AN ERDÖS-MORDELL TYPE INEQUALITY ON THE TRIANGLE

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Abstract. In this short note, we give a new Erdös-Mordell type inequality on the triangle: for any point P inside the triangle ABC, then

$$\frac{r_a}{r_1}x^2+\frac{r_b}{r_2}y^2+\frac{r_c}{r_3}z^2\geqslant 2\left(yz\frac{r_a}{R_a}+zx\frac{r_b}{R_b}+xy\frac{r_c}{R_c}\right)$$

where x,y,z are three real numbers, r_a,r_b,r_c the radiuses of escribed circle for the sides BC,CA,AB, R_a,R_b,R_c the circumradiuses of triangles BPC,CPA,APB and r_1,r_2,r_3 the distances from P to BC,CA,AB. With equality holding if and only if $x=y=z$, and P is the circumcenter of equilateral triangle ABC.

1. Introduction

Throughout the paper we assume A,B,C the angles of triangle ABC, a,b,c the sides, s the semi-perimeter, r_a,r_b,r_c the radius of escribed circle for the sides BC,CA,AB, R_a,R_b,R_c the circumradiuses of triangles BPC,CPA,APB, R_1,R_2,R_3 the distances from P to A,B,C and r_1,r_2,r_3 the distances from P to BC,CA,AB on any point P inside the triangle ABC.

The following inequality (1.1) is somewhat more sophisticated than the ones we have seen so far, but is nonetheless useful. It was conjectured by the Hungarian mathematician and problemist P.Erdös in 1935 and first proved by L.Mordell in the same year.

Theorem 1.1. *For any point P inside the triangle ABC, the sum of the distances from P to A,B,C is at least twice the sum of the distances from P to BC,CA,AB. that is:*

$$R_1+R_2+R_3\geqslant 2\left(r_1+r_2+r_3\right), \tag{1.1}$$

with equality holding if and only if triangle ABC is equilateral and P is its center.

Inequality (1.1) is called Erdös-Mordell inequality [1].

In the paper [2], J.Wolstenolme gave a well-known three-variable quadratic inequality (1.3), there is broader applications in geometric inequality, it is a forceful tool of research geometric inequality, and it in geometric inequality as AM-GM inequality in analytic inequality as important.

Theorem 1.2. *If x,y,z are three real numbers, then in every triangle ABC, we have*

$$x^2+y^2+z^2\geqslant 2yz\cos A+2zx\cos B+2xy\cos C, \tag{1.2}$$

with equality holding if and only if $x:y:z=\sin A:\sin B:\sin C$.

By using Wolstenlme's inequality, D.S.Mitrnović etc noted some generalizations of Erdös-Mordell inequality in 1989. Among their results are the following theorem for three-variable quadratic Erdös-Mordell type inequality in [3]:

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Theorem 1.3. *If x, y, z are three real numbers, then for any point P inside the triangle ABC, we have*

$$x^2R_1 + y^2R_2 + z^2R_3 \geqslant 2\left(yzr_1 + zxr_2 + xyr_3\right), \tag{1.3}$$

with equality holding if and only if triangle $x = y = z$ and P is the center of equilateral triangle ABC.

In this short note, we give a new three-variable quadratic Erdös-Mordell type inequality.

2. Main Result

In order to prove Theorem2.1 below, we require the following two lemmas. Lemma2.1 is corollary of Wolstenlme's inequality (1.2), and Lemma2.2 is noted by O.Bottema in [1].

Lemma 2.1. *If x, y, z are three real numbers, then for any point P inside the triangle ABC, we have*

$$bcx^2 + cay^2 + abz^2 \geqslant 4\left[yz(s-b)(s-c) + zx(s-c)(s-a) + xy(s-a)(s-b)\right], \tag{2.1}$$

with equality holding if and only if $x = y = z$ and triangle ABC is equilateral one.

Proof. Firstly, alter $A \to \dfrac{\pi - A}{2}, B \to \dfrac{\pi - B}{2}, C \to \dfrac{\pi - C}{2}$ in Theorem1.2, we obtain

$$x^2 + y^2 + z^2 \geqslant 2yz\sin\frac{A}{2} + 2zx\sin\frac{B}{2} + 2xy\sin\frac{C}{2}, \tag{2.2}$$

with equality holding if and only if $x : y : z = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$.

Secondly, alter $x \to \dfrac{x}{\sqrt{a}}, y \to \dfrac{y}{\sqrt{b}}, z \to \dfrac{z}{\sqrt{c}}$ in inequality (2.2), and using the facts that $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $2\sqrt{(s-b)(s-c)} \leqslant a$, and another four formulas for B, C, b and c, we have

$$\begin{aligned}
\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} &\geqslant 2\left[\frac{yz\sin\frac{A}{2}}{\sqrt{bc}} + \frac{zx\sin\frac{B}{2}}{\sqrt{ca}} + \frac{xy\sin\frac{C}{2}}{\sqrt{ab}}\right]\\
&= 2\left[\frac{yz\sqrt{(s-b)(s-c)}}{bc} + \frac{zx\sqrt{(s-c)(s-a)}}{ca} + \frac{xy\sqrt{(s-a)(s-b)}}{ab}\right]\\
&\geqslant \frac{4}{abc}\left[yz(s-b)(s-c) + zx(s-c)(s-a) + xy(s-a)(s-b)\right],
\end{aligned}$$

Rearranging we get inequality (2.1), with equality holding if and only if $x = y = z$ and triangle ABC is equilateral one. The proof of Lemma2.1 is completed. ∎

Remark 2.1. *Let $a = v + w, b = w + u, c = u + v$, then we have inequality*

$$(w+u)(u+v)x^2 + (u+v)(v+w)y^2 + (v+w)(w+u)z^2 \geqslant 4\left[vwyz + wuzx + uvxy\right], \tag{2.3}$$

where x, y, z are three real numbers, and $u, v, w > 0$, and with equality holding if and only if $x = y = z$ and $u = v = w$.

