INTEGRAL CHARACTERIZATIONS FOR EXPONENTIAL STABILITY OF SEMIGROUPS AND EVOLUTION FAMILIES ON BANACH SPACES

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Abstract. Let $X$ be a real or complex Banach space and $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ be a strongly continuous and exponentially bounded evolution family on $X$. Let $J$ be a non-negative functional on the positive cone of the space of all real-valued locally bounded functions on $\mathbb{R}_+ := [0, \infty)$. We suppose that $J$ satisfies some extra-assumptions. Then the family $\mathcal{U}$ is uniformly exponentially stable provided that for every $x \in X$ we have:

$$\sup_{s \geq 0} J(||U(s + \cdot, s)x||) < \infty.$$ 

This result is connected to the uniform asymptotic stability of the well-posed linear and non-autonomous abstract Cauchy problem

$$\begin{cases}
  \dot{u}(t) = A(t)u(t), & t \geq s \geq 0, \\
  u(s) = x & x \in X.
\end{cases}$$

In the autonomous case, i.e. when $U(t,s) = T(t-s)$ for some strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ we obtain the well-known theorems of Datko, Littman, Neerven, Pazy and Rolewicz.

1. Introduction

Let $X$ be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on $X$. The norm of vectors in $X$ and operators in $\mathcal{L}(X)$ will be denoted by $|| \cdot ||$. Let $\mathcal{T} := \{T(t)\}_{t \geq 0}$ be a semigroup of operators acting on $X$, that is, $T(t) \in \mathcal{L}(X)$ for every $t \geq 0$, $T(0) = I$ the identity operator in $\mathcal{L}(X)$ and $T(t + s) = T(t) \circ T(s)$ for every $t \geq 0$ and $s \geq 0$. The semigroup $\mathcal{T}$ is called strongly continuous if for each $x \in X$ the map $t \mapsto T(t)x : [0, \infty) \rightarrow X$ is continuous. Every strongly continuous semigroup is locally bounded, that is, there exist $h > 0$ and $M \geq 1$ such that $||T(t)|| \leq M$ for all $t \in [0, h]$. It is easy to see that every locally bounded semigroup is exponentially bounded, that is, there exist $\omega \in \mathbb{R}_+$ and $M \geq 1$ such that

$$||T(t)|| \leq Me^{\omega t} \text{ for all } t \geq 0.$$ 

It is well-known that if $\mathcal{T} = \{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup on a Banach space $X$ and there exists $p \in [1, \infty)$ such that for each $x \in X$ one has

$$\int_0^\infty ||T(t)x||^p dt = M(p, x) < \infty,$$

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then \( T \) is exponentially stable, that is, its uniform growth bound
\[
\omega_0(T) := \inf_{t > 0} \frac{\ln \|T(t)\|}{t},
\]
is negative. This result is usually referred to as the Datko-Pazy theorem, see [6, 12]. An important application of the Datko-Pazy theorem can be found in [16]. A quantitative version of this theorem states that if \( M(p, x) \) from (1.1) is equal to \( C\|x\|^p \), where \( C \) is some positive constant, then \( \omega_0(T) < -\frac{1}{pC} \). See [10, Theorem 3.1.8] for details. An important generalization of the Datko-Pazy theorem was given by S. Rolewicz [13]. In the autonomous case the Rolewicz theorem reads as follows. Let \( T = \{T(t)\}_{t \geq 0} \) be a strongly continuous semigroup on a Banach space \( X \).

If there exists a continuous non-decreasing function \( \phi : [0, \infty) \to [0, \infty) \) such that \( \phi(t) > 0 \) for each \( t > 0 \) and if
\[
\int_0^\infty \phi(\|T(t)x\|)dt := M_\phi(x) < \infty \text{ for each } x \in X,
\]
then the semigroup \( T \) is exponentially stable. The same result was obtained independently by Littman [8]. In particular, from Rolewicz’s theorem it follows that the Datko-Pazy theorem remains valid for \( p \in (0, 1) \). The condition (1.1) indicates that for each \( x \in X \) the map \( t \mapsto T(t)x \) belongs to \( L^p(\mathbb{R}_+) \). Jan van Neerven has shown in [9] that a strongly continuous semigroup \( T \) on \( X \) is uniformly exponentially stable if there exists a Banach function space over \( \mathbb{R}_+ := [0, \infty) \) with the property that
\[
\lim_{t \to \infty} \|1_{[0, t]}\|_E = \infty,
\]
such that
\[
\|T(\cdot)x\| \in E \text{ for every } x \in X.
\]
He has also shown that the autonomous variant of the Rolewicz theorem can be derived from his result by taking for \( E \) a suitable Orlicz space over \( \mathbb{R}_+ \). In another paper, [11], Jan van Neerven has come to the same conclusion by replacing either (1.1), (1.2) or (1.4) by the hypothesis that the set of all \( x \in X \) for which the following inequality holds
\[
J(\|T(\cdot)x\|) < \infty,
\]
is of the second category in \( X \). Here \( J \) is a certain lower semi-continuous functional as defined in Theorem 2 from [11]. The proof of this latter result is based on a non-trivial result from operator theory given by V. Mićić, see Lemma 1 from [11], for further details. We give here a surprisingly simple proof for a result of the same type, moreover, we do not require the lower semi-continuity of \( J \).

