AN ERDÖS-MORDELL TYPE INEQUALITY ON THE TRIANGLE

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Abstract. In this short note, we give a new Erdős-Mordell type inequality on the triangle: for any point \( P \) inside the triangle \( ABC \), then

\[
\frac{r_a}{r_1}x^2 + \frac{r_b}{r_2}y^2 + \frac{r_c}{r_3}z^2 \geq 2 \left( yz \frac{r_a}{R_a} + zx \frac{r_b}{R_b} + xy \frac{r_c}{R_c} \right)
\]

where \( x, y, z \) are three real numbers, \( r_a, r_b, r_c \) the radii of escribed circle for the sides \( BC, CA, AB \), \( R_a, R_b, R_c \) the circumradiuses of triangles \( BPC, CPA, APB \) and \( r_1, r_2, r_3 \) the distances from \( P \) to \( BC, CA, AB \). With equality holding if and only if \( x = y = z \), and \( P \) is the circumcenter of equilateral triangle \( ABC \).

1. Introduction

Throughout the paper we assume \( A, B, C \) the angles of triangle \( ABC \), \( a, b, c \) the sides, \( s \) the semi-perimeter, \( r_a, r_b, r_c \) the radius of escribed circle for the sides \( BC, CA, AB \), \( R_a, R_b, R_c \) the circumradiuses of triangles \( BPC, CPA, APB \) and \( r_1, r_2, r_3 \) the distances from \( P \) to \( BC, CA, AB \). With equality holding if and only if \( x = y = z \), and \( P \) is the circumcenter of equilateral triangle \( ABC \).

Theorem 1.1. For any point \( P \) inside the triangle \( ABC \), the sum of the distances from \( P \) to \( A,B,C \) is at least twice the sum of the distances from \( P \) to \( BC,CA,AB \). that is:

\[
R_1 + R_2 + R_3 \geq 2 \left( r_1 + r_2 + r_3 \right),
\]

with equality holding if and only if triangle \( ABC \) is equilateral and \( P \) is its center.

Inequality (1.1) is called Erdős-Mordell inequality [1].

In the paper [2], J.Wolstenolme gave a well-known three-variable quadratic inequality (1.2), there is broader applications in geometric inequality, it is a forceful tool of research geometric inequality, and it in geometric inequality as AM-GM inequality in analytic inequality as important.

Theorem 1.2. If \( x, y, z \) are three real numbers, then in every triangle \( ABC \), we have

\[
x^2 + y^2 + z^2 \geq 2yz \cos A + 2zx \cos B + 2xy \cos C,
\]

with equality holding if and only if \( x : y : z = \sin A : \sin B : \sin C \).

By using Wolstenolme’s inequality, D.S.Mitrnović etc noted some generalizations of Erdős-Mordell inequality in 1989. Among their results are the following theorem for three-variable quadratic Erdős-Mordell type inequality in [3]:

\[
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Theorem 1.3. If $x, y, z$ are three real numbers, then for any point $P$ inside the triangle $ABC$, we have
\begin{equation}
(1.3) \quad x^2R_1 + y^2R_2 + z^2R_3 \geq 2(yzr_1 + zxr_2 + xyr_3),
\end{equation}
with equality holding if and only if triangle $x = y = z$ and $P$ is the center of equilateral triangle $ABC$.

In this short note, we give a new three-variable quadratic Erdős-Mordell type inequality.

2. Main Result

In order to prove Theorem 2.1 below, we require the following two lemmas. Lemma 2.1 is corollary of Wolstenholme's inequality (1.2), and Lemma 2.2 is noted by O. Bottema in [1].

Lemma 2.1. If $x, y, z$ are three real numbers, then for any point $P$ inside the triangle $ABC$, we have
\begin{equation}
(2.1) \quad bcx^2 + cay^2 + abz^2 \geq 4[yz(s-b)(s-c) + zx(s-c)(s-a) + xy(s-a)(s-b)],
\end{equation}
with equality holding if and only if $x = y = z$ and triangle $ABC$ is equilateral one.

Proof. Firstly, alter $A \to \frac{\pi - A}{2}, B \to \frac{\pi - B}{2}, C \to \frac{\pi - C}{2}$ in Theorem 1.2, we obtain
\begin{equation}
(2.2) \quad x^2 + y^2 + z^2 \geq 2yz \frac{A}{2} + 2zx \frac{B}{2} + 2xy \frac{C}{2},
\end{equation}
with equality holding if and only if $x : y : z = \cos \frac{A}{2} : \cos \frac{B}{2} : \cos \frac{C}{2}$.

Secondly, alter $x \to \frac{x}{\sqrt{a}}, y \to \frac{y}{\sqrt{b}}, z \to \frac{z}{\sqrt{c}}$ in inequality (2.2), and using the facts that $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, 2\sqrt{(s-b)(s-c)} \leq a$, and another four formulas for $B, C, b$ and $c$, we have
\[
\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} \geq 2 \left[ \frac{yz \sin \frac{A}{2}}{\sqrt{bc}} + \frac{zx \sin \frac{B}{2}}{\sqrt{ca}} + \frac{xy \sin \frac{C}{2}}{\sqrt{ab}} \right]
\geq 2 \left[ \frac{yz \sqrt{(s-b)(s-c)}}{bc} + \frac{zx \sqrt{(s-c)(s-a)}}{ca} + \frac{xy \sqrt{(s-a)(s-b)}}{ab} \right]
\geq \frac{4}{abc} [yz(s-b)(s-c) + zx(s-c)(s-a) + xy(s-a)(s-b)],
\]
Rearranging we get inequality (2.1), with equality holding if and only if $x = y = z$ and triangle $ABC$ is equilateral one. The proof of Lemma 2.1 is completed.

