leq d(x, y), \forall x, y \in X \text{ and } d(x, y) = 0_{\mathbb{R}^n} \Leftrightarrow x = y$$

$$d(x, y) = d(y, x), \forall x, y \in X$$

$$d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X,$$

where for $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ from $\mathbb{R}^n$ we have $x \leq y \Leftrightarrow x_i \leq y_i$, for any $i = 1, n$. The pair $(X, d)$ will be called generalized metric space.

2 Existence and uniqueness

We suppose that $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ and exists $\varpi > 0$ such that

$$f(t + \varpi, x, y) = f(t, x, y), \quad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

We consider the following functional spaces

$$X(\varpi) = \{y \in C(\mathbb{R}) : y(t + \varpi) = y(t), \quad \forall t \in \mathbb{R}\}$$

$$X_1(\varpi) = \{x \in C^1(\mathbb{R}) : x(t + \varpi) = x(t), \quad \forall t \in \mathbb{R}\}$$

$$X_+(\varpi) = \{x \in X_1(\varpi) : x(t) \geq 0, \quad \forall t \in \mathbb{R}\}.$$
and denote by $X$ the product space $X = X_+ (\varpi) \times X (\varpi)$ which is generalized metric space with

$$d_C : X \times X \to \mathbb{R}^2, \quad d_C ((x_1, y_1), (x_2, y_2)) = (\|x_1 - x_2\|, \|y_1 - y_2\|),$$

where

$$\|u\| = \max \{|u(t)| : t \in [0, \varpi]\}$$

for any $u \in X (\varpi)$.

To obtain the existence and uniqueness result for the integro-differential equation (1) we use the following Perov’s fixed point theorem [7] (see also [3], [6])

**Theorem 1** (Perov, see [7]) Let $(X, d)$ a complete generalized metric space with $d(x, y) \in \mathbb{R}^n$. If $T : X \to X$ is a map for which exists a matrix $A \in M_n (\mathbb{R})$ such that

$$d(T(x), T(y)) \leq Ad(x, y), \quad \forall x, y \in X$$

and the eigenvalues of $A$ lies in the open unit disc from $\mathbb{R}^2$, then $T$ has a unique fixed point $x^*$ and the sequence of successive approximations $x_m = T^m(x_0)$ converges to $x^*$ for any $x_0 \in X$. Moreover, the following estimation holds

$$d (x_m, x^*) \leq A^m (I_2 - A)^{-1} d(x_0, x_1), \quad \forall m \in \mathbb{N}^*.$$

If we derive (1) with respect $t$ and denoting $y(t) = x'(t)$ we obtain

$$y(t) = f(t, x(t), y(t)) - f(t - \tau, x(t - \tau), y(t - \tau)), \quad \forall t \in \mathbb{R}.$$ 

which lead to

$$\begin{cases}
  x(t) = \int_{t-\tau}^t f(s, x(s), y(s)) \, ds \\
y(t) = f(t, x(t), y(t)) - f(t - \tau, x(t - \tau), y(t - \tau))
\end{cases} \quad (3)$$

Let $T : X \to C(\mathbb{R}) \times C(\mathbb{R})$ the map given by

$$T(x, y) = (T_1(x, y), T_2(x, y))$$

$$\begin{pmatrix} T_1(x, y)(t) \\ T_2(x, y)(t) \end{pmatrix} = \begin{pmatrix} \int_{t-\tau}^t f(s, x(s), y(s)) \, ds \\ f(t, x(t), y(t)) - f(t - \tau, x(t - \tau), y(t - \tau)) \end{pmatrix} \quad (4)$$

We impose the following conditions:

(i) $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ and exists $m, M \geq 0$ such that

$$m \leq f(t, x, y) \leq M, \quad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$ 

(ii) $f$ has the property

$$f(t + \varpi, x, y) = f(t, x, y), \quad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$
(iii) exists $\alpha, \beta > 0$ such that

$$|f(t, u, v) - f(t, u', v')| \leq \alpha |u - u'| + \beta |v - v'|$$

$\forall t \in \mathbb{R}, \forall u, u' \in \mathbb{R}_+, \forall v, v' \in \mathbb{R}$.

From condition (i) we see that $T_1(X) \subseteq C^1(\mathbb{R})$ and

$$T_1(x, y)(t) = \int_{t-\tau}^{t+\tau} f(s, x(s), y(s)) ds \geq \int_{t-\tau}^{t+\tau} m ds = m\tau$$

$\forall t \in \mathbb{R}$. It is obvious that $T_1(x, y)(t) \leq M\tau \forall t \in \mathbb{R}$, $\forall (x, y) \in X$.

