SEIFFERT MEANS IN A TRIANGLE

ALFRED WITKOWSKI

Abstract. Simple geometric proofs of some old and new inequalities between the Seiffert mean and classical means.

Seiffert introduced his first mean in [3] as

\[ P(x, y) = \begin{cases} \frac{x - y}{2 \arcsin \frac{x - y}{x + y}} & x \neq y, \\ \frac{x}{x + y} & x = y. \end{cases} \]

and proved in [4, 3] that for \( x \neq y \)

\[ G \leq L \leq P \leq I \leq A \]

where

\[ G(x, y) = \sqrt{xy}, \]
\[ L(x, y) = \frac{x - y}{\log x - \log y}, \]
\[ I(x, y) = \frac{1}{e} \left( \frac{x^y}{y^x} \right)^{1/y}, \]
\[ A(x, y) = \frac{x + y}{2} \]

are the geometric, logarithmic, iden tric and arithmetic means. Later in [6] he used series representation to show that

\[ P < A < \frac{\pi}{2} P. \]

and

\[ \frac{3}{P} < \frac{2}{A} + \frac{1}{G}. \]

Sándor in [2] obtained further refinement. Using Pfaff's algorithm he proved that

\[ \frac{A + G}{2} < P < \sqrt{\frac{A + G}{2}} \]

The second Seiffert mean [5] is defined by

\[ T(x, y) = \begin{cases} \frac{x - y}{2 \arctan \frac{x - y}{x + y}} & x \neq y, \\ \frac{x}{x + y} & x = y. \end{cases} \]

The goal of this paper is to give simple geometric proofs of (2), (7), (8) and sharpen the inequality (9). We also use obtain similar inequalities for \( T \).

Date: 9 September 2004.

2000 Mathematics Subject Classification. 26D15.

Key words and phrases. Seiffert mean, power means.
Consider a right triangle $\Delta ABC$ with sides

$$|AB| = \frac{x + y}{2} = A, \quad |AC| = \frac{|x - y|}{2}, \quad |BC| = \sqrt{xy} = G$$

Let $P$ be the intersection point of $AB$ and the circle of radius $|BC|$ centered at $B$. Then

$$\angle B = \arcsin \frac{x - y}{x + y}$$

and

$$P = \frac{|AC|}{\angle B} = \frac{|AC||BC|}{|PC|} \quad \text{(11)}$$

The following equations will be useful:

$$\sin \frac{B}{2} = \sqrt{\frac{1 - \cos B}{2}} = \sqrt{\frac{|AB| - |BC|}{2|AB|}} = \frac{|AC|}{2\sqrt{|AB||AB| + |BC|}} \quad \text{(12)}$$

$$\tan \frac{B}{2} = \sqrt{\frac{1 - \cos B}{1 + \cos B}} = \sqrt{\frac{|AB| - |BC|}{|AB| + |BC|}} = \frac{|AC|}{|AB| + |BC|} \quad \text{(13)}$$

Now we are ready to prove the first theorem:

**Theorem 1.** For $x \neq y$

$$G < P \quad \text{(14)}$$

and there is no constant $c$ satisfying $P < cG$ for all $x, y$. 
Proof. As $|\widehat{PC}| < |AC|$ and $|BC| = G$ (14) follows form (11). On the other hand the ratio

$$\frac{|AC|}{|PC|} > \frac{2 |AC|}{\pi |BC|} = \frac{x - y}{\pi \sqrt{xy}} = \frac{1}{\pi} \left( \sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} \right)$$

can be made as large as we wish, so the ratio $P/G$ cannot be bounded from above. □

Let $PQ$ be the height of the triangle $\Delta PBC$. Then the following inequalities hold:

(15)

$$1 < \frac{|\widehat{PC}|}{|PQ|} < \frac{\pi}{2}$$

which implies

Theorem 2.

$$\frac{2}{\pi} A < P < A$$

Proof. From (15) and (11) we get

$$\frac{2 |AC||BC|}{\pi |PC||PQ|} < P < \frac{|AC||BC|}{|PC||PQ|}$$

and

$$\frac{|AC||BC|}{|PC||PQ|} = \frac{|AC||BP|}{|PQ|} = |AB| = A.$$ 

Another obvious inequality

(16)

$$1 < \frac{|\widehat{PC}|}{|PC|} < \frac{\pi}{2\sqrt{2}}$$

gives

Theorem 3.