In order to introduce some non-autonomous results of this type we recall the notion of an evolution family.

A family \( \mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0} \) of bounded linear operators on a Banach space \( X \) is a strongly continuous evolution family if

1. \( U(t, t) = I \) and \( U(r, s) = U(t, s) \) for \( t \geq r \geq s \geq 0 \).
2. The map \( t \mapsto U(t, s)x : [s, \infty) \to X \) is continuous for every \( s \geq 0 \) and every \( x \in X \).

The family \( \mathcal{U} \) is exponentially bounded if there exist \( \omega \in \mathbb{R} \) and \( M_\omega \geq 0 \) such that
\[
\|U(t, s)\| \leq M_\omega e^{\omega(t-s)} \text{ for } t \geq s \geq 0.
\]
Then \( \omega(\mathcal{U}) := \inf \{ \omega \in \mathbb{R} : \text{there is } M_{\omega} \geq 0 \text{ such that } (1.5) \text{ holds} \} \) is called the growth bound of \( \mathcal{U} \). The family \( \mathcal{U} \) is uniformly exponentially stable if its growth bound is negative.

In [1] it is proved that an exponentially bounded evolution family \( \mathcal{U} \) is uniformly exponentially stable if there exists a solid space \( E \) satisfying (1.3) such that for each \( s \geq 0 \) and each \( x \in X \) the map \( ||U(s + t, s)x|| \) belongs to \( E \) and

\[
\sup_{t \geq 0} ||U(s + t, s)x|| := K(x) < \infty.
\]

The non-autonomous Datko theorem, [7], follows from this by taking \( E = L^p(\mathbb{R}_+) \). The theorem of Rolewicz, [14], can be derived as well by taking for \( E \) a suitable Orlicz space over \( \mathbb{R}_+ \), see Theorem 2.10 from [1]. New guidelines about the proof of the Datko theorem can be found in [5] and [15]. In this paper we propose a more natural generalization of the theorems of Datko and Rolewicz which can also be extended to the general non-autonomous case. For some recently obtained autonomous or periodic versions of the above; see [4], [11].

2. A Generalization of the Datko-Pazy Theorem

We begin by stating and proving two lemmas which are useful later.

**Lemma 1.** Let \( \mathcal{T} = \{T(t) : t \geq 0\} \) be a locally bounded semigroup on a Banach space \( X \). If for each \( x \in X \) there exists \( t(x) > 0 \) such that \( T(t(x))x = 0 \), then \( \mathcal{T} \) is uniformly exponentially stable.

**Proof.** It is easy to see that \( \mathcal{T} \) is uniformly bounded. Indeed, if not, then there exists a sequence \( (t_n) \) of positive real numbers with \( t_n \to \infty \) such that \( ||T(t_n)|| \to \infty \). By the Uniform Boundedness Theorem it follows that there exists \( x \in X \) such that \( ||T(t_n)x|| \to \infty \). This is in contradiction to the hypothesis. Now let \( \nu > 0 \). The semigroup \( \{e^{\nu t}T(t)\} \) verifies the hypothesis of the present Lemma and it is uniformly bounded. Finally, we deduce that \( \mathcal{T} \) is uniformly exponentially stable.

**Lemma 2.** Let \( \mathcal{T} = \{T(t)\}_{t \geq 0} \) be a locally bounded semigroup such that for each \( x \in X \) the map \( t \mapsto ||T(t)x|| \) is continuous on \( (0, \infty) \). If there exist a positive \( h \) and \( 0 < q < 1 \) such that for all \( x \in X \) there exists \( t(x) \in (0, h] \) with

\[
||T(t(x))x|| \leq q||x||, \tag{2.1}
\]

then the semigroup \( \mathcal{T} \) is uniformly exponentially stable.

