Periodic solution for a delay integro-differential equation in biomathematics

Alexandru Bica
Department of Mathematics, University of Oradea,
Str. Armatei Romane no. 5, 410087 Oradea, Romania
smbica@yahoo.com, abica@uoradea.ro

Sorin Muresan
Department of Mathematics, University of Oradea,
Str. Armatei Romane no. 5, 410087 Oradea, Romania
smuresan@uoradea.ro

January 19, 2004

Abstract

Sufficient conditions for the existence and uniqueness of periodic solution of a delay integro-differential equation which arise in biomathematics are given. The results use a bidimensional variant of the Perov's fixed point theorem.

AMS Mathematics Subject Classification: Primary 45J05, Secondary 34A12, 65R20, 65D32.

Keywords and phrases: delay integro-differential equation, Perov's fixed point theorem, successive approximations.

1 Introduction

In this paper we consider a model for the spread of certain infections disease with a contact rate that varies seasonally. This model is govern by the following integro-differential equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s), x'(s))\, ds \tag{1}$$

where:
(i) $x(t)$ is the proportion of infectious in population at time t;
(ii) $\tau > 0$ is the length of time in which an individual remains infectious;
(iii) $x'(t)$ is the speed of infectivity;

(iv) $f(t, x(t), x'(t))$ is the proportion of new infections on unit time.

We study the existence and uniqueness of a positive and periodic solution for equation (1).

A similary integral equation which models the same problem

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds \tag{2}$$

has been considered in [4], [5], [9], [8], [13] and [10] where sufficient conditions for the existence of nontrivial periodic nonnegative and continous solutions for this equation are given in the case of a periodic contact rate: $f(t+\omega, x) = f(t, x)$, $\forall t \in \mathbb{R}$. The tools were: Banach fixed point principle in [10], topological fixed point theorems in [4], [5], [8], [13], fixed point index theory in [5] and monotone technique in [5], [8], [9]. Also, a system of integral equations in the form (2) has been studied in [2] and [11] using: the monotone technique in [2] and the Perov's fixed point theorem for differentiable dependence by the parameter of the solution in [11]. In [1], sufficient conditions for the existence and uniqueness of a positive, continuous solution of the following initial value problem

$$x(t) = \begin{cases} \int\limits_{t-\tau}^{t} f(s, x(s), x'(s))\, ds, & t \in [0, T] \\ \varphi(t), & t \in [-\tau, 0] \end{cases}$$

are obtained.

In the following, if X is a nonempty set then by a generalized metric d on X we understand a function $d : X \times X \to \mathbb{R}^n$ which fulfils the following:

$$\begin{gathered} 0_{\mathbb{R}^n} \le d(x, y), \forall x, y \in X \text{ and } d(x, y) = 0_{\mathbb{R}^n} \Leftrightarrow x = y \\ d(x, y) = d(y, x), \forall x, y \in X \\ d(x, y) \le d(x, z) + d(z, y), \forall x, y, z \in X, \end{gathered}$$

where for $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ from $\mathbb{R}^n$ we have $x \le y \Leftrightarrow x_i \le y_i$, for any $i = \overline{1, n}$. The pair (X, d) will be called generalized metric space.

2 Existence and uniqueness

We suppose that $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ and exists $\varpi > 0$ such that

$$f(t + \varpi, x, y) = f(t, x, y),\ \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

We consider the following functional spaces

$$X(\varpi) = \{y \in C(\mathbb{R}) : y(t + \varpi) = y(t),\ \forall t \in \mathbb{R}\}$$

$$X_1(\varpi) = \{x \in C^1(\mathbb{R}) : x(t + \varpi) = x(t),\ \forall t \in \mathbb{R}\}$$

$$X_+(\varpi) = \{x \in X_1(\varpi) : x(t) \ge 0,\ \forall t \in \mathbb{R}\}.$$

and denote by X the product space $X = X_+(\varpi) \times X(\varpi)$ which is generalized metric space with

$$d_C : X \times X \to \mathbb{R}^2,\ d_C\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = \left(\|x_1 - x_2\|, \|y_1 - y_2\|\right),$$

where

$$\|u\| = \max\{|u\left(t\right)| : t \in [0, \varpi]\}$$

for any $u \in X(\varpi)$.