Lemma 2.2. *For any point P inside the triangle ABC, we have*

$$r_2 + r_3 \leqslant 2R_1\sin\frac{A}{2}, \tag{2.4}$$

with equality holding if and only if P is the center of equilateral triangle ABC.

Remark 2.2. *By using AM-GM inequality, and from*

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

and

$$R_1^2=\frac{2R_bR_cr_2r_3}{R_ar_1},$$

then inequality (2.4) *become*

$$bcr_2r_3\leqslant(s-b)(s-c)R_1^2 \tag{2.5}$$

or

$$bcR_ar_1\leqslant 2(s-b)(s-c)R_bR_c, \tag{2.6}$$

with both equalities holding if and only if P is the center of equilateral triangle ABC.

Theorem 2.1. *If x, y, z are three real numbers, then for any point P inside the triangle ABC, we have*

$$\frac{r_a}{r_1}x^2+\frac{r_b}{r_2}y^2+\frac{r_c}{r_3}z^2\geqslant 2\left(yz\frac{r_a}{R_a}+zx\frac{r_b}{R_b}+xy\frac{r_c}{R_c}\right), \tag{2.7}$$

with equality holding if and only if $x=y=z$, and P is the center of equilateral triangle ABC.

Proof. Utilizing the facts that $r_a=\frac{rs}{s-a}$, (2.6) and another four formulas for r_2, r_3, r_b and r_c, we have

$$\begin{aligned}&2\,(rs)^2\left[\frac{r_a}{r_1}x^2+\frac{r_b}{r_2}y^2+\frac{r_c}{r_3}z^2\right]\\ &=2r_ar_br_c\left[\frac{(s-b)(s-c)}{r_1}x^2+\frac{(s-c)(s-a)}{r_2}y^2+\frac{(s-a)(s-b)}{r_3}z^2\right]\\ &\geqslant r_ar_br_c\left[\frac{bcR_a}{R_bR_c}x^2+\frac{caR_b}{R_cR_a}y^2+\frac{abR_c}{R_aR_b}z^2\right],\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&(rs)^2\left(yz\frac{r_a}{R_a}+zx\frac{r_b}{R_b}+xy\frac{r_c}{R_c}\right)\\ &=r_ar_br_c\left[\frac{(s-b)(s-c)}{R_a}yz+\frac{(s-c)(s-a)}{R_b}zx+\frac{(s-a)(s-b)}{R_c}xy\right].\end{aligned} \tag{2.9}$$

Alter

$$x\to x\sqrt{\frac{R_a}{R_bR_c}},\, y\to y\sqrt{\frac{R_b}{R_cR_a}},\, z\to z\sqrt{\frac{R_c}{R_aR_b}}$$

in inequality (2.1), we get

$$\frac{bcR_a}{R_bR_c}x^2+\frac{caR_b}{R_cR_a}y^2+\frac{abR_c}{R_aR_b}z^2\geqslant 4\left[\frac{(s-b)(s-c)}{R_a}yz+\frac{(s-c)(s-a)}{R_b}zx+\frac{(s-a)(s-b)}{R_c}xy\right] \tag{2.10}$$

Combining expression (2.8)-(2.10), the inequality (2.7) are proved, and with equality holding if and only if $x=y=z$, and P is the center of equilateral triangle ABC. The proof of Theorem2.1 is completed. ∎

3. Two Open Questions

We conclude the paper by asking the following two open questions that we have proved by using computer:

Open Question 3.1. *For any point P inside the triangle ABC, if $0 < k \leqslant 4$, then prove or disprove that*

$$\frac{r_a}{r_1^k}x^2 + \frac{r_b}{r_2^k}y^2 + \frac{r_c}{r_3^k}z^2 \geqslant 2^k\left(yz\frac{r_a}{R_a^k} + zx\frac{r_b}{R_b^k} + xy\frac{r_c}{R_c^k}\right) \tag{3.1}$$

Open Question 3.2. *For any point P inside the triangle ABC, if $0 < k \leqslant 5$, then prove or disprove that*

$$\left(\frac{r_a}{r_1}\right)^k + \left(\frac{r_b}{r_2}\right)^k + \left(\frac{r_c}{r_3}\right)^k \geqslant 2^k\left[\left(\frac{r_a}{R_a}\right)^k + \left(\frac{r_b}{R_b}\right)^k + \left(\frac{r_c}{R_c}\right)^k\right] \tag{3.2}$$

References

[1] O.Bottema, R.Z.Djordjević, R.R.Janić, D.S. Mitrinović, and P.M. Vasić. *Geometric Inequalities.* Wolters-Noordho-Publishing, Groningen, 1969.

[2] J.Wolstenolme. *A Book of Mathenmatical Problems on Subjects Included in the Cambridge Course.* London and Cambridge, 1867.

[3] D.S.Mitrinovič, J.E.Pečarć and V.Voloneć. *Recent Advances in Geometric Inequalities.* 1989.

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