**Proof.** Let \( x \in X \) be fixed and \( t_1 \in (0, h] \) such that \( ||T(t_1)x|| \leq q||x|| \), then there exists \( t_2 \in (0, h] \) such that

\[
||T(t_2 + t_1)x|| \leq q||T(t_1)x|| \leq q^2||x||.
\]

By mathematical induction it is easy to see that there exists a sequence \( (t_n) \), with \( 0 < t_n \leq h \) such that \( ||T(s_n)x|| \leq q^n||x|| \), where \( s_n := t_1 + t_2 + \cdots + t_n \).

If \( s_n \to \infty \), then for each \( t \in [s_n, s_{n+1}] \) we have that \( t < (n + 1)h \) and

\[
||T(t)x|| \leq Mq^n||x|| \leq Me^{-\ln(q)e^{\ln(q)}}||x||,
\]

that is, \( \mathcal{T} \) is exponentially stable.

If the sequence \( (s_n) \) is bounded, let \( t(x) \) be the limit of \( (s_n) \). By the assumption of continuity it follows that \( T(t(x)) = 0 \) and then application of Lemma 2 completes the proof.

We can now state the main result of this section.
Theorem 1. Let $\mathcal{M}_{\text{loc}}([0, \infty))$ be the space of all real valued locally bounded functions on $\mathbb{R}_+ = [0, \infty)$ endowed with the topology of uniform convergence on bounded sets and $\mathcal{M}_{\text{loc}}^+(\mathbb{R}_+)$ its positive cone.

Let $J : \mathcal{M}_{\text{loc}}^+(\mathbb{R}_+) \to [0, \infty]$ be a map with the following properties:

1. $J$ is nondecreasing.
2. For each positive real number $\rho$,
   $$\lim_{t \to \infty} J(\rho \cdot 1_{[0,t]}) = \infty.$$

If $T$ is a semigroup on a Banach space $X$ as in Lemma 2 such that
$$\sup_{||x|| \leq 1} J(||T(\cdot)x||) := K_J < \infty,$$
then $T$ is exponentially stable.

Proof. Suppose that $T$ is not exponentially stable. For all $h > 0$ and all $0 < q < 1$ then there exists $x_0 \in X$ of norm one such that
$$||T(t)x_0|| > q$$
for every $t \in [0, h]$, as proved in Lemma 2. It follows then that
$$K_J \geq J(||T(\cdot)x_0||) \geq J(q \cdot 1_{[0,h]})$$
which contradicts (2.2).

Corollary 1. Let $T = \{T(t)\}_{t \geq 0}$ be a semigroup on a Banach space $X$ as in Lemma 2 and $1 \leq p < \infty$. If (1.1) holds for all $x \in X$ then the semigroup $T$ is exponentially stable.

Proof. For each fixed positive $h$ consider the bounded linear operator
$$x \mapsto T_hx : X \to L^p(\mathbb{R}_+, X)$$
defined by
$$(T_hx)(t) = \begin{cases} T(t)x, & \text{if } 0 \leq t \leq h \\ 0, & \text{if } t > h. \end{cases}$$
For each $x \in X$ we have:
$$||T_hx||_{L^p(\mathbb{R}_+, X)} = \left(\int_0^h ||T(t)x||^p dt\right)^{\frac{1}{p}} \leq M(p, x)^{\frac{1}{p}}.$$
From the Uniform Boundedness Theorem it follows that there exists a positive constant $C_p$ such that
$$||T_hx||_{L^p(\mathbb{R}_+, X)} \leq C_p ||x||$$
for every $x \in X$.

Now it is easy to derive the inequality
$$\sup_{||x|| \leq 1} \int_0^\infty ||T(t)x||^p dt \leq K_p < \infty,$$
where $K_p$ is a positive constant. Choose $J(f) := \int_0^\infty f(t)^p dt$, apply Theorem 1 and the proof is complete.

Corollary 2. Let $T = \{T(t)\}_{t \geq 0}$ be a semigroup on a Banach space $X$ as in the above Lemma 2. If there exists a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) > 0$ for each $t > 0$ and (1.2) holds then the semigroup $T$ is exponentially stable.
Proof. Seemingly we could proceed as in the proof of Corollary 1 but, however, we cannot directly apply the Uniform Boundedness Theorem. First we prove that the semigroup $T$ is uniformly bounded. In fact, this has been done in [2] in the general framework of the evolution families. For the sake of completeness we mention some steps of that proof for this particular case. We may assume that $\phi(0) = 0$, $\phi(1) = 1$ and that $\phi$ is strictly increasing on $\mathbb{R}_+$, if not, we replace $\phi$ by some multiple of the function

$$t \mapsto \tilde{\phi}(t) := \begin{cases} \int_0^t \phi(u)du, & \text{if } 0 \leq t \leq 1 \\ \frac{\int_0^t \phi(u)du}{at + b}, & \text{if } t > 1, \end{cases}$$

where $a := \int_0^1 \phi(u)du$.