.To obtain the existence and uniqueness result for the integro-differential equation (1) we use the following Perov's fixed point theorem [7] (see also [3], [6])

Theorem 1 *(Perov, see [7]) Let (X, d) a complete generalized metric space with $d\left(x, y\right) \in \mathbb{R}^n$. If $T : X \to X$ is a map for which exists a matrix $A \in \mathcal{M}_n\left(\mathbb{R}\right)$ such that*

$$d\left(T\left(x\right), T\left(y\right)\right) \leq Ad\left(x, y\right), \forall x, y \in X$$

and the eigenvalues of A lies in the open unit disc from $\mathbb{R}^2$, then T has a unique fixed point x^* and the sequence of successive approximations $x_m = T^m\left(x_0\right)$ converges to x^* for any $x_0 \in X$. Moreover, the following estimation holds

$$d\left(x_m, x^*\right) \leq A^m\left(I_2 - A\right)^{-1} d\left(x_0, x_1\right), \forall m \in \mathbb{N}^*.$$

If we derive (1) with respect t and denoting $y\left(t\right) = x'\left(t\right)$ we obtain

$$y\left(t\right) = f\left(t, x\left(t\right), y\left(t\right)\right) - f\left(t - \tau, x\left(t - \tau\right), y\left(t - \tau\right)\right),\ \forall t \in \mathbb{R}.$$

which lead to

$$\begin{cases} x(t) = \int\limits_{t-\tau}^{t} f\left(s, x\left(s\right), y\left(s\right)\right) ds \\ y(t) = f\left(t, x\left(t\right), y\left(t\right)\right) - f\left(t - \tau, x\left(t - \tau\right), y\left(t - \tau\right)\right) \end{cases} \tag{3}$$

Let $T : X \to C(\mathbb{R}) \times C(\mathbb{R})$ the map given by

$$T(x, y) = (T_1(x, y), T_2(x, y))$$

$$\begin{pmatrix} T_1(x, y)(t) \\ T_2(x, y)(t) \end{pmatrix} = \begin{pmatrix} \int\limits_{t-\tau}^{t} f\left(s, x\left(s\right), y\left(s\right)\right) ds \\ f\left(t, x\left(t\right), y\left(t\right)\right) - f\left(t - \tau, x\left(t - \tau\right), y\left(t - \tau\right)\right) \end{pmatrix} \tag{4}$$

We impose the following conditions:

(i) $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ and exists $m, M \geq 0$ such that

$$m \leq f(t, x, y) \leq M,\ \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

(ii) f has the property

$$f(t + \varpi, x, y) = f(t, x, y),\ \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

(iii) exists $\alpha, \beta > 0$ such that

$$|f(t,u,v) - f(t,u',v')| \leq \alpha |u-u'| + \beta |v-v'|$$

$\forall t \in \mathbb{R}, \forall u, u' \in \mathbb{R}_+, \forall v, v' \in \mathbb{R}$.

From condition (i) we see that $T_1(X) \subseteq C^1(\mathbb{R})$ and

$$T_1(x,y)(t) = \int\limits_{t-\tau}^{t} f(s, x(s), y(s))\, ds \geq \int\limits_{t-\tau}^{t} m ds = m\tau$$

$\forall t \in \mathbb{R}$. It is obvious that $T_1(x,y)(t) \leq M\tau$ $\forall t \in \mathbb{R}$, $\forall (x,y) \in X$.

Theorem 2 *If the conditions (i)-(iii) are satisfied and $\alpha\tau + 2\beta < 1$ then the integro-differential equation (1) have in $X_+(\varpi)$ an unique solution.*

