

# A COMPLETE MONOTONICITY OF THE GAMMA FUNCTION

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ABSTRACT. The function  $\frac{1}{x} \ln \Gamma(x+1) - \ln x + 1$  is strictly completely monotonic on  $(0, \infty)$ .

The classical gamma function is usually defined for  $\operatorname{Re} z > 0$  by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \quad (1)$$

The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [6, p. 16] as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt, \quad (2)$$

$$\psi^{(m)}(x) = (-1)^{m+1} \int_0^\infty \frac{t^m}{1 - e^{-t}} e^{-xt} dt \quad (3)$$

for  $x > 0$  and  $m \in \mathbb{N}$ , where  $\gamma = 0.57721566490153286 \dots$  is the Euler-Mascheroni constant.

In 1985, D. Kershaw and A. Laforgia [5] showed that the function  $x[\Gamma(1 + \frac{1}{x})]^x$  is strictly increasing on  $(0, \infty)$ , which is equivalent to the function  $\frac{[\Gamma(x+1)]^{1/x}}{x}$  being strictly decreasing on  $(0, \infty)$ . In addition, it was proved that the function  $x^{1-\gamma}[\Gamma(1 + \frac{1}{x})]^x$  decreases for  $0 < x < 1$ , which is equivalent to  $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$  being increasing on  $(1, \infty)$ .

In [2, 10], it is proved that the function  $f(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1}$  is strictly decreasing and strictly logarithmically convex in  $(0, \infty)$  and the function  $g(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$  is strictly increasing and strictly logarithmically concave in  $(0, \infty)$ . Some new proofs for the monotonicity of the function  $x^r[\Gamma(x+1)]^{1/x}$  on  $(0, \infty)$  are given for  $r \notin (0, 1)$ . In addition, if  $s$  is a positive real number, then for all real numbers  $x > 0$ ,

$$\frac{e^{-\gamma}}{[\Gamma(s+1)]^{1/s}} < \frac{[\Gamma(x+1)]^{1/x}}{[\Gamma(x+s+1)]^{1/(x+s)}} < 1, \quad (4)$$

$\lim_{x \rightarrow 0} f(x) = e^{-\gamma}$  and  $\lim_{x \rightarrow \infty} f(x) = e^{-1}$ .

Using monotonicity and inequalities of the generalized weighted mean values [1, 7, 8, 12], the first author proved [9] that the functions  $\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}$ ,  $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$

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and  $\left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$  are increasing in  $r > 0$ ,  $s > 0$  and  $x > 0$ . For any given  $x > 0$ , the function  $\frac{s\gamma(s,x)}{x^s}$  is decreasing in  $s > 0$ .

In [3], N. Elezović, C. Giordana and J. Pečarić, among others, verified the convexity with respect to variable  $x$  of the function  $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}$  for  $|t-s| < 1$ .

Recall that a function  $f$  is said to be completely monotonic on an interval  $I$  if  $f$  has derivatives of all orders on  $I$  which alternate successively in sign, that is

$$(-1)^n f^n(x) \geq 0 \quad (5)$$

for  $x \in I$  and  $n \geq 0$ . If inequality (5) is strict for all  $x \in I$  and for all  $n \geq 0$ , then  $f$  is said to be strictly completely monotonic. See [11] and references therein.

In this short note, we are about to prove a complete monotonicity result of a function involving the gamma function.

**Theorem 1.** *The function  $\frac{1}{x} \ln \Gamma(x+1) - \ln x + 1$  is strictly completely monotonic on  $(0, \infty)$  and tends to  $\infty$  as  $x \rightarrow 0$  and to 0 as  $x \rightarrow \infty$ .*

*Proof.* It has been shown in [5] that the function  $\frac{[\Gamma(x+1)]^{1/x}}{x}$  is strictly decreasing on  $(0, \infty)$ , then  $f(x) = \frac{1}{x} \ln \Gamma(x+1) - \ln x + 1$  is strictly decreasing on  $(0, \infty)$ . From the asymptotic expansion in [4]:

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12x} + O(x^{-3}) \quad \text{as } x \rightarrow \infty,$$

we conclude that  $\lim_{x \rightarrow \infty} f(x) = 0$  and  $\lim_{x \rightarrow 0} f(x) = \infty$ . This implies  $f(x) > 0$  for  $x > 0$ .

Using Leibniz' rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x) \quad (6)$$

we obtain

$$\begin{aligned} f^{(n)}(x) &= \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{x}\right)^{(n-k)} [\ln \Gamma(x+1)]^{(k)} - \frac{(-1)^{n-1}(n-1)!}{x^n} \\ &= \left(\frac{1}{x}\right)^{(n)} \ln \Gamma(x+1) + \sum_{k=1}^n \binom{n}{k} \left(\frac{1}{x}\right)^{(n-k)} \psi^{(k-1)}(x+1) + \frac{(-1)^n n!}{nx^n} \\ &= \frac{(-1)^n n!}{x^{n+1}} \ln \Gamma(x+1) + \sum_{k=1}^n \frac{n!}{k!} \frac{(-1)^{n-k}}{x^{n-k+1}} \psi^{(k-1)}(x+1) + \frac{(-1)^n n!}{nx^n} \\ &\triangleq (-1)^n \frac{n!}{x^{n+1}} g(x), \end{aligned} \quad (7)$$

and

$$g'(x) = \frac{(-1)^n}{n!} x^n \psi^{(n)}(x+1) + \frac{1}{n}. \quad (8)$$

Using (3) and  $\frac{(n-1)!}{x^n} = \int_0^\infty t^{n-1} e^{-xt} dt$  for  $x > 0$  and  $n \in \mathbb{N}$ , we conclude

$$\frac{1}{x^n} g'(x) = \frac{1}{n!} \int_0^\infty \left(1 - \frac{t}{e^t - 1}\right) t^{n-1} e^{-xt} dt > 0, \quad (9)$$

since  $0 < \frac{t}{e^t-1} < 1$  for  $x > 0$ . Thus, the function  $g$  is strictly increasing and  $g(x) > g(0) = 0$  on  $(0, \infty)$ , which implies  $(-1)^n f^{(n)}(x) > 0$  for  $x > 0$  and  $n = 0, 1, 2, \dots$ . The proof is complete.  $\square$

*Remark 1.* After this paper was finalized, a similar result in [13] was found: The function  $1 + \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1)$  is strictly completely monotone in  $(-1, \infty)$  and tends to 1 as  $x \rightarrow -1$  and to 0 as  $x \rightarrow \infty$ . This property is derived from the following integral representation:

$$\ln \Gamma(x+1) = x \ln(x+1) - x + \int_0^\infty \left( \frac{1}{t} - \frac{1}{e^t-1} \right) e^{-t} \frac{1-e^{-xt}}{t} dt. \quad (10)$$

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