Remark 2.1. Let $a = v + w, b = w + u, c = u + v$, then we have inequality
\begin{equation}
(2.3) \quad (w+u)(u+v)x^2 + (u+v)(v+w)y^2 + (v+w)(w+u)z^2 \geq 4[uvw + wuz + wxy],
\end{equation}
where $x, y, z$ are three real numbers, and $u, v, w > 0$, and with equality holding if and only if $x = y = z$ and $u = v = w$.

Lemma 2.2. For any point $P$ inside the triangle $ABC$, we have
\begin{equation}
(2.4) \quad r_2 + r_3 \leq 2R_1 \sin \frac{A}{2},
\end{equation}
with equality holding if and only if $P$ is the center of equilateral triangle $ABC$. 

Remark 2.2. By using AM-GM inequality, and from
\[ \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \]
and
\[ R_1^2 = \frac{2R_bR_c r_2 r_3}{R_a r_1}, \]
then inequality (2.4) become
\[ (2.5) \quad bc r_2 r_3 \leq (s-b)(s-c) R_1^2 \]
or
\[ (2.6) \quad bc R_a r_1 \leq 2(s-b)(s-c) R_b R_c, \]
with both equalities holding if and only if \( P \) is the center of equilateral triangle \( ABC \).

Theorem 2.1. If \( x, y, z \) are three real numbers, then for any point \( P \) inside the triangle \( ABC \), we have
\[ (2.7) \quad \frac{r_a x^2}{r_1} + \frac{r_b y^2}{r_2} + \frac{r_c z^2}{r_3} \geq 2 \left( yz \frac{r_a}{R_a} + zx \frac{r_b}{R_b} + xy \frac{r_c}{R_c} \right), \]
with equality holding if and only if \( x = y = z \), and \( P \) is the center of equilateral triangle \( ABC \).

Proof. Utilizing the facts that \( r_a = \frac{rs}{s-a}, (2.6) \) and another four formulas for \( r_2, r_3, r_b \) and \( r_c \), we have
\[ (2.8) \quad 2 \left( rs \right)^2 \left[ \frac{r_a x^2}{r_1} + \frac{r_b y^2}{r_2} + \frac{r_c z^2}{r_3} \right] \]
\[ = 2r_a r_b r_c \left[ \frac{(s-b)(s-c)}{r_1} x^2 + \frac{(s-c)(s-a)}{r_2} y^2 + \frac{(s-a)(s-b)}{r_3} z^2 \right] \]
\[ \geq r_a r_b r_c \left[ \frac{bc R_a}{R_b R_c} x^2 + \frac{ca R_b}{R_c R_a} y^2 + \frac{ab R_c}{R_a R_b} z^2 \right], \]
and
\[ (2.9) \quad \left( rs \right)^2 \left( yz \frac{r_a}{R_a} + zx \frac{r_b}{R_b} + xy \frac{r_c}{R_c} \right) \]
\[ = r_a r_b r_c \left[ \frac{(s-b)(s-c)}{R_a} yz + \frac{(s-c)(s-a)}{R_b} zx + \frac{(s-a)(s-b)}{R_c} xy \right]. \]
Alter
\[ x \to x \sqrt{\frac{R_a}{R_b R_c}}, \quad y \to y \sqrt{\frac{R_b}{R_c R_a}}, \quad z \to z \sqrt{\frac{R_c}{R_a R_b}} \]
in inequality (2.1), we get
\[ (2.10) \quad \frac{bc R_a}{R_b R_c} x^2 + \frac{ca R_b}{R_c R_a} y^2 + \frac{ab R_c}{R_a R_b} z^2 \geq 4 \left[ \frac{(s-b)(s-c)}{R_a} yz + \frac{(s-c)(s-a)}{R_b} zx + \frac{(s-a)(s-b)}{R_c} xy \right] \]
Combining expression (2.8)-(2.10), the inequality (2.7) are proved, and with equality holding if and only if \( x = y = z \), and \( P \) is the center of equilateral triangle \( ABC \). The proof of Theorem 2.1 is completed.
3. TWO OPEN QUESTIONS

We conclude the paper by asking the following two open questions that we have proved by using computer:

Open Question 3.1. For any point $P$ inside the triangle $ABC$, if $0 < k \leq 4$, then prove or disprove that

$$\frac{r_a}{r_1} x^2 + \frac{r_b}{r_2} y^2 + \frac{r_c}{r_3} z^2 \geq 2^k \left( \frac{y z r_a}{R_a} + z x \frac{r_b}{R_b} + x y \frac{r_c}{R_c} \right)$$

Open Question 3.2. For any point $P$ inside the triangle $ABC$, if $0 < k \leq 5$, then prove or disprove that

$$\left( \frac{r_a}{r_1} \right)^k + \left( \frac{r_b}{r_2} \right)^k + \left( \frac{r_c}{r_3} \right)^k \geq 2^k \left[ \left( \frac{r_a}{R_a} \right)^k + \left( \frac{r_b}{R_b} \right)^k + \left( \frac{r_c}{R_c} \right)^k \right]$$

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