Proof. From condition (ii) follows that $T_1(X) \subset X_+(\varpi)$. Indeed,

$$T_1(x,y)(t+\varpi) = \int\limits_{t+\varpi-\tau}^{t+\varpi} f(s, x(s), y(s))\, ds =$$

$$= \int\limits_{t-\tau}^{t} f(u-\varpi, x(u-\varpi), y(u-\varpi))\, du =$$

$$= \int\limits_{t-\tau}^{t} f(u-\varpi+\varpi, x(u-\varpi+\varpi), y(u-\varpi+\varpi))\, du =$$

$$= T_1(x,y)(t), \ \forall t \in \mathbb{R}, \forall (x,y) \in X.$$

In adition

$$T_2(x,y)(t+\varpi) = f(t+\varpi, x(t+\varpi), y(t+\varpi)) -$$
$$-f(t+\varpi-\tau, x(t+\varpi-\tau), y(t+\varpi-\tau)) =$$
$$= f(t+\varpi, x(t), y(t)) - f(t+\varpi-\tau, x(t-\tau), y(t-\tau))$$
$$= f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau)) = T_2(x,y)(t)$$

$\forall t \in \mathbb{R}, \forall (x,y) \in X$. Consquently, $T(X) \subset X$. Let $(x_1,y_1), (x_2,y_2) \in X$.

$$|T_1(x_1,y_1)(t) - T_1(x_2,y_2)(t)| =$$

$$= \left| \int\limits_{t-\tau}^{t} f(s, x_1(s), y_1(s))\, ds - \int\limits_{t-\tau}^{t} f(s, x_2(s), y_2(s))\, ds \right| \leq$$

$$\leq \int\limits_{t-\tau}^{t} |f(s, x_1(s), y_1(s)) - f(s, x_2(s), y_2(s))|\, ds \leq$$

$$\leq \int\limits_{t-\tau}^{t} [\alpha \left| x_1(s) - x_2(s) \right| + \beta \left| y_1(s) - y_2(s) \right|] ds \leq$$

$$\leq \alpha\tau \|x_1 - x_2\| + \beta\tau \|y_1 - y_2\|, \forall t \in [0, \varpi].$$

and

$$|T_2(x_1, y_1)(t) - T_2(x_2, y_2)(t)| = \mid f(t, x_1(t), y_1(t)) -$$
$$- f(t - \tau, x_1(t - \tau), y_1(t - \tau)) - f(t, x_2(t), y_2(t)) +$$
$$+ f(t - \tau, x_2(t - \tau), y_2(t - \tau)) \mid \leq \mid f(t, x_1(t), y_1(t)) -$$
$$- f(t, x_2(t), y_2(t)) \mid + \mid f(t - \tau, x_1(t - \tau), y_1(t - \tau))$$
$$- f(t - \tau, x_2(t - \tau), y_2(t - \tau)) \mid \leq \alpha \left| x_1(t) - x_2(t) \right| +$$
$$+ \beta \left| y_1(t) - y_2(t) \right| + \alpha \left| x_1(t - \tau) - x_2(t - \tau) \right| +$$
$$+ \beta \left| y_1(t - \tau) - y_2(t - \tau) \right| \leq 2\alpha \|x_1 - x_2\| +$$
$$+ 2\beta \|y_1 - y_2\|, \forall t \in [0, \varpi], \forall (x_1, y_1), (x_2, y_2) \in X$$

Then

$$\begin{pmatrix} \|T_1(x_1, y_1) - T_1(x_2, y_2)\| \\ \|T_2(x_1, y_1) - T_2(x_2, y_2)\| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} \alpha\tau \|x_1 - x_2\| + \beta\tau \|y_1 - y_2\| \\ 2\alpha \|x_1 - x_2\| + 2\beta \|y_1 - y_2\| \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{pmatrix} \cdot \begin{pmatrix} \|x_1 - x_2\| \\ \|y_1 - y_2\| \end{pmatrix}$$

that is

$$d_C(T(x_1, y_1), T(x_2, y_2)) \leq A \cdot d_C((x_1, y_1), (x_2, y_2))$$

The matrix

$$A = \begin{pmatrix} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{pmatrix} \tag{5}$$

has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2\beta + \alpha\tau$. Since $\alpha\tau + 2\beta < 1$, by the Perov's fixed point theorem we infer that T has in X an unique fixed point,.denoted by $x_* = (x^*, y^*)$. It is easy to see that $(x^*)'(t) = y^*(t)$, $\forall t \in \mathbb{R}$. Indeed,

$$y^*(t) = f(t, x^*(t), y^*(t)) - f(t - \tau, x^*(t - \tau), y^*(t - \tau))$$

$$x^*(t) = \int\limits_{t-\tau}^{t} f(s, x^*(s), y^*(s)) \, ds$$

and after derivation,

$$(x^*)'(t) = f(t, x^*(t), y^*(t)) - f(t - \tau, x^*(t - \tau), y^*(t - \tau))$$

Then, $x^* \in X_+(\varpi)$ is the solution of the equation (1). $\blacksquare$

From the above result, the solution x^* of (1) and his derivative are ϖ-periodic.

