Monotonicity Of Generalized Weighted Mean Values

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Abstract

The author gives a new simple proof of monotonicity of the generalized extended mean values $M(r,s) = \left(\frac{\int f^s d\mu}{\int f^r d\mu}\right)^{1/(s-r)}$ introduced by F. Qi.

Means and inequalities for them have a very long history and a rich literature. The basic inequality between the geometric and arithmetic means has been proved in many ways. More than fifty proofs can be found in [1]. The generalizations have gone into different directions. The power (or Hölder) mean $M(r) = ((x^r + y^r)/2)^{1/r}, r \neq 0, M(0) = \sqrt{xy} = G(x,y)$, has been extended to the weighted power means

$$M(r) = \left(\sum pa^r / \sum p\right)^{1/r}, M(0) = \exp\left(\sum pa\log a / \sum p\right), \qquad (1)$$

and further to the weighted integral means where sums are replaced by integrals. The monotonicity of $M(r)$ has been proved in many ways (see [1, 3, 6]).

Another family of means arises from the logarithmic mean $L(x,y) = (x-y)/(\log x - \log y)$ by putting

$$S_p(x,y) = \left(\frac{y^p - x^p}{p(y-x)}\right)^{1/(p-1)},\ S_0(x,y) = L(x,y),\ S_1(x,y) = e^{-1}\left(\frac{y^y}{x^x}\right)^{1/(y-x)},$$

see Galvani [2].

Stolarsky [7] extended this family to the two-parameter extended mean values defined by

$$E(r,s;x,y) = \begin{cases} \left(\frac{r}{s}\frac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)} & sr(s-r)(x-y) \neq 0, \\ \left(\frac{1}{r}\frac{y^r - x^r}{\log y - \log x}\right)^{1/r} & r(x-y) \neq 0,\ s = 0, \\ e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r - x^r)} & r = s,\ r(x-y) \neq 0, \\ \sqrt{xy} & r = s = 0,\ x - y \neq 0, \\ x & x = y. \end{cases} \qquad (2)$$

Leach and Scholander [4, 5] have shown that E is increasing in all variables. In 1998 Qi [8] extended these notions by defining the generalized weighted mean values M as follows

$$M(r,s) = M(r,s;x,y) = \begin{cases} \left(\frac{\int_x^y p(t)f^s(t)dt}{\int_x^y p(t)f^r(t)dt}\right)^{1/(s-r)} & r \neq s, \\ \exp\left(\frac{\int_x^y p(t)f^r(t)\log f(t)dt}{\int_x^y p(t)f^r(t)dt}\right) & r = s, \end{cases} \tag{3}$$

where p and f are positive, integrable functions. Obviously $M(r,0)$ is the weighted power mean (1) and $M(r-1,s-1) = E(r,s)$ for $p \equiv 1$ and $f(t) = t$.

Qi [9] proved that for continuous p and f, M is increasing in variables p and s. He has also shown in [8] that if f is monotone then M is of the same monotonicity in variables x and y.

In this note we extend and generalize these results by showing that the monotonicity of $M(r,s)$ is a straightforward consequence of the Cauchy-Schwarz inequality and holds also in case of integrable functions. We also show that monotonicity of f is a necessary and sufficient condition for M to be monotone in x and y. We believe that our proofs are also simpler than the original reasoning.

Theorem 1 *Let $f : X \to \mathbb{R}$ be a measurable, positive function on a measure space (X,μ) and*

$$M(r,s) = \begin{cases} \left(\frac{\int_X f^s d\mu}{\int_X f^r d\mu}\right)^{1/(s-r)} & r \neq s, \\ \exp\left(\frac{\int_X f^s \log f d\mu}{\int_X f^s d\mu}\right) & r = s. \end{cases} \tag{4}$$

Then M is increasing in both parameters r and s.

Proof. Let $I(r) = \int_X f^r d\mu$. The Cauchy-Schwarz inequality applied to $f^{s/2}$ and $f^{r/2}$ gives

$$I(\tfrac{r+s}{2}) \leq \sqrt{I(r)}\sqrt{I(s)},$$

which shows that $\log I$ is convex in the sense of Jensen, hence being continuous is convex.

Let us recall now the following property of convex functions: If h is convex then the function $g(x,y) = \frac{h(x)-h(y)}{x-y}, x \neq y$, is increasing in both variables. This property applied to $\log I$ shows that $\log M(r,s)$ is increasing for $s \neq r$. As M is continuous, the monotonicity extends to the whole plane of parameters (r,s).

Theorem 2 *If $p, f : [a,b] \to \mathbb{R}$ are continuous and positive then the following conditions are equivalent:*

i) *the function f is increasing (decreasing, respectively).*

ii) *for every r, s the function $M(r,s;x,y)$ is increasing (decreasing, respectively) in x and y.*

Proof. As in the proof of Theorem 1 it is easier to consider monotonicity of $\log M$. For $r \neq s$ we have

$$\begin{aligned}
\frac{\partial \log M}{\partial x} &= (s-r)^{-1}\left(\frac{-p(x)f^s(x)}{\int_x^y p(t)f^s(t)dt} - \frac{-p(x)f^r(x)}{\int_x^y p(t)f^r(t)dt}\right) \\
&= H\frac{\int_x^y p(t)\left(\left(\frac{f(t)}{f(x)}\right)^s - \left(\frac{f(t)}{f(x)}\right)^r\right)dt}{s-r} \\
&= H\frac{\int_x^y p(t)\left(\frac{f(t)}{f(x)}\right)^r\left(\left(\frac{f(t)}{f(x)}\right)^{s-r} - 1\right)dt}{s-r},
\end{aligned} \tag{5}$$

where $H = H(r,s;x,y) = \frac{p(x)f^{r+s}(x)}{\int_x^y p(t)f^r(t)dt \int_x^y p(t)f^r(t)dt}$ is positive.
Observe that

$$\text{if} \quad f(x) = \min_{t\in[x,y]} f(t) \quad \text{then} \quad \frac{\left(\frac{f(t)}{f(x)}\right)^{s-r} - 1}{s-r} \geq 0, \tag{6}$$

$$\text{if} \quad f(x) = \max_{t\in[x,y]} f(t) \quad \text{then} \quad \frac{\left(\frac{f(t)}{f(x)}\right)^{s-r} - 1}{s-r} \leq 0. \tag{7}$$

From (5), (6) and (7) we conclude that if f is increasing then $\log M$ is increasing in x. Similar reasoning shows monotonicity in y, so the implication i)$\Rightarrow$ii) holds.

If f is not monotone then one can choose an interval $[z,y]$ and $x_1, x_2 \in [z,y]$ such that

$$f(x_1) = \min_{t\in[z,y]} f(t) < f(y) < \max_{t\in[z,y]} f(t) = f(x_2)$$

and from (6) and (7)

$$\frac{\partial \log M}{\partial x}(r,s;x_1,y) < 0 < \frac{\partial \log M}{\partial x}(r,s;x_2,y),$$

which completes the proof in case $r \neq s$.
To prove the case $s = r$ it is enough to replace $(s-r)^{-1}\left(\left(\frac{f(t)}{f(x)}\right)^{s-r} - 1\right)$ with $(\log f(t) - \log f(x))$ in (5), (6) and (7).

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