$$\frac{2\sqrt{2}}{\pi} \sqrt{\frac{A}{2} \frac{A + G}{2}} < P < \sqrt{\frac{A}{2} \frac{A + G}{2}}$$

Proof. From (16) and (11) we get

$$\frac{2\sqrt{2} |AC||BC|}{\pi |PC|} < P < \frac{|AC||BC|}{|PC|}$$

and

(17)

$$\frac{|AC||BC|}{|PC|} = \frac{|AC|}{2 \sin \frac{B}{2}} = \sqrt{|AB| \frac{|AB| + |BC|}{2}} = \sqrt{\frac{A}{2} \frac{A + G}{2}}$$

by (12). □

In the middle of $\widehat{PC}$ draw a tangent line that meets $BA$ at $F$ and $BC$ at $E$. It is obvious that

(18)

$$\frac{\pi}{4} < \frac{|\widehat{PC}|}{|EF|} < 1$$

and this implies
Theorem 4. (see [1], Cor. 1.11)
\[
\frac{A + G}{2} < P < \frac{4}{\pi} \frac{A + G}{2}
\]

Proof. From (18) and (11) we get
\[
\frac{|AC||BC|}{|EF|} < P < \frac{4}{\pi} \frac{|AC||BC|}{|EF|}
\]
and
\[
\frac{|AC||BC|}{|EF|} = \frac{|AC|}{2\tan \frac{B}{2}} = \frac{|AB| + |BC|}{2} = \frac{A + G}{2}
\]
by 13.

In order to show other inequalities for the Seiffert means we need the following lemma:

Lemma 1. Let \( \phi_t(x) = (1 - t) \sin x + t \tan x - x, 0 \leq t \leq 1 \). Then
(a) \( \phi_t(x) > 0 \) for \( x \in (0, \frac{\pi}{2}) \) if and only if \( t \geq \frac{1}{3} \).
(b) \( \phi_t(x) < 0 \) for \( x \in (0, \frac{\pi}{4}) \) if and only if \( t \leq \frac{\pi - \sqrt{2}}{\frac{3\sqrt{2}}{2}} \).
(c) \( \phi_t(x) < 0 \) for \( x \in (0, \frac{\pi}{4}) \) if and only if \( t \leq \frac{\pi - \sqrt{2}}{\frac{3\sqrt{2}}{2}} \).

Proof. \( \phi_t'(x) = (1 - t) \cos x + t \cos^{-2} x - 1 \), so \( \phi_t'(0) = 0 \).

\[
\phi_t''(x) = 2t \sin x \left( \cos^{-3} x - \frac{1 - t}{2t} \right).
\]
From (20) we see that if \( t \geq \frac{1}{3} \) then \( \phi_t'' > 0 \) so \( \phi_t \) is convex, so from \( \phi_t(0) = 0 \) and \( \phi_t'(0) = 0 \) we deduce that \( \phi > 0 \). On the other hand if \( t < \frac{1}{3} \) then \( \phi \) is concave for small \( x \) hence is negative.

To prove (b) note that if \( t < \frac{1}{3} \) then \( \phi_t \) is concave and negative for \( x < x_0 \) and then becomes convex, so \( \phi \) has exactly one zero in \( (0, \frac{\pi}{4}) \). So \( \phi_t < 0 \) in \( (0, \frac{\pi}{4}) \) if and only if \( \phi_t(\frac{\pi}{4}) < 0 \), which holds for \( t \leq \frac{\pi - \sqrt{2}}{\frac{3\sqrt{2}}{2}} \).

Proof of (c) is exactly the same with \( \pi/4 \) replaced by \( \pi/8 \).

