THE SOLUTION OF AN OPEN QUESTION

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Abstract. In this short note, we obtain the solution of OQ.1277:
The inequality
\[ I > \frac{\alpha A + \beta G}{\alpha + \beta} > (A^\gamma G^\delta)^{\frac{1}{\gamma + \delta}} > \sqrt{AG} \]
holding if and only if \( 1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2 \) for \( \alpha > \beta > 0 \), and \( \gamma > \delta > 0 \), where \( a > b > 0 \), and
\[ I = I(a, b) = \frac{1}{e} \left( \frac{a^a}{b^b} \right)^{\frac{1}{a-b}}, A = A(a, b) = \frac{a+b}{2}, G = G(a, b) = \sqrt{ab}. \]

1. Introduction

In [1], an interesting and open question is posed by Mihály Bencze. That is
**OQ.1277.** Determine all \( \alpha > \beta > 0 \) and \( \gamma > \delta > 0 \), such that
\[ (1.1) \quad I > \frac{\alpha A + \beta G}{\alpha + \beta} > (A^\gamma G^\delta)^{\frac{1}{\gamma + \delta}} > \sqrt{AG}, \]
where \( a > b > 0 \), and
\[ (1.2) \quad I = I(a, b) = \frac{1}{e} \left( \frac{a^a}{b^b} \right)^{\frac{1}{a-b}}, A = A(a, b) = \frac{a+b}{2}, G = G(a, b) = \sqrt{ab}. \]

In this short note, we obtain the solution of OQ.1277.

**Theorem 1.1.** The inequalities [1.1] holding if and only if \( 1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2 \) for \( \alpha > \beta > 0 \), and \( \gamma > \delta > 0 \).

2. Lemma

In order to prove Theorem 1.1 above, we require some lemmas.

**Lemma 2.1.** If \( x > y > 0 \), and \( \gamma > \delta > 0 \). We then have the inequality
\[ (2.1) \quad \left( x^\gamma y^\delta \right)^{\frac{1}{\gamma + \delta}} > \sqrt{xy}. \]

**Lemma 2.2.** (2) If \( x > y > 0 \), and \( \frac{\alpha}{\beta} > \frac{\gamma}{\delta} \) for \( \alpha > \beta > 0, \gamma > \delta > 0 \). Then the following inequality holds
\[ (2.2) \quad \frac{\alpha x + \beta y}{\alpha + \beta} > \left( x^\gamma y^\delta \right)^{\frac{1}{\gamma + \delta}}. \]

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Lemma 2.3. If \( x > y > 0 \), and \( \frac{\alpha}{\beta} < 2 \) for \( \alpha > \beta > 0 \). We then have the inequality

\[
\frac{2x + y}{3} > \frac{\alpha x + \beta y}{\alpha + \beta}.
\]

Lemma 2.4. \( (2) \)Heinz-Seiffert’s inequality

\[
(2.4) \quad I > \frac{2A + G}{3}
\]
holds for any \( a > 0 \), and \( b > 0 \).

3. The Proof of Theorem 1.1

Now, we prove Theorem 1.1

**Proof.** Firstly, for \( 1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2 \), \( \alpha > \beta > 0 \), and \( \gamma > \delta > 0 \), let \( x = A, y = G \), and combining Lemma 2.1-2.4, we obtain the inequalities (1.1).

Next, we prove that \( 1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2 \) (\( \alpha > \beta > 0 \), \( \gamma > \delta > 0 \)) are the best possible for (1.1). Assume the following inequalities have holden for any \( x > 1 \):

\[
(3.1) \quad I(x, 1) > \frac{\alpha A(x, 1) + \beta G(x, 1)}{\alpha + \beta} > \left( A^\gamma(x, 1)G^\delta(x, 1) \right)^{\frac{1}{\gamma + \delta}} > \sqrt{A(x, 1)G(x, 1)}.
\]

There is no harm in supposing \( 1 < x \leq 2 \) (In fact, if \( n < x \leq n + 1 \), we can take \( x = t + n \), where \( n \) is a positive integer). Setting \( x = t + 1 \), applying Taylor’s Theorem to the functions \( G(x, 1), A(x, 1) \), and \( I(x, 1) \), we have

\[
(3.2) \quad G(x, 1) = G(t + 1, 1) = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \cdots,
\]
\[
(3.3) \quad A(x, 1) = A(t + 1, 1) = 1 + \frac{1}{2}t,
\]
\[
(3.4) \quad I(x, 1) = I(t + 1, 1) = 1 + \frac{1}{2}t - \frac{1}{24}t^2 + \cdots.
\]

and

\[
(3.5) \quad \frac{\alpha A(x, 1) + \beta G(x, 1)}{\alpha + \beta} = 1 + \frac{1}{2}t - \frac{\beta}{8(\alpha + \beta)}t^2 + \cdots,
\]
\[
(3.6) \quad \left( A^\gamma(x, 1)G^\delta(x, 1) \right)^{\frac{1}{\gamma + \delta}} = 1 + \frac{1}{2}t - \frac{\delta}{8(\gamma + \delta)}t^2 + \cdots,
\]
\[
(3.7) \quad \sqrt{A(x, 1)G(x, 1)} = 1 + \frac{1}{2}t - \frac{1}{16}t^2 + \cdots.
\]

With simple manipulations (3.4)-(3.7), together with (3.1), yield

\[
(3.8) \quad -\frac{1}{24} > -\frac{\beta}{8(\alpha + \beta)} > -\frac{\delta}{8(\gamma + \delta)} > -\frac{1}{16}.
\]

From (3.8), it immediately follows that

\[
1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2(\alpha > \beta > 0, \gamma > \delta > 0).
\]

The proof of Theorem 1.1 is completed. □
References


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