The sum-of-divisors minimum and maximum functions

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1. Let \( f : \mathbb{N}^* \to \mathbb{N} \) be a given arithmetic function, and \( A \subset \mathbb{N}^* \) a given set. The arithmetic function

\[
F_f^A(n) = \min\{k \in A : n | f(k)\}
\]

has been introduced in [7] and [6]. For \( A = \mathbb{N}^* \), \( f(k) = k! \) one obtains the Smarandache function; for \( A = \mathbb{N}^* \), \( A = P = \{2, 3, 5, \ldots\} = \) set of all primes, one obtains a function

\[
P(n) = \min\{k \in P : n | k!\}
\]

For properties of this function, see [7], [6].

For \( A = \{k^2 : k \in \mathbb{N}^*\} = \) set of perfect squares, and \( f(k) = k! \) one obtains the function

\[
Q(n) = \min\{m^2 \geq 1 : n | (m^2)!\},
\]

while for \( A = \) set of squarefree numbers \( \geq 1 \), \( f(k) = k! \) we get

\[
Q_1(n) = \min\{m \geq 1 \text{ squarefree}: n | m!\}
\]

For properties of \( Q(n) \) and \( Q_1(n) \), see [11].
The "dual" function of (1) has been defined by

\[ G_g^A(n) = \max \{ k \in A : g(k) | n \} \]  \hspace{1cm} (5)

where \( g : \mathbb{N}^* \to \mathbb{N} \) is a given function. Particularly for \( A = \mathbb{N}^* \), \( g(k) = k! \) one obtains the dual of the Smarandache function

\[ S_*(n) = \max \{ k \geq 1 : k! | n \} \]  \hspace{1cm} (6)

For properties of this function, see [7], [6]. F. Luca [4], K. Atanassov [1] and M. Le [2] have proved in the affirmative a conjecture of the author stated in [7] and [6].

For \( A = \mathbb{N}^* \), \( f(k) = g(k) = \varphi(k) \) (where \( \varphi \) is Euler’s totient) in (1), resp. (5) one obtains the Euler minimum, resp. maximum-functions, defined by

\[ E(n) = \min \{ k \geq 1 : n | \varphi(k) \} \]  \hspace{1cm} (7)
\[ E_*(n) = \max \{ k \geq 1 : \varphi(k) | n \} \]  \hspace{1cm} (8)

For properties of these functions, see [5], [8].

When \( A = \mathbb{N}^* \), \( f(k) = d(k) = \) number of divisors of \( k \), one has the divisor minimum function (see [7], [6], [9]):

\[ D(n) = \min \{ k \geq 1 : n | d(k) \} \]  \hspace{1cm} (9)

It is interesting to note that the divisor maximum function (i.e. the "dual" of \( D(n) \)) given by

\[ D_*(n) = \max \{ k \geq 1 : d(k) | n \} \]  \hspace{1cm} (10)

is not well-defined! Indeed, for any prime \( p \) we have \( d(p^{n-1}) = n \), and \( p^{n-1} \) is unbounded as \( p \to \infty \). When \( A \) is a finite set, however,

\[ D_*(n) = \max \{ k \in A : d(k) | n \} \]  \hspace{1cm} (11)
does exist.

When \( A = \mathbb{N}^* \), \( f(k) = g(k) = S(k) = \min\{m \geq 1 : k | m!\} \) (Smarandache function) one obtains the Smarandache minimum and maximum functions, given by

\[
S_{\min}(n) = \min\{k \geq 1 : n | S(k)\}, \tag{12}
\]
\[
S_{\max}(n) = \max\{k \geq 1 : S(k) | n\}. \tag{13}
\]

These functions have been introduced and studied recently in [10].

2. Let \( \sigma(n) \) be the sum of divisors of \( n \). The function

\[
\Sigma(n) = \min\{k \geq 1 : n | \sigma(k)\} \tag{14}
\]

has been introduced in [7], [6] (denoted there by \( F_\sigma \)). Let \( k \) be a prime of the form \( k = an - 1 \), where \( n \geq 1 \) is given. By Dirichlet’s theorem on arithmetical progressions, such a prime does exist. Then clearly \( \sigma(k) = an \), so \( n | \sigma(k) \), and \( \Sigma(n) \) is well defined.

The dual of \( \Sigma(n) \) is

\[
\Sigma_*(n) = \max\{k \geq 1 : \sigma(k) | n\} \tag{15}
\]

Since \( \sigma(1) = 1 | n \) and \( \sigma(k) \geq k \), clearly \( \Sigma_*(n) \leq n \), so this function is correctly defined.