**Theorem 2** If the conditions (i)-(iii) are satisfied and $\alpha \tau + 2\beta < 1$ then the integro-differential equation (1) have in $X_+(\mathbb{R})$ an unique solution.

**Proof.** From condition (ii) follows that $T_1(X) \subset X_+(\mathbb{R})$. Indeed,

$$T_1(x, y)(t + \tau) = \int_{t+\tau-\tau}^{t+\tau} f(s, x(s), y(s)) ds =$$

$$= \int_{t-\tau}^{t} f(u - \tau, x(u - \tau), y(u - \tau)) du =$$

$$= \int_{t-\tau}^{t} f(u - \tau, x(u - \tau + \tau), y(u - \tau + \tau)) du =$$

$$= T_1(x, y)(t), \forall t \in \mathbb{R}, \forall (x, y) \in X.$$
The matrix
\[
(\alpha |x_1 (s) - x_2 (s)| + \beta |y_1 (s) - y_2 (s)|) ds \leq 0
\]
and
\[
|T_2 (x_1, y_1) (t) - T_2 (x_2, y_2) (t)| = |f (t, x_1 (t), y_1 (t)) - f (t, x_2 (t), y_2 (t)) + + f (t - \tau, x_2 (t - \tau), y_2 (t - \tau))| \leq |f (t, x_1 (t), y_1 (t)) - f (t, x_2 (t), y_2 (t))| + + |f (t - \tau, x_2 (t - \tau), y_2 (t - \tau))| \leq \alpha |x_1 (t) - x_2 (t)| + + \beta |y_1 (t) - y_2 (t)| \leq 2 \alpha |x_1 - x_2| + + 2 \beta |y_1 - y_2|, \forall t \in [0, \infty].
\]

Then
\[
\left\| \frac{T_1 (x_1, y_1) - T_1 (x_2, y_2)}{\|T_2 (x_1, y_1) - T_2 (x_2, y_2)\|} \right\| \leq \frac{\alpha \tau \beta \tau}{2 \alpha 2 \beta} \left( \frac{|x_1 - x_2|}{|y_1 - y_2|} \right)
\]
that is
\[
d_C (T (x_1, y_1), T (x_2, y_2)) \leq A \cdot d_C ((x_1, y_1), (x_2, y_2))
\]
The matrix
\[
A = \begin{pmatrix}
\alpha \tau \\
2 \alpha \\
\beta \tau \\
2 \beta
\end{pmatrix}
\]
has the eigenvalues \( \lambda_1 = 0 \) and \( \lambda_2 = 2 \beta + \alpha \tau \). Since \( \alpha \tau + 2 \beta < 1 \), by the Perov’s fixed point theorem we infer that \( T \) has in \( X \) an unique fixed point, denoted by \( x_* = (x^*, y^*) \). It is easy to see that \( (x^*)' (t) = y^* (t), \forall t \in \mathbb{R} \). Indeed,
\[
y^* (t) = f (t, x^* (t), y^* (t)) - f (t - \tau, x^* (t - \tau), y^* (t - \tau)) = \int_{t-\tau}^{t} f (s, x^* (s), y^* (s)) ds
\]
and after derivation,
\[
(x^*)' (t) = f (t, x^* (t), y^* (t)) - f (t - \tau, x^* (t - \tau), y^* (t - \tau))
\]
Then, \( x^* \in X_+ (\infty) \) is the solution of the equation (1).

From the above result, the solution \( x^* \) of (1) and its derivative are \( \varpi \)-periodic.
Theorem 3 In the conditions of Theorem 2 the solution \( x_\ast \) of (3), which is obtained by the successive approximations method starting from any \( x^0 = (x_0, y_0) \in X \), verify the following estimation
\[
d_C(x^m, x_\ast) \leq \frac{\lambda^m_2}{1 - \lambda^m_2} \left( \frac{\alpha \tau}{2\alpha} \right) d_C(x^1, x^0).
\]
where \( x^m = T(x^{m-1}) \), \( x^m = (x_m, y_m) \), \( \forall m \in \mathbb{N}^* \).

Proof. From Theorem 1, in conditions of Theorem 2 we have that
\[
d_B(x^m, x_\ast) \leq A^m (I - A)^{-1} d_B(x^1, x^0), \forall m \in \mathbb{N}^*.
\]
For the matrix \( A \) given in (5) we have \( A^m = \lambda^m_2 A \), \( \forall m \in \mathbb{N}^* \) and \( (I - A)^{-1} = \frac{1}{1 - \lambda^m_2} \left( \begin{array}{cc} 1 - 2\beta \beta \tau & \beta \tau \\ 2\alpha & 1 - \alpha \tau \end{array} \right) \). ■

The solution of (1) and his derivative can be obtained by the successive approximations method starting from any element of \( X \).

References