Let $x \in X$ be fixed, $N$ be a positive integer such that $M_\phi(x) < N$ and let $t \geq N$. For each $\tau \in [t - N, t]$ and all $u \geq 0$ we have:

$$e^{-\omega N} \mathbf{1}_{[t-N,t]}(u)||T(t)x|| \leq e^{-\omega(t-\tau)} \mathbf{1}_{[t-N,t]}(u)||T(t-\tau)T(\tau)x|| \leq M||T(u)x||$$

and then

$$(N\phi)(\frac{||T(t)x||}{Me^{\omega N}}) \leq \int_{t-N}^t \phi \left(\frac{||T(s)x||}{Me^{\omega N}}\right) du \leq M\phi(x).$$

Hence $||T(t)x|| \leq Me^{\omega N}M\phi(x)$ for every $t \geq N$, and so the semigroup $T$ is uniformly bounded.

From [11] Lemma 3.2.1 it follows that there exists an Orlicz’s space $E$ satisfying (1.3) such that for each $x \in X$ which satisfies (1.2), the map $t \mapsto T(t)x$ belongs to $E$. For each non-negative, bounded and measurable real-valued function $f$ we put $J(f) := \sup_{t \geq 0} \|1_{[0,t]}(s)\|_E$, giving,

$$J(||T(\cdot)x||) = \sup_{t \geq 0} \|1_{[0,t]}||T(\cdot)x||_E \leq ||T(\cdot)x||_E < \infty,$$

for every $x \in X$.

Arguing as in Corollary 1 it follows that there exists a positive constant $K_\phi$, independent of $x$, such that

$$\sup_{||x|| \leq 1} J(||T(\cdot)x||) < K_\phi < \infty.$$

Application of Theorem 1 completes the proof. $\blacksquare$

3. The Non-autonomous Case

We state and prove two lemmas that will be used in the sequel.

**Lemma 3.** Let $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ be an exponentially bounded evolution family on a Banach space $X$. If for each $x \in X$ there exists $t(x) > 0$ such that $U(s+t(x),s)x = 0$ for every $s \geq 0$ then the family $\mathcal{U}$ is uniformly exponentially stable.

**Proof.** First we prove that there exists $M > 0$ such that

$$\sup_{s \geq 0} ||U(s+t,s)|| \leq M \text{ for all } t \geq 0.$$

Indeed, if we suppose the contrary then there exists a sequence $(t_n)$ of positive real numbers with $t_n \to \infty$ such that $\lim_{n \to \infty} ||U(s+t_n,s)|| = \infty$. From the Uniform Boundedness Theorem it follows that there exists $x \in X$ such that $||U(s+t_n,s)x|| \to \infty$ when $n \to \infty$ which is in contradiction to the hypothesis. We now observe that
the family \( \{ e^{\nu(t-s)}U(t,s) \}_{t \geq s \geq 0} \) verifies the hypothesis of the present lemma and then
\[
||U(t,s)|| \leq Me^{-\nu(t-s)} \text{ for all } t \geq s,
\]
i.e. the assertion holds.

**Lemma 4.** Let \( \mathcal{U} = \{ U(t,s) \}_{t \geq s \geq 0} \) be an exponentially bounded evolution family on a Banach space \( X \) such that for each \( y \in X \) and each \( s \geq 0 \) the map
\[
t \mapsto ||U(s+t,s)y|| : \mathbb{R}_+ \to \mathbb{R}_+
\]
is continuous on \((0,\infty)\). If there exist positive real numbers \( h \) and \( q < 1 \) such that for every \( x \in X \) there exists \( t(x) \in (0,h] \) with the property that
\[
\sup_{s \geq 0} ||U(s+t(s),s)x|| \leq q||x||,
\]
then the family \( \mathcal{U} \) is exponentially stable.

**Proof.** Is similar to that of Lemma 2 and so we omit the details.

**Theorem 2.** Let \( \mathcal{U} = \{ U(t,s) \}_{t \geq s \geq 0} \) be an evolution family on a Banach space \( X \) as in the above Lemma 4 and let \( J \) be a functional as in Theorem 1. If there exists \( r > 0 \) such that
\[
(3.1) \quad \sup_{s \geq 0} \sup_{||x|| \leq r} J(||U(s,\cdot,s)x||) := L(J,r) < \infty,
\]
then the evolution family \( \mathcal{U} \) is uniformly exponentially stable.