The aim of this note is the initial study of these functions \( \Sigma(n) \) and \( \Sigma_*(n) \).

Some values of \( \Sigma(n) \) are: \( \Sigma(1) = 1, \Sigma(2) = 3, \Sigma(3) = 2, \Sigma(4) = 3, \Sigma(5) = 8, \Sigma(6) = 5, \Sigma(7) = 4, \Sigma(8) = 7, \Sigma(9) = 10, \Sigma(11) = 43, \Sigma(12) = 6, \Sigma(13) = 9, \Sigma(14) = 12, \Sigma(15) = 8, \Sigma(16) = 21, \Sigma(17) = 67, \Sigma(18) = 10, \Sigma(19) = 37, \Sigma(20) = 19, \Sigma(21) = 20, \Sigma(22) = 43, \Sigma(23) = 137, \Sigma(24) = 14, \Sigma(25) = 149, \Sigma(26) = 45, \Sigma(27) = 34, \Sigma(28) = 12, \Sigma_*(1) = 1, \Sigma_*(2) = 1, \Sigma_*(3) = 2, \Sigma_*(4) = 3, \Sigma_*(5) = 1, \Sigma_*(6) = 5, \)

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\(\Sigma_s(7) = 4, \Sigma_s(8) = 7, \Sigma_s(9) = 2, \Sigma_s(10) = 1, \Sigma_s(11) = 1, \Sigma_s(12) = 11,\)
\(\Sigma_s(13) = 9, \Sigma_s(14) = 13, \Sigma_s(15) = 8, \Sigma_s(16) = 7, \Sigma_s(17) = 1, \Sigma_s(18) = 17, \Sigma_s(19) = 1, \Sigma_s(20) = 19, \Sigma_s(21) = 4, \Sigma_s(22) = 1, \Sigma_s(23) = 1,\)
\(\Sigma_s(24) = 23, \Sigma_s(25) = 1, \Sigma_s(26) = 9, \Sigma_s(27) = 2, \Sigma_s(28) = 12.\)

3. The first theoretical result gives informations on values of these functions at \(n = p + 1\), where \(p\) is a prime:

**Theorem 1.** If \(p\) is a prime, then

\[
\Sigma(p + 1) \leq p \leq \Sigma_s(p + 1)
\]  

(16)

**Proof.** Since \((p + 1)|\sigma(p) = p + 1\), by definition (14) one can write
\(\Sigma(p + 1) \leq p.\) Similarly, definition (15) gives (by \(\sigma(p) = (p + 1)|(p + 1)\))
\(\Sigma_s(p + 1) \geq p.\)

**Remark.** On the left side of (16) one can have equality, e.g. \(\Sigma(3) = 2,\)
\(\Sigma(6) = 5, \Sigma(8) = 7.\) But the inequality can be strict, as \(\Sigma(12) = 6 < 11,\)
\(\Sigma(18) = 10 < 17.\) For the right side of (16) however, one can prove the more precise result:

**Theorem 2.** For all primes \(p\), one has

\[
\Sigma_s(p + 1) = p
\]  

(17)

**Proof.** First we prove that for all \(n \geq 2\) we have

\[
\Sigma_s(n) \leq n - 1
\]  

(18)

Indeed, since \(\sigma(k)|n\), clearly we must have \(\sigma(k) \leq n.\) On the other hand, for all \(k \geq 2\) we have \(\sigma(k) \geq k + 1\) (with equality only for \(k =\) prime), so \(k \leq n - 1,\) and this is true for all \(k;\) so (18) follows.

Let now \(n = p + 1 \geq 3\) in (18). Then \(\Sigma_s(p + 1) \leq p,\) which combined
with (16) implies relation (17).
Theorem 3. Let $p$ be a prime and suppose that

$$(p + 1)|n$$

Then

$$\Sigma_*(n) \geq p$$

Proof. Indeed, by $\sigma(p) = (p + 1)|n$, and definition (15), relation (20) follows. By letting $p = 2, 3, 5, 7, 11$ one gets:

- Corollary. If $3|n$, then $\Sigma_*(n) \geq 2$.
- If $4|n$, then $\Sigma_*(n) \geq 3$.
- If $6|n$, then $\Sigma_*(n) \geq 5$.
- If $8|n$, then $\Sigma_*(n) \geq 7$.
- If $12|n$, then $\Sigma_*(n) \geq 11$.

Remark. If $7|n$, then $\Sigma_*(n) \geq 4$.

Indeed, $\sigma(4) = 7|n$.