Consider now the points \( M_t = (1 - t)Q + tC \) and \( N_t = (1 - t)P + tA \). We have
\[
|M_tN_t| = (1 - t)|QP| + t|CA| = |BC|((1 - t) \sin B + t \tan B).
\]

Theorem 5.
\[
\frac{3}{P} < \frac{2}{A} + \frac{1}{G}
\]

Proof. From Lemma 1(a) we see that \( |M_tN_t| > |PC| \) holds for every triangle if and only if \( t \geq \frac{1}{3} \). (11) and (21) give
\[
\frac{1}{P} = \frac{|PC|}{|AC||BC|} < \frac{|M_tN_t|}{|AC||BC|} = \frac{(1 - t)|QP| + t|CA|}{|AC||BC|} = \frac{1 - t}{A} + \frac{t}{G}.
\]
The right hand side of this expression increases with \( t \), so the inequality in theorem is the strongest one.
Similarly let $R_t = (1 - t)C + tE$ and $S_t = (1 - t)P + tF$. Then

$$(22) \quad |R_tS_t| = (1 - t)|CP| + t|EF| = 2|BC|((1 - t)\sin\frac{B}{2} + t \tan\frac{B}{2}).$$

The formula is similar to (21) but $B/2$ varies from $0$ to $\pi/4$ and we can improve the inequalities (9)

**Theorem 6.**

$$(23) \quad \frac{1 - r_1}{\sqrt{A_A^2 + G_2^2}} + \frac{r_1}{A^2} < \frac{1}{\sqrt{\frac{2}{A_A^2 + G_2^2}}} + \frac{1/3}{A^2}$$

where $r_1 = \frac{\pi - 2\sqrt{2}}{4 - 2\sqrt{2}} \approx 0.2673035$.

**Proof.** As in the proof of the previous theorem we see from (22) and lemma 1(b), that for $t > \frac{1}{3}$ $|R_tS_t| > |PC|$ and for $t < r_1$ $|R_tS_t| < |PC|$.

Using (11), (22),(17) and (19) we obtain the desired estimations. \hfill \Box

Similar inequalities for the second Seiffert mean $T$ can be obtained in the same way by considering a triangle with sides

$|AB| = \sqrt{\frac{x^2 + y^2}{2}} = A_2 \quad |AC| = \frac{|x - y|}{2}, \quad |BC| = \frac{x + y}{2} = A$.

$A_2$ is called the root-square-mean.

In this case

$$(24) \quad T = \frac{|AC|}{\angle B} = \frac{|AC||BC|}{|PC|}$$

and we obtain similar results with $G$ and $A$ replaced with $A$ and $A_2$. Important difference between the two cases is that now $|AC| < |BC|$, so $0 < \angle B < \pi/4$ hence the constants in inequalities are different:

$$\frac{\pi}{4} < \frac{|PC|}{|AC|} < 1$$

$$1 < \frac{|PC|}{|PQ|} < \frac{\pi \sqrt{2}}{4}$$

$$1 < \frac{|PC|}{|PC|} < \frac{\pi}{4\sqrt{2} - \sqrt{2}}$$

$$\frac{\pi}{8(\sqrt{2} - 1)} < \frac{|PC|}{|EF|} < 1$$

which leads to
Theorem 7.

\[
A < T < \frac{4}{\pi}A
\]

\[
\frac{2\sqrt{2}}{\pi}A_2 < T < A_2
\]

\[
\frac{4\sqrt{2} - \sqrt{2}}{\pi} \sqrt{\frac{A_2 + A}{2}} < T < \sqrt{\frac{A_2 + A}{2}}
\]

\[
\frac{A_2 + A}{2} < T < \frac{8(\sqrt{2} - 1)A_2 + A}{\pi}
\]

\[
\frac{1 - r_1}{A_2} + \frac{r_1}{A} < T < \frac{2}{3} + \frac{1}{3}
\]

\[
\frac{1 - r_2}{\sqrt{A_2 A_2 + A}} + \frac{r_2}{A_2 + A} < T < \frac{2}{3} + \frac{1}{3}
\]

where \( r_1 = \frac{\pi - 2\sqrt{2}}{4 - 2\sqrt{2}} \approx 0.2673035 \) and \( r_2 = \frac{\pi - 4\sqrt{2 - \sqrt{2}}}{4[2(\sqrt{2} - 1) - \sqrt{2 - \sqrt{2}}]} \approx 0.3176533 \)

References