**Proof.** Suppose that the family \( \mathcal{U} \) is not uniformly exponentially stable. Under such circumstances as proved in Lemma 4, for every positive real number \( h \) and every \( q \in (0,1) \) there exist \( x_0 \in X \) of norm one and \( s_0 \geq 0 \) such that
\[
||U(s_0 + t, s_0)x_0|| > q \text{ for all } t \in [0,h].
\]
Thus
\[
L(J,r) \geq J(||U(s_0 + t, s_0)x_0||) \geq J(qr \cdot 1_{[0,h]})
\]
for each \( h > 0 \), which contradicts (3.1).

**Theorem 3.** Let \( J \) be as in the above Theorem 1. We suppose, in addition, that \( J \) is lower semi-continuous and convex in the sense of Jensen (or sub-additive, that is, \( J(f+g) \leq J(f) + J(g) \) for every \( f \) and \( g \) in \( \mathcal{M}_{loc}(\mathbb{R}_+) \)). Let \( \mathcal{U} \) be an evolution family as in the Lemma 4. If the set \( X \) of all \( x \in X \) for which
\[
\sup_{s \geq 0} J(||U(s,\cdot,s)x||) < \infty
\]
is of the second category in \( X \), then the family \( \mathcal{U} \) is uniformly exponentially stable.

**Proof.** Let \( s \geq 0 \), be fixed. The map \( x \mapsto ||U(s,\cdot,s)x|| : X \to \mathcal{M}_{loc}(\mathbb{R}_+) \) is continuous. As a consequence, the map
\[
x \mapsto \Phi_{s}(x) := J(||U(s,\cdot,s)x||) : X \to [0,\infty]
\]
is lower semi-continuous as well. For each positive integer \( k \), the set
\[
X_k(s) := \{ x \in X : J(||U(s,\cdot,s)x||) \leq k \}
\]
is closed, because it is the reverse image of the real closed interval \([0, k]\) by the map \(\Phi_s\). It is clear that the set

\[
X_k := \left\{ x \in X : \sup_{s \geq 0} J(||U(s + \cdot, s)x||) \leq k \right\} = \bigcap_{s \geq 0} X_k(s)
\]

is also closed and moreover that \(X\) is the union of all sets \(X_k\). Because \(X\) is of the second category in \(X\), there exists a set \(X_{k_0}\) whose interior is non empty. Let \(x_0 \in X\) and \(r_0 > 0\) such that \(B(x_0, r_0)\) belongs to \(X_{k_0}\). It is easy to see that \(B(0, \frac{1}{2} r_0)\) belongs to \(X_{k_0}\), that is,

\[
\sup_{s \geq 0} \sup_{||x|| \leq \frac{1}{2} r_0} J(||U(s + \cdot, s)x||) \leq k_0.
\]

Indeed for every \(x \in X\) with \(||x|| \leq r_0\) we have:

\[
J\left(\frac{1}{2} ||U(s + \cdot, s)(x + x_0)||\right) = J\left(\frac{1}{2} ||U(s + \cdot, s)((x + x_0) - x_0)||\right)
\leq J\left(\frac{1}{2} ||U(s + \cdot, s)(x + x_0) + ||U(s + \cdot, s)x_0||\right)
\leq \frac{1}{2} J(||U(s + \cdot, s)(x + x_0)|| + \frac{1}{2} J(||U(s + \cdot, s)x_0||)
\leq k_0.
\]

Application of Theorem 2 completes the proof.

**Corollary 3.** Let \(U = \{U(t, s)\}_{t \geq s \geq 0}\) be an exponentially bounded evolution family on a Banach space \(X\) such that for each \(x \in X\) the map \(t \mapsto ||U(s + t, s)x||\) is continuous on \((0, \infty)\) for every \(s \geq 0\). Consider the following three inequalities:

1. There exists \(p \in [1, \infty)\) such that
\[
\sup_{s \geq 0} \int_0^\infty ||U(s + t, s)x||^p dt < \infty
\]
   for every \(x \in X\).

2. There exists a Banach function space \(E\) satisfying (1.3) such that for each \(s \geq 0\) and each \(x \in X\) the map \(U(s + \cdot, s)x\) belongs to \(E\) and for every \(x \in X\) we have
\[
\sup_{s \geq 0} |||U(s + \cdot, s)x|||_E < \infty.
\]

3. There exists a non-decreasing function \(\phi : [0, \infty) \to [0, \infty)\) with \(\phi(t) > 0\) for each \(t > 0\) such that
\[
\sup_{s \geq 0} \int_0^\infty \phi(||U(s + t, s)x||) dt < \infty
\]
   for every \(x \in X\).

If any one of these statements is true then the family \(U\) is exponentially stable.

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