If $15|n$, then $\Sigma_*(n) \geq 8$.

Indeed, $\sigma(8) = 15|n$.

It is immediate that $\Sigma(n) = 1$ only for $n = 1$. On the other hand, there exist many integers $m$ with $\Sigma_*(m) = 1$.

Theorem 4. Let $p$ be a prime such that

$$p \notin \sigma(\mathbb{N}^*)$$

Then

$$\Sigma_*(p) = 1$$

Proof. Remark that $\sigma(k)|p \iff \sigma(k) = 1$ or $\sigma(k) = p$. Now, if (28) is true, then the equation $\sigma(k) = p$ is impossible for all $k \geq 1$, so $\sigma(k) = 1$, i.e. $k = 1$, giving relation (29).
For example, \( p = 17, 19, 23 \) satisfy relation (28).

**Theorem 5.** If for all \( d > 1 \), \( d \mid n \) one has

\[
d \not\in \sigma(\mathbb{N}^*),
\]

then

\[
\Sigma_n(n) = 1
\]

**Proof.** Let \( d > 1 \), \( d \mid n \). If \( d \not\in \sigma(\mathbb{N}^*) \), then the equation \( \sigma(k) = d \) is impossible. But then \( \sigma(k) \mid n \) is also impossible for \( \sigma(k) > 1 \), yielding (31).

For example, \( n = 10, 22, 25 \) satisfy relation (30).

**Theorem 6.** Let \( n \) be odd and suppose that \( \Sigma_n(n) \neq 1, 2 \). Then

\[
\Sigma_n(n) \leq \left( \frac{-1 + \sqrt{3 + 4n}}{2} \right)^2
\]

**Proof.** We use the following well-known results:

**Lemma 1.** \( \sigma(k) \) is odd iff \( k = m^2 \) or \( k = 2^\alpha m^2 \), where \( \alpha \geq 1 \) and \( m \) is an odd integer.

**Proof.** Let \( k = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \). Then

\[
\sigma(k) = (1 + p_1 + \cdots + p_1^{\alpha_1}) \cdots (1 + p_r + \cdots + p_r^{\alpha_r}).
\]

If \( k \) is odd, the \( \sigma(k) \) is odd if each term \( 1+p_1+\cdots+p_1^{\alpha_1}, \ldots, 1+p_r+\cdots+p_r^{\alpha_r} \) is odd, and since \( p_i (1 = 1, r) \) are all odd numbers, we must have \( \alpha_1 = \text{even}, \ldots, \alpha_r = \text{even} \). This gives \( k = m^2 \), with \( m = \text{odd} \). When \( k \) is even, then \( k = 2^\alpha p_1^{\alpha_1} \cdots p_r^{\alpha_r} \), and since \( \sigma(2^\alpha) = 2^\alpha + 1 = \text{odd} \), by the same argument as above, \( k = 2^\alpha m^2 \), with \( m = \text{odd} \).

**Lemma 2.** If \( k \) is composite, then

\[
\sigma(k) \geq k + \sqrt{k} + 1
\]
Proof. Write $k = ab$, where $1 < a \leq b < k$. Then $k \leq b^2$, so $b \geq \sqrt{k}$, implying $\sigma(k) \geq 1 + b + k \geq 1 + \sqrt{k} + k$, i.e. relation (34). When $k = p^2$, with $p$ an odd prime, one has equality since $\sigma(p^2) = p^2 + p + 1$.

Now, if $\sigma(k)|n$ and $n$ is odd, then clearly $\sigma(k)$ must be odd, too. Now, by (33) this is possible only when $k = m^2$ or $k = 2^a m^2$, with $m \geq 1$ odd. If $m > 1$, then $k = m^2$ is composite, while if $m = 1$ in $k = 2^a m^2$, then $k = 2^a$ is prime only if $a = 1$, i.e. if $k = 2$. Supposing $k \neq 1, 2$ then $k$ is always composite, so $\sigma(k) \geq k + \sqrt{k} + 1$. Since $\sigma(1) \leq n$, we get $k + \sqrt{k} + 1 - n \leq 0$ so $\sqrt{k} \leq \frac{-1 + \sqrt{-3 + 4n}}{2}$, and this gives (32).

Remark. For example, by (26), for $7|n$, $n$ odd, (32) is true.