Theorem 3 *In the conditions of Theorem 2 the solution x_* of (3), which is obtained by the successive approximations method starting from any $x^0 = (x_0, y_0) \in X$, verify the following estimation*

$$d_C\left(x^m, x_*\right) \leq \frac{\lambda_2^{m-1}}{1-\lambda_2}\begin{pmatrix} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{pmatrix} d_C\left(x^1, x^0\right).$$

where $x^m = T\left(x^{m-1}\right)$, $x^m = (x_m, y_m), \forall m \in \mathbb{N}^*$.

Proof. From Theorem 1, in conditions of Theorem 2 we have that

$$d_B\left(x^m, x^*\right) \leq A^m \left(I - A\right)^{-1} d_B\left(x^1, x^0\right), \forall m \in \mathbb{N}^*.$$

For the matrix A given in (5) we have $A^m = \lambda_2^{m-1} A$, $\forall m \in \mathbb{N}^*$ and $\left(I - A\right)^{-1} = \frac{1}{1-\lambda_2}\begin{pmatrix} 1-2\beta & \beta\tau \\ 2\alpha & 1-\alpha\tau \end{pmatrix}$. ■

The solution of (1) and his derivative can be obtained by the successive approximations method starting from any element of X.

References

[1] A.Bica, S.Muresan, H.Oros, Existence result and numerical method for a delay integro-differential equation arising in infectious diseases, *Proceedings of the 11-th Conference on Applied and Industrial Mathematics (2003), 34-41,* Oradea, Romania..

[2] A.Canada, A.Zertiti, Systems of nonlinear delay integral equations modelling population growth in a periodic environment, *Coment. Math. Carolinae* 35, 4 (1994), 633-644.

[3] B.Ćirić, Fixed and periodic points in Kurepa spaces, *Math. Balcanika*, 2 (1972), 13-20.

[4] K.L.Cooke, J.L.Kaplan, A periodicity threshold theorem for epidemics and population growth, *Math.Biosciences*, 31 (1976), 87-104.

[5] D.Guo, V.Lakshmikantham, Positive solutions of nonlinear integral equations arising in infectious diseases, *J. Math. Anal. Appl.*, 134(1988), 1-8.

[6] A.I.Perov, On the Cauchy's problem for a system of ordinary differential equations (in Russian), *Priblizhen. Metod. Reshenia Urav.*, Kiev, 1964.

[7] A.I.Perov, A.V.Kidenko, On a general method to study the boundary value problems, *Iz. Akad. Nauk.*, 30(1966), pp249-264.

[8] R.Precup, Periodic solutions for an integral equation from biomathematics via Leray-Schauder principle, *Studia Univ. "Babes-Bolyai", Mathematica*, vol. XXXIX, 1, 1994, 47-58.

[9] R.Precup, Monotone technique to the initial value problem for a delay integral equation from biomathematics, *Studia Univ. "Babes-Bolyai", Mathematica*, XL,2, (1995), 63-73.

[10] I.A.Rus, A delay integral equation from biomathematics, *Seminar of Differential Equations, Preprint3*, Cluj-Napoca, 1989, p. 87-90.

[11] I.A.Rus, Fiber Picard operators on generalized metric spaces and an application, *Scripta Scientiarum Mathematicarum*, Tomus I,Fasc. II,Anno MCMXCIX, 355-363.

[12] I.A.Rus, *Principles and applications of the fixed point theory*, Ed. Dacia, Cluj-Napoca, 1979 (in Romanian).

[13] L.R.Williams, R.W.Leggett, Nonzero solutions of nonlinear integral equations modeling infectious disease, *SIAM J. Math. Anal.*, 13(1982), 112-121.