Theorem 7. If $n \geq 4$, then $\Sigma(n) \geq 3$. For all $n \geq 4$,
\[ \Sigma(n) > n^{2/3} \] (35)

Proof. $\Sigma(n) = 1$ iff $n|1$, when $n = 1$. For $\Sigma(n) = 2$ we have $\sigma(2) = 3$ so $n|3 \iff n = 1, 3$. Thus for $n \geq 4$, we have $k = \Sigma(n) \geq 3$. Now, if $n|\sigma(k)$, then clearly $n \leq \sigma(k)$. Let $k \geq 3$. Then, it is known (see [3]) that
\[ \sigma(k) < k\sqrt{k} \] (36)

By $n < k\sqrt{k} = k^{3/2}$, inequality (35) follows.

Corollary. For all $m \geq 2$ (left side), and $m \geq 1$ (right side):
\[ (2^{m+1} - 1)^{2/3} < \Sigma(2^{m+1} - 1) \leq 2^m \] (37)

Proof. $2^{m+1} - 1 > 4$ for $m \geq 2$, and the left side is a consequence of (35). Now, the right side follows by $(2^{m+1} - 1)|\sigma(2^m)$, since $\sigma(2^m) = 2^{m+1} - 1$, and apply definition (14).
Theorem 8. Let \( f : [1, \infty) \to [1, \infty) \) be given by \( f(x) = x + x \log x \). Then for all \( n \geq 1 \),

\[
\Sigma(n) \geq f^{-1}(n),
\]  

where \( f^{-1} \) is the inverse function of \( f \).

Proof. \( \sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{n}{d} = n \sum_{d|n} \frac{1}{d} \leq n \sum_{1 \leq d \leq n} \frac{1}{d} \leq n(1 + \log n) \)
as it is well known that \( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \log n \) for all \( n \geq 1 \). Thus if \( n|\sigma(k) \), then \( n \leq \sigma(k) \leq f(k) \), so (38) follows. The function \( f \) is strictly increasing and continuous, so it is bijective, having an inverse function \( f^{-1} : [1, \infty) \to [1, \infty) \).

Remark. The inequality \( f(x) < x\sqrt{x} \), i.e. \( \log x < \sqrt{x} - 1 \) is true for \( x \) sufficiently large (e.g. \( x \geq e^3 \)). Indeed, let \( g(x) = \sqrt{x} - \log x - 1 \), when \( g(e^3) = e^{3/2} - 4 > 0 \) by \( e^3 \approx 19.6 > 4^2 = 16 \), and \( g'(x) = \frac{\sqrt{x} - 2}{2x} > 0 \) for \( x > 4 \). So \( g(x) \geq g(e^3) > 0 \) for \( x \geq e^3 \). Thus \( x + x \log x < x\sqrt{x} \). By putting \( x = n^{2/3} \) we get \( f(n^{2/3}) < n \), i.e. for \( n^{2/3} \geq e^3 \) (\( m \geq e^{9/2} \)) we get:

\[
f^{-1}(n) > n^{2/3} \text{ for } n \geq e^{9/2}
\]

which improves, by (38), inequality (35).

For values of \( \Sigma(n) \) and \( \Sigma_*(n) \) at primes \( n = p \) the following is true:

Theorem 9. For all primes \( p \geq 5 \),

\[
1 \leq \Sigma_*(p) \leq p - 2
\]

and

\[
\Sigma_*(p) \leq \left(\frac{-1 + \sqrt{-3 + 4p}}{2}\right)^2
\]

Proof. The inequality \( \Sigma_*(n) \geq 1 \) is true for all \( n \) (but remains an Open Problem the determination of all \( n \) with equality). Now, remark that \( \sigma(k)|p \iff \sigma(k) = 1 \) or \( \sigma(k) = p \). If \( \sigma(k) > 1 \), then by \( \sigma(k) \geq k + 1 \)
we get \( k \leq p - 1 \). But we cannot have equality, since then \( k = q = \text{prime} \), when \( \sigma(q) = q + 1 = p \geq 5 \) and this is impossible, since \( q + 1 \) is even for \( q \geq 3 \), while for \( q = 2 \), \( q + 1 = 3 < 5 \). Thus \( k \leq p - 2 \), so (40) follows. By applying the inequality \( \sigma(k) \geq k + \sqrt{k} + 1 \) (see (34)) then one arrives at (41), which is sharp, since e.g. \( \Sigma_s(7) = 4 \leq 4 \).

**Theorem 10.** For all Mersenne primes \( p \) one has

\[
\Sigma(p) \leq \frac{p + 1}{2}
\]  

(42)

**Proof.** This follows from the right side of (37), by remarking that when \( p = 2^{m+1} - 1 \) is a prime, by \( \Sigma(2^{m+1} - 1) \leq 2^m = \frac{p + 1}{2} \) we get (